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SUMAR

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NOTES ON SUBDIRECT PRODUCTS OF MODULES

BY

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§ 1. **Introduction.** Among the submodules of cartesian products of modules, the subdirect products play an important role, although their structure is rather complicated, since a subdirect product of modules $M_i (i \in I)$ is not uniquely determined by the components M_i. Starting with some well known results, we give some properties of subdirectly irreducible R-modules and of an irredundant representation of a module M as subdirect product of subdirectly irreducible modules, if M is Noetherian. In § 3 we give a necessary and sufficient condition therefore that a subdirect product is a special subdirect product. In § 4 we give an application of special subdirect products, while § 5 is devoted to relations between subdirect product and injectivity.

§ 2. **General properties of subdirect products.** In the following, only unitary left R-modules are considered.

A subdirect product of R-modules $\{M_i\}_{i \in I}$ is a submodule M of the direct product $\prod_{i \in I} M_i$, with the property that the canonical projections $\pi_i : M \to M_i$ are epimorphisms. The usual notation: $M = \underset{i \widetilde{\in} I}{\times} M_i$. The direct sum and the direct product of a set of modules M_i $(i \in I)$ are both subdirect products of the M_i, proving that a subdirect product of the $M_i (i \in I)$ is not uniquely determined by the M_i.

Let $\{M_i\}_{i \in I}$ be a family of R-modules, F an R-module and $\alpha_i : M_i \to F (i \in I)$ an epimorphism: then we define the *special subdirect product* $M = \underset{i \widetilde{\in} I}{\times} M_i(\alpha_i ; F)$ as the set of all elements $(m_i) \in \prod_{i \in I} M_i$, such that $\alpha_i m_i = \alpha_j m_j$ for all $i, j \in I$. A special subdirect product is therefore a subdirect product of the $M_i (i \in I)$ with *common factor module*. The direct product of a set of modules $M_i (i \in I)$ is always a special subdirect product (with $F = (0)$), the direct sum $\bigoplus_{i \in I} M_i$ is (if I is not finite!) not a special subdirect product of the M_i.

It is well known that a subdirect product of *two* modules is a special subdirect product, i. e. there exists a module F and two epimorphisms α_1 and α_2, such that $M = M_1 \mathbin{\widetilde{\times}} M_2$ is exactly the set of all pairs $(m_1, m_2) \in M_1 \times M_2$, with $\alpha_1 m_1 = \alpha_2 m_2$ [1].

If $M = \underset{i \widetilde{\in} I}{\times} M_i$, $\pi_i : M \to M_i$ and Ker $\pi_i = K_i (i \in I)$, then $\bigcap_{i \in I} K_i = (0)$; if conversely an R-module M contains a system $\{K_i\}_{i \in I}$ of submodules such that $\bigcap_{i \in I} K_i = (0)$, then M is isomorphic to a subdirect product of the factor modules $\{M/K_i\}_{i \in I}$.

A representation of an R-module M as a subdirect product $M = \underset{i \widetilde{\in} I}{\times} M_i$ is called trivial, if at least one of the natural projections $\pi_i : M \to M_i$ is an R-isomorphism. The R-module M is called *subdirectly irreducible*, if any representation of M as a subdirect product is trivial. An R-module $M(\neq (0))$ is subdirectly irreducible if and only if the intersection of all non-zero submodules of M is itself a non-zero submodule of M.

The importance of the subdirectly irreducible R-modules appears from the fact that every R-module M can be represented as a subdirect product of subdirectly irreducible R-modules.

Theorem 2. 1. *The R-module M is subdirectly irreducible if and only if M is an essential extension of a simple R-module.*

Proof. If M is an essential extension of a simple R-module S, then any non-zero submodule M' of M must contain S, hence M is subdirectly irreducible. Conversely, if M is subdirectly irreducible, then the intersection of all non-zero submodules M_i of M must be a non-zero simple submodule S of M; any non-zero submodule $M' \subset M$ contains S, hence M is essential extension of S. We have therefore the result: if M is subdirectly irreducible and containing the (only) simple submodule S, then

$$S \subset M \subset \hat{S}.$$

where $\hat{S}$ is the injective hull of S. From the last result it also follows that a subdirectly irreducible R-module M is indecomposable; in particular, an injective subdirectly irreducible module is injective and indecomposable.

In the case of an Artinean ring (resp. Artinean R-module) we have:

Theorem 2.2. *If R is an Artinean ring then an indecomposable injective R-module is subdirectly irreducible.*

Proof. If R is Artinean, then every R-module $(\neq (0))$ has a simple submodule. If M is moreover indecomposable and injective, then $M = \hat{S}$ for some simple submodule S of M, i. e. M is subdirectly irreducible.

As a consequence, in the case of an Artinean ring R (resp. Artinean R-module) we have the equivalence.

