Теорема 2. *Предположим, что правая часть F уравнения (2) устойчива по Пуассону по переменной t равномерно относительно φ на компакте Φ. Если это уравнение допускает решение f, которое равномерно устойчиво по Ляпунову, то всякое уравнение вида (3) обладает почти периодическим решением, являющимся ω—предельным образом функции f и принимающим значения в компакте $f(T_r)$.*

Доказательство. Пусть f — равномерно устойчивое по Ляпунову решение уравнения (2). Согласно следствию из теоремы 1, найдется устойчивое по Пуассону решение g этого уравнения, принадлежащее множеству Ω_f и удовлетворяющее условию $g(T_r) \subset f(T_r)$. Так как решение f устойчиво по Лагранжу и равномерно устойчиво по Ляпунову, то Ω_f состоит из почти периодических функций (см. [1], стр. 108). Значит решение g уравнения (2) является почти периодическим.

Пусть $Q \in \bar{\Sigma}_F$. Согласно лемме 2, найдется решение $q \in \bar{\Sigma}_g$ уравнения (3), удовлетворяющее условию $q(\overline{T_r}) \subset g(\overline{T_r})$. Множество Σ_g состоит из почти периодических функций, поэтому решение q почти периодично. Кроме того, $q \in \bar{\Sigma}_g \subset \Omega_f$ и $q(T_r) \subset \subset g(T_r) \subset f(T_r)$. Теорема доказана.

Заметим, что в теореме 2 не требуется, чтобы правая часть F уравнения (2) была почти периодической по t — условие, которое в подобных случаях обычно предполагается выполненным. В работе Й о ш и з а в а [10] установлен признак, аналогичный теореме 2, для конечномерного случая при условии, что правая часть уравнения почти периодична по t и это уравнение допускает асимптотически почти периодическое решение.

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EXISTENȚA SOLUȚIILOR STABILE DUPĂ POISSON A ECUAȚIILOR DIFERENȚIALE CU ARGUMENT ÎNTÎRZIAT

Rezumat

În acest articol se stabilesc condițiile de existență a soluțiilor stabile după Poisson a ecuațiilor diferențiale cu argument întîrziat în spații Banach. Se demonstrează în particular un criteriu de existență a soluțiilor aproape periodice pentru acest tip de ecuații. Se studiază ecuațiile neautonome fără ipoteza unicității soluțiilor.

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ON PERTURBED STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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1. In recent years stochastic differential equations have been studied more and more intensively. This is apparently due, in part, to the importance of such systems in various branches of technology. Stochastic or random differential equations arise often in the engineering, biological, and physical sciences, and recent attempts have been made to develop and unify the theory of such equations, see [1], [3], [5], [6] and some of the references given there. In this paper we shall consider the nonlinear stochastic differential system including an operator T of the form

$$x'(t, \omega) = A(\omega)x(t, \omega) + f(t, x(t, \omega), (Tx)(t, \omega)), \quad x(0, \omega) = x_0(\omega). \tag{1}$$

The system will be studied as a perturbation of the linear stochastic differential system

$$y'(t, \omega) = A(\omega)\, y(t, \omega), \qquad y(0