M is injective subdirectly irreducible $\leftrightarrow$ M is injective indecomposable. If R is an Artinean ring, then every injective R-module is a direct sum of a family of injective hulls of simple modules. Since an injective hull of a simple module S is subdirectly irreducible, we have

Theorem 2.3. *If R is an Artinean ring, then any injective R-module is a direct sum of injective subdirectly irreducible R-modules.*

Remark. If $R = Z$, then the simple abelian groups are of type $Z(p)$, p being a prime: the subdirectly irreducible abelian groups therefore are $Z(p^k)$, $k = 1, 2, \ldots, \infty$, p a prime. If the system $\{K_i\}_{i\in I}$ of submodules of M, with

$$(1) \qquad \bigcap_{i\in I} K_i = 0$$

has the property that no K_i contains the intersection of the remaining ones, the system $\{K_i\}_{i\in I}$ is called *irredundant*, and then $\bigcap_{j\neq i} K_j \neq 0 (\forall i \in I)$, i. e. none of the K_i's can be omitted in (1). In that case we say that the corresponding representation of M as subdirect product of the $M_i = M/K_i (i \in I)$ is an *irredundant representation* of M as subdirect product. It is clear that if M can be represented as a subdirect product by means of the finitely many modules $M_i = M/K_i (i = 1, \ldots, n)$, then the corresponding set of submodules $K_1, \ldots, K_n$ can be replaced by an irredundant system $K_{i_1}, \ldots, K_{i_j}$ by the simple process of successively striking out those K_i which contain the intersection of the remaining ones. Indeed, it is clear that in this situation

$$M \cong M/K_1 \tilde{\times} \ldots \tilde{\times} M/K_n \cong M/K_{i_1} \tilde{\times} \ldots \tilde{\times} M/K_{i_j};$$

this can be shown by mapping

$$\varphi : (m + K_1, \ldots, m + K_n) \to (m + K_{i_1}, \ldots, m + K_{i_j}), \quad m \in M.$$

According to the fact that we have replaced the set $\{K_1, \ldots, K_n\}$ by the irredundant system $\{K_{i_1}, \ldots, K_{i_j}\}$, φ is an isomorphism. Now we can say something of the number of submodules $K_i (i \in I)$ used for an irredundant representation of the R-module M as a subdirect product of subdirectly irreducible modules, assuming that M is a Noetherian R-module. This results from the more general case of an algebra whose congruence relations are permutable and satisfy the ascending chain condition (see: G. Birkhoff, Lattice Theory, A.M.S. Coll. Publ.). Therefore we need the notion of irreducibility: if A is a submodule of M, A is said to be an „irreducible submodule" of M, if (a) $A \neq M$, and (b) there do not exist submodules A_1, A_2 of M, $A_1 \supsetneq A$, $A_2 \supsetneq A$ and $A_1 \cap A_2 = A$. A simple argument shows that if M satisfies the ascending chain condition for submodules, then every submodule $A (\neq M)$ has a representation as an intersection of finitely many irreducible submodules A_i.

If the R-module M is subdirectly irreducible, the zero submodule is irreducible.

Now suppose that the module M satisfies the ascending chain condition for submodules and that we have for the submodule $A (\neq M)$ two irredundant representations as intersections of irreducible components:

$$(2) \qquad A = B_1 \cap B_2 \cap \ldots \cap B_n = B'_1 \cap B'_2 \cap \ldots \cap B'_{n'}.$$

Then we have the following two statements:

(i) one can substitute for any B_i a suitable B'_j and get a new irredundant representation of A as intersection of irreducible components;

(ii) the number of factors in the representations (2) is the same for all such representations.

As a consequence we have the

Theorem 2.4. *If M is a Noetherian R-module, then the number of factors in any irredundant representation of M as a subdirect product of subdirectly irreducible modules is the same for all such representations.*

This theorem is a direct consequence of the mentioned statement of G. Birkhoff.

§ 3. Special subdirect products. We want to derive necessary and sufficient conditions under which a subdirect product is special. If $M = \tilde{\times}_{i\in I} M_i$, we denote by $C_i = M \cap M_i (i \in I)$: therefore $(0) \subseteq C_i \subseteq M_i (i \in I)$. If $M = \prod_{i\in I} M_i$, or $M = \bigoplus_{i\in I} M_i$, we have in both cases $C_i = M_i (\forall i \in I)$; if M is the submodule of $\bigoplus_{i\in I} M_i$ consisting of those elements $m = (\ldots, m_i, \ldots)$ with $\sum_{i\in I} m_i = 0$, then $C_i = 0 (\forall i \in I)$. If $M = \tilde{\times}_{i\in I} M_i$, $C_i = M \cap M_i$, define $C = M \cap \prod_{i\in I} C_i$; then $(0) \subseteq C \subseteq M$. In the cases $M = \prod_{i\in I} M_i$ or $M = \bigoplus_{i\in I} M_i$, we have $C = M$.

If $M = \tilde{\times}_{i\in I} M_i$, define $D_i = M \cap \prod_{j\neq i} M_j$. If $M \cap M_i = M_i$ for some $i \in I$, then we have $M = C_i \oplus D_i$. This happens e. g. in the cases of the direct product and the direct sum. Conversely: if for a certain $i \in I$ we have $M = C_i \oplus D_i$, then $C_i = M \cap M_i$.

In the case of a special subdirect product $M = \tilde{\times}_{i\in I} M_i (\alpha_i : F)$ we have $C_i = \mathrm{Ker}(\alpha_i)$, $(i \in I)$, furthermore $C = \prod_{i\in I} \mathrm{Ker}(\alpha_i)$ (remark that M is the submodule of $\prod_{i\in I} M_i$ of all elements $m = (\ldots, m_i, \ldots) \in \prod_{i\in I} M_i$ with the projection property. Moreover we have for a special subdirect product: $D_i \subseteq C (\forall i \in I)$. But for $M = \bigoplus_{i\in I} M_i$ we have also $D_i \subseteq C$, therefore this condition is not a sufficient condition for a subdirect product to be a special subdirect product.

Theorem 3.1. *Let $M = \tilde{\times}_{i\in I} M_i$ be a subdirect product with the following properties:*

(1) $D_i \subseteq C (\forall i \in I)$; (2) not for all $i \in I$ we have $C_i = M_i$; then there exists an R-module $F \neq (0)$ and epimorphisms $\alpha_i : M_i \to F (i \in I)$ such that M is exactly the submodule of $\prod_{i\in I} M_i$ of the elements $m = (\ldots, m_i, \ldots, m_j, \ldots)$ with the property

$$\dots = \alpha_i m_i = \dots = \alpha_j m_j = \dots$$

Remark. The condition (2) excludes the cases of the direct product and the direct sum; the only special subdirect product with $F=(0)$ is the direct product.

Proof. From $\pi_i m = m_i$ it follows $M/D_i \cong M_i$ $(i\in I)$, where the isomorphism is given by $m_i \leftrightarrow m+D_i$, if $m\in M$, $\pi_i m = m_i$. Now we have

$$M/C \cong M/D_i/C/D_i \cong M_i/C_i \ (\forall i\in I).$$

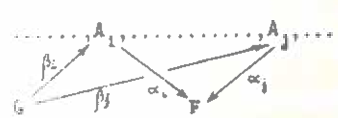
Since C_i is not equal M_i for all $i\in I$, we have that C_i is a *proper* submodule of M_i for at least one $i\in I$. But then (2) implies that $M/C\neq(0)$, and moreover that $C_i \underset{\neq}{\subset} M_i$ for *all* $\in I$. If we define $M/C=F$, then $F\neq(0)$ and $M_i/C_i\cong F(\forall i\in I)$. Defining $\alpha_i : M_i\to F$ by $\alpha_i(m_i)=m+C$, if $m\in M$, $\pi_i m = m_i$, we see that α_i is well defined, for $\alpha_i(m_i)=m'+C$ implies $m-m'\in D_i\subset C$. For an element $m=(\dots, m_i, \dots, m_j, \dots)\in M$, we have therefore $\alpha_i m_i = \alpha_j m_j$ for all $i, j\in I$. Conversely, suppose $\bar m = (\dots, \bar m_i, \dots, \bar m_j, \dots)\in \prod_{i\in I} M_i$ has the projection property $\alpha_i \bar m_i = \alpha_j \bar m_j$ for all $i, j\in I$. Let $\alpha_i \bar m_i = m + C$, where $m\in M$, $\pi_i m = \bar m_i$ (and such an element $m\in M$ exists). Then $\alpha_j \bar m_j = \alpha_i \bar m_i = m + C$ and $\pi_j m = \bar m_j$, therefore $m = (\dots, \bar m_i, \dots, \bar m_j, \dots) = \bar m\in M$.

This proves that the conditions (1) and (2) of the theorem are necessary and sufficient conditions for a subdirect product to be a special subdirect product $\neq \prod_{i\in I} M_i$.

§ 4. Relations between $\mathrm{Hom}_R(M, N)$ and subdirect products. If $\{A_i\}_{i\in I}$ is a family of R-modules, $\alpha_i : A_i \to F (i\in I)$ a family of R-homomorphisms *into* a fixed R-module F, then the *subproduct* $A = \underset{i\in I}{\tilde\times} A_i(\alpha_i ; F)$ is the submodule of $\prod_{i\in I} A_i$ of all $a=(\dots, a_i, \dots)\in \prod_i A_i$ with $\alpha_i a_i = \alpha_j a_j$ (for all $i, j\in I$). It is clear that A is a special subdirect product if all α_i are epimorphisms onto F. We will study for any module G and a subproduct $A = \underset{i\in I}{\tilde\times} A_i(\alpha_i ; F)$ the structure of $\mathrm{Hom}_R(G, A)$.

Suppose that the $\beta_i : G\to A_i$ $(i\in I)$ satisfy

$$\alpha_i \,.\, \beta_i = \alpha_j \,.\, \beta_j \ (\forall i, j\in I);$$



then the $\{\beta_i\}_{i\in I}$ determine in a unique way an R-homomorphism $\beta\in \mathrm{Hom}_R(G, A)$ such that $\pi_i \,.\, \beta = \beta_i$ $(\forall i\in I)$, where $\pi_i : A\to A_i$ is the canonical projection of A onto A_i. Indeed, define $\beta g = (\dots, \beta_i g, \dots)\in A$. It is easy to prove that β is uniquely determined by the condition $\pi_i\beta = \beta_i (\forall i\in I)$.

Conversely, if $\beta\in \mathrm{Hom}_R(G, A)$, then β defines $\beta_i : G\to A_i$ by $\beta_i = \pi_i\beta$ $(i\in I)$, and we have $\alpha_i\beta_i = \alpha_j\beta_j$ $(\forall i, j\in I)$.

Let us denote by $\alpha_i : \mathrm{Hom}_R(G, A_i)\to \mathrm{Hom}_R(G, F)$, defined by $\alpha_i(\beta_i) = \alpha_i\beta_i$ $(i\in I)$, then a system $\{\beta_i\}_{i\in I}$, $\beta_i\in \mathrm{Hom}_R(G, A_i)$ determines uniquely an R-homomorphism $\beta : G\to A$ if and only if $\alpha_i(\beta_i) = \alpha_j(\beta_j)$ $(\forall i, j\in I)$. We have therefore the following result:

$$\mathrm{Hom}_R(G, \underset{i\in I}{\tilde\times} A_i(\alpha_i ; F)) \cong \underset{i\in I}{\tilde\times} \mathrm{Hom}_R(G, A_i)(\alpha_i, \mathrm{Hom}_R(G, F)). \tag{1}$$

With the same notation as in (1) we state the following

Theorem 4.1. *The right hand side of (1) is a special subdirect product for all special subdirect products $A = \underset{i\in I}{\tilde\times} A_i(\alpha_i ; F)$, if and only if G is a projective R-module.*

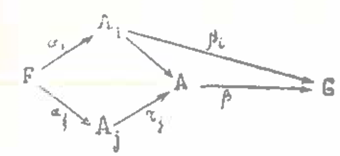
Proof. If G is projective, then the exactness of $A_i \xrightarrow{\alpha_i} F\to 0$ $(i\in I)$ implies the exactness of

$$\mathrm{Hom}_R(G, A_i) \xrightarrow{\alpha_i} \mathrm{Hom}_R(G, F)\to 0 (i\in I). \tag{2}$$

If $\beta_i\in \mathrm{Hom}_R(G, A_i)$, then $\alpha_i(\beta_i) = \alpha_i\beta_i\in \mathrm{Hom}_R(G, F)$, and by the exactness of (2) for all $i\in I$, there is a $\beta_j\in \mathrm{Hom}_R(G, A_j)$ with $\alpha_j(\beta_j) = \alpha_i(\beta_i)$. If therefore G is projective then the right hand side of (1) is a special subdirect product for all special subdirect products $A = \underset{i\in I}{\tilde\times} A_i(\alpha_I ; F)$.

Conversely, if G is such, that the right hand side of (1) is a special subdirect product for all special subdirect products $A = \underset{i\in I}{\tilde\times} A_i(\alpha_i ; F)$, then any $\beta_i\in \mathrm{Hom}_R(G, A_i)$ induces $\alpha_i\beta_i\in \mathrm{Hom}_R(G, F)$ and for all $j\in I$ there is a $\beta_j\in \mathrm{Hom}_R(G, A_j)$ with $\alpha_i\beta_i = \alpha_j\beta_j$. That implies that the sequence $\mathrm{Hom}_R(G, A_j) \xrightarrow{\alpha_j} \mathrm{Hom}_R(G, F)\to 0$ must be exact for all possible exact sequences $A_j \xrightarrow{\alpha_j} F\to 0$, hence G must be projective.

We now come to the dual case! Let $\{A_i\}_{i\in I}$ and F be R-modules and $\alpha_i : F\to A_i$ $(i\in I)$; then the *generalized pushout* (or *amalgamated sum*)



$$A = \underset{i\in I}{\times\!\!\!\!\!\times} A_i(\alpha_i : F)$$

is defined as the R-module $A = \bigoplus_{i\in I} A_i/H$, where H consists of the elements of $\bigoplus_{i\in I} A_i$ of the form $(\dots, \alpha_i(z_i), \dots, \dots, \alpha_j(z_j), \dots)$ with $z_i\in F (i\in I)$, such that $\sum_i z_i = 0$. If all α_i are monomorphisms, A is called a *special amalgamated sum*. If G is any module, we want to study $\mathrm{Hom}_R(A, G)$. Let $\beta_i : A_i\to G (i\in I)$ such that $\beta_i\alpha_i = \beta_j\alpha_j (\forall i, j\in I)$; then there is an R-homomorphism $\beta : A\to G$ induced by the $\{\alpha_i\}$ and $\{\beta_i\}$ with the property $\beta\tau_i = \beta_i (i\in I)$, where $\tau_i(a_i) = (0, \dots 0, a_i, 0\dots) + H$, $a_i\in A_i$. Indeed, define for $a = (\dots, a_i, \dots) + H$ the map β by $\beta(a) = \sum_i \beta_i(a_i)$; this definition of $\beta(a)$ does not depend upon the representation of $a\in A$. For if we add an element of H, e. g. $(\dots, \alpha_i(z_i), \dots)\in H$, $\sum_i z_i = 0$, then we have $\sum_i \beta_i\alpha_i(z_i) = 0$, since $\beta_i\alpha_i = \beta_j\alpha_j (\forall i, j\in I)$. Moreover we have $\beta\tau_i(a_i) = \beta_i(a_i)$, $a_i\in A_i$. Therefore the set $\{\beta_i\}_{i\in I}$, $\beta_i\in \mathrm{Hom}_R(A_i, G)$, with $\beta_i\alpha_i = \beta_j\alpha_j (\forall i, j\in I)$ defines in a unique way an element $\beta\in \mathrm{Hom}_R(A, G)$.

If, conversely, $\beta\in \mathrm{Hom}_R(A, G)$, we define $\beta_i : A_i\to G$ by

$$\beta_i(a_i) = \beta\tau_i(a_i) = \beta((0, \dots, a_i, \dots) + H),\ a_i\in A_i.$$

If $z\in F$, then (using the definition of H) $\beta_i\alpha_i(z) = \beta((0, \dots, \alpha_i(z), 0, \dots) + H) = \beta((0, \dots, \alpha_j(z), \dots) + H) = \beta_j \,.\, \alpha_j(z)$, hence $\beta_i \,.\, \alpha_i = \beta_j \,.\, \alpha_j (\forall i, j\in I)$. Denoting by α_i the map

$$\bar\alpha_i : \mathrm{Hom}_R(A_i, G)\to \mathrm{Hom}_R(F, G),$$

defined by

$$\bar\alpha_i(\beta_i) = \beta_i\alpha_i (i\in I),$$

we see that for $A = \underset{i\in I}{\times\!\!\!\!\!\times} A_i(\alpha_i ; F)$ we have

$$\mathrm{Hom}_R(A, G) \cong \underset{i\in I}{\tilde\times} \mathrm{Hom}_R(A_i, G)(\bar\alpha_i : \mathrm{Hom}(F, G)). \tag{3}$$

With the same notations as in (3) we prove the

Theorem 4.2. *The right hand side of (3) is a special subdirect product for all special amalgamated sums* $A=\underset{i\in I}{\dot{+}}A_i(\alpha_i\,;\,F)$, *if and only if G is an injective R-module.*

Proof. If G is injective, the diagram shows that for any $\eta\in \mathrm{Hom}_R(F, G)$ there exists

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\alpha_i"] \arrow[d, "\eta"'] & A_i \arrow[dl, dashed, "\beta_i"] \\ & G & \end{tikzcd}$$

a $\beta_i\in \mathrm{Hom}_R(A_i, G)$, such that $\bar{\alpha}_i(\beta_i)=\beta_i\alpha_i=\eta(\forall i\in I)$; moreover any set $\{\beta_i\}_{i\in I}$ with $\beta_i\in\mathrm{Hom}_R(A_i, G)$ determines in a unique way an R-homomorphism $\beta: A\to G$. If therefore G is injective, the right hand side of (3) is a special subdirect product. Conversely, if the right hand side is a special subdirect product of the $\mathrm{Hom}_R(A_i, G)\,(i\in I)$ for all special amalgamated sums $A=\underset{i\in I}{\dot{+}}A_i(\alpha_i\,;\,F)$, then for any $\eta\in\mathrm{Hom}_R(F, G)$ there exist R-homomorphisms $\beta_i\in\mathrm{Hom}_R(A_i, G)$, such that $\beta_i\alpha_i=\eta$ and this implies that G must be injective.

§ 5. Relations between subdirect product and injectivity. There are some well known relations between injectivity and direct products (resp. direct sums) of modules in the literature. We now pay attention to the relation between injectivity and subdirect products.

Theorem 5.1. *If the injective R-module M is a non-trivial subdirect product $M=A\underset{\sim}{\times}B$, then M is directly decomposable.*

Proof. Denoting the canonical epimorphisms by $\pi_1: M\to A$, $\pi_2: M\to B$, we have

$$\mathrm{Ker}\,\pi_1=M\cap B\neq 0,\quad \mathrm{Ker}\,\pi_2=M\cap A\neq 0.$$

Moreover $M\cap A\neq M$, $M\cap B\neq M$. We consider the injective hull $\widehat{M\cap A}$ in M, $E_1=\widehat{M\cap A}$. Then

$$M=E_1\oplus F_1, \tag{1}$$

where $E_1\neq 0$. Moreover $E_1\neq M$, for, otherwise M was an essential extension of $M\cap A$, but from

$$(M\cap A)\cap(M\cap B)=0$$

it followed that $M\cap B=0$! Therefore the decomposition (1) is a proper one, and both components are injective. We prove moreover: E_1 (as a submodule of M) projects monomorphically into A, and F_1 projects monomorphically into B. Indeed: let us denote the corresponding projections by π_1', resp. π_2'. Then $\mathrm{Ker}\,\pi_1'\subset M\cap B$, hence $\mathrm{Ker}\,\pi_1'\cap(M\cap A)=0$ and since $\widehat{M\cap A}$ is an essential extension of $M\cap A$, we have $\mathrm{Ker}\,\pi_1'=0$. Therefore E_1 is isomorphic with a submodule A_1 of A, or $M\cap A$ has an injective hull in A. Let $y\in F_1$, $\pi_2'(y)=0$, then $y\in A$, hence $y\in A\subset F_1\cup A\subset M\cup E_1$, and since $E_1\cap F_1=0$, we have $y=0$. Summarizing, we have: if $(a', b')\in\widehat{M\cap A}$, and $a'=0$, then $(a', b')=(0, 0)$; if $(a'', b'')\in F_1$, $b''=0$, then $(a'', b'')=(0, 0)$. An analogous result is

$$M=E_2\oplus F_2,\quad \text{where}\quad F_2=\widehat{M\cap B}, \tag{2}$$

where — in the same way — E_2 projects monomorphically into A, and F_2 into B.

We prove that $\widehat{M\cap A}\cap\widehat{M\cap B}=0$. Suppose that $(a, b)\in\widehat{M\cap A}\cap\widehat{M\cap B}$, $(a, b)\neq(0, 0)$, then it follows from the last results, that $a\neq 0$, $b\neq 0$. Using the properties of an essential extension, there must be elements $\lambda_1\neq 0$, $\lambda_2\neq 0$ in R such that

$$(0, 0)\neq(\lambda_1 a, \lambda_1 b)\in M\cap A,\quad (0, 0)\neq(\lambda_2 a, \lambda_2 b)\in M\cap B.$$

This gives a contradiction, for $(\lambda_1 a,\lambda_1 b)\in\widehat{M\cap A}\cap\widehat{M\cap B}$ and from $(\lambda_1 a,\lambda_1 b)\in M\cap A$ it follows $\lambda_1 b=0$ and therefore we must have $\lambda_1 a=0$, and similarly from $\lambda_2 a=0$ it follows $\lambda_2 b=0$. Therefore $\widehat{M\cap A}\cap\widehat{M\cap B}=0$. For the injective, non-trivial subdirect product $M=A\underset{\sim}{\times}B$ we have therefore the decomposition

$$M=\widehat{M\cap A}\oplus\widehat{M\cap B}\oplus K \tag{3}$$

for some (injective) R-module K. From (3) it follows

$$B\cong M/M\cap A\cong\widehat{M\cap A}/M\cap A\oplus\widehat{M\cap B}\oplus K, \tag{4}$$

proving once more that $M\cap B$ has an injective hull in B, etc.

Theorem 5.2. *If M is an injective non-trivial subdirect product $M=A\underset{\sim}{\times}B$, and B is torsionfree, then $M\cong(M\cap A)\oplus B$, and B is injective.*

Proof. If $(a, b)\in\widehat{M\cap A}$, $(a, b)\neq(0, 0)$, then $a\neq 0$. If $b=0$, then $(a, b)\in M\cap A$. If $b\neq 0$, then there exists an element $\lambda\in R$, $\lambda\neq 0$ such that $(0, 0)\neq(\lambda a, \lambda b)\in M\cap A$, hence $\lambda b=0$ and therefore $b=0$, contradiction! This implies that $(a, b)\in M\cap A$, or $\widehat{M\cap A}=M\cap A$, proving that $M\cap A$ is an injective direct summand of M. From $M/M\cap A\cong B$ it follows that B is an injective direct summand and from (4): $B\cong\widehat{M\cap B}\oplus K$.

Corollary 1. If M is an injective non-trivial subdirect product, $M=A\underset{\sim}{\times}B$, A and B being torsionfree, then $M=A\oplus B$, both A and B are injective. Indeed, from thm. 5.2. it follows that A and B are direct summands of M. Moreover (3) implies $M=A_0\oplus B_0\oplus K$, $A_0=M\cap A$, $B_0=M\cap B$, hence the kernels of the subdirect product are both injective modules.

Theorem 5.3. *If M is an injective non-trivial subdirect product $M=A\underset{\sim}{\times}B$, and A and B are indecomposable R-modules, then $M=A\oplus B$, and A and B are injective.*

Proof. From (4) it follows $A\cong\widehat{M\cap A}$, so A is direct summand of M. The same holds for B, hence $M=A\oplus B$.

Corollary 2. If M is an indecomposable injective R-module, then any representation of M as a subdirect product $M=A\underset{\sim}{\times}B$ is trivial.

Theorem 5.4. *Let M be a special subdirect product, $M=\underset{i\in I}{\underset{\sim}{\times}}M_i(\alpha_i\,;\,F)$, where the M_i are injective and F torsionfree; then M is injective.*

Proof. Let L be a left ideal of R, $\varphi: L\to M$ an R-homomorphism, and $\pi_i: M\to M_i$ the usual projection, then $\pi_i\varphi: L\to M_i$ can be extended to an R-homomorphism $\varphi_i: R\to M_i$, extending $\pi_i\varphi(i\in I)$. Let $\varphi_i(1)=m_i$, then for $\lambda\in L$ we have $\varphi_i(\lambda)=\pi_i\varphi(\lambda)=\lambda m_i\,(i\in I)$. Since $\varphi(\lambda)\in M$, we have $\varphi(\lambda)=(\ldots,\lambda m_i,\ldots,\lambda m_j,\ldots)$ with $\alpha_i(\lambda m_i)=\alpha_j(\lambda m_j)$, $\forall i, j\in I$; hence $\lambda(\alpha_i m_i-\alpha_j m_j)=0$ for all $\lambda\in L$, therefore $\alpha_i m_i=\alpha_j m_j$, since F is torsionfree and $L\neq(0)$. This means that $m=(\ldots, m_i,\ldots, m_j,\ldots)\in M$ and moreover $\varphi(\lambda)=\lambda m$, proving the injectivity of M.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\varphi"'] & R \arrow[ddl, dashed, "\varphi_i"] \\ & M \arrow[d, "\pi_i"'] & \\ & M_i & \end{tikzcd}$$

Theorem 5.5. *Let $M=M_1\underset{\sim}{\times}M_2\underset{\sim}{\times}\ldots\underset{\sim}{\times}M_k(\alpha_i\,;\,F)$ be a special subdirect product of the k modules $M_1,\ldots, M_k$, where F is torsionfree and injective. Then we can extend the epimorphisms $\alpha_i: M_i\to F$ to epimorphisms $\bar{\alpha}_i:\hat{M}_i\to F(i=1,\ldots, k)$, and we prove that $\hat{M}\cong\hat{M}_1\underset{\sim}{\times}\ldots\underset{\sim}{\times}\hat{M}_k(\bar{\alpha}_i\,;\,F)$.*

Proof. $N=\hat{M}_1\underset{\sim}{\times}\ldots\underset{\sim}{\times}\hat{M}_k(\bar{\alpha}_i\,;\,F)$ is an injective R-module N, containing M (thm. 5.4.); therefore we only have to prove that N is an essential extension of M. Let $\bar{m}=(\bar{m}_1,\ldots,\bar{m}_k)\in$

$\in N$, $\bar{m} \neq 0$, then $\bar{\alpha}_1 \bar{m}_1 = \ldots = \bar{\alpha}_k \bar{m}_k$. Suppose $\bar{m} \notin M$, then at least one of the components, e.g. $\bar{m}_1 \notin M_1$. Then there exists an element $\lambda_1 \in R$, $\lambda_1 \neq 0$, such that $0 \neq \lambda_1 \bar{m}_1 = m_1' \in M_1$. Then $\lambda_1 \bar{m} = (\lambda_1 \bar{m}_1, \ldots, \lambda_1 \bar{m}_k)$ satisfies

$$\bar{\alpha}_1(\lambda_1 \bar{m}_1) = \ldots = \bar{\alpha}_k(\lambda_1 \bar{m}_k),$$

or

$$\alpha_1(m_1') = \bar{\alpha}_2(\lambda_1 \bar{m}_2) = \ldots = \bar{\alpha}_k(\lambda_1 \bar{m}_k).$$

If $\lambda_1 \bar{m}_2 \notin M_2$, then we continue by multiplying with a suitable $\lambda_2 \in R$, $\lambda_2 \neq 0$, such that

$$0 \neq \lambda_2 \lambda_1 \bar{m}_2 = m_2' \in M_2,$$

hence

$$\lambda_2 \lambda_1 \bar{m} = (\lambda_2 m_1', m_2', \lambda_2 \lambda_1 \bar{m}_3, \ldots, \lambda_2 \lambda_1 \bar{m}_k).$$

Continuing this process we obtain, after a finite number of steps, an element $m \in M$,

$$m = \lambda_k \ldots \lambda_1 \bar{m} = (m_1, m_2, \ldots, m_k) \neq 0,$$

such that

$$\alpha_1 m_1 = \alpha_2 m_2 = \ldots = \alpha_k m_k.$$

That proves, that

$$N = \hat{M}_1 \underset{\sim}{\times} \ldots \underset{\sim}{\times} \hat{M}_k (\bar{\alpha}_1, \ldots, \alpha_k ; F)$$

is an injective essential extension of M, therefore N is an injective hull of M.

Theorem 5.6. *If M is an indecomposable injective R-module, then any representation of M as a subdirect product of the finitely many R-modules $M_1, \ldots, M_n$, $M = M_1 \underset{\sim}{\times} \ldots \times M_n$, is trivial.*

Proof. Suppose there existed a non-trivial representation $M = M_1 \underset{\sim}{\times} \ldots \underset{\sim}{\times} \widetilde{M}_n$ of M, then M contained a set of non-zero submodules $N_i, \ldots, N_n$ with $\bigcap_i N_i = 0$. We may suppose that the system $\{N_i\}$ is irredundant (no N_i contains the intersection of the remaining N_j). Denoting by $N_1' = \bigcap_{i \neq 1} N_i$, we have $N_1 \neq 0$, $N_1' \neq 0$ and $N_1 \cap N_1' = 0$. Then $M \cong M/N_1 \underset{\sim}{\times} M/N_1' = M_1 \underset{\sim}{\times} M_1'$, and the kernels N_1, N_1' are non-zero. Using the result of thm. 5.1. we find $M \cong \widehat{M \cap M_1} \oplus \widehat{M \cap M_1'} \oplus K$, contradicting the hypothesis that M is indecomposable. Therefore a non-trivial representation of M as a subdirect product of finitely many modules cannot exist.

Remark. If M is indecomposable and injective there may be a representation of M as a non-trivial subdirect product, but not with a finite number of components. Take e. g. the injective Z-module Q and define for the prime number $p : Q_p = \{ m/p^n \mid m, n \in Z \}$, then $Q = \bigoplus_p Q_p$.

LITERATURE

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NOTE DESPRE PRODUSE SUBDIRECTE DE MODULE

Rezumat

În lucrare autorul stabileşte unele proprietăţi ale R-modulelor subdirect ireductibile şi dă o reprezentare iredundantă a unui modul M ca produs subdirect de module subdirect ireductibile, dacă M este noetherian. În § 3 se dă o condiţie necesară şi suficientă pentru ca un produs subdirect să fie un produs subdirect special. Paragraful 4 este consacrat unei aplicaţii a produselor subdirecte speciale iar paragraful 5 studiului relaţiei dintre produsul subdirect şi injectivitate.

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REMARKS ON TWO GEOMETRIC CONJECTURES OF L. FEJES TÓTH

BY

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Recently Professor L. F e j e s T ó t h discussed in a Seminar on Intuitive Geometry at the University of Wisconsin-Madison the following two conjectures. I quote in full his note [1]:

„Starting with an arbitrary convex polygon p_0, we successively construct the polygons $p_1, p_2, \ldots$ so that the vertices of p_n are the sidemidpoints of p_{n-1}. Again, starting with an arbitrary convex polygon P_0, we successively construct the polygons $P_1, P_2, \ldots$ in such a way that P_n arises by drawing a line through each vertex of P_{n-1} parallel to the line joining the two adjacent vertices of P_{n-1}. Some years ago I made two conjectures [2] which may be considered as dual counterparts of each other.

Conjecture 1. *There are affine images of $p_1, p_2, \ldots$ which converge to a regular polygon.*

Conjecture 2. *There are affine images of $P_1, P_2, \ldots$ which converge to a regular polygon.*

In my paper [2], I proved the correctness of Conjecture 1 for pentagons and hexagons. This paper started some further papers (see [1] for nine further references) giving, among others, various complete proofs for the correctness of Conjecture 1. On the other hand, the question whether Conjecture 2 is correct or not is not solved as yet".

The main results of the present note are as follows.

1. Twenty four years ago the author showed in [3] how the finite Fourier Series can solve a number of (mostly extremum) problems of elementary geometry, in particular that the polygons p_n, of Conjecture 1, shrink to a point. In section 1 it is shown how this approach yields in a few lines the results of H. R e i c h a r d t and G. L ü k ö (see [1] for references) concerning Conjecture 1.

2. It is shown in section 2 by a counter-example that Conjecture 2 is false for hexagons. The counter-example generalizes to polygons of an *even* number k of vertices (third paragraph of section 4). However, *the question whether Conjecture 2 is correct or not remains undecided if the number k of vertices is odd* ($k = 5, 7, 9, \ldots$). Computations done by my colleague C. de B o o r ($n = 1, 2, \ldots, 20$) show beyond a shadow of a doubt that Conjecture 2 is correct for pentagons having an axis of symmetry.

3. Starting from a convex polygon $\bar{P}_0 = A_1 A_2 \ldots A_k$ we obtain a new convex polygon $\bar{P}_1$ by removing from $\bar{P}_0$ all k corner triangles $A_{i-1} A_i A_{i+1}$ ($A_1 = A_{k+1}$, $A_0 = A_k$). Iterating this „peeling" process we obtain the polygons $\bar{P}_2, \bar{P}_3, \ldots$. I conjecture that the intersection

$$\bigcap_{n=1}^{\infty} \bar{P}_n$$

reduces to a point. Computer evidence, again obtained by d e B o o r, strongly supports this conjecture for pentagons having an axis of symmetry. I am much indebted to C. de B o o r for his help.

1. **On Conjecture 1.** The sequence of polygons (p_n) of Conjecture 1 was the subject of Problem No. 3547, American Mathematical Monthly, Vol. 39 (1932), p. 239, proposed by Martin R o s e n m a n. The objective was to show that the polygon p_n shrinks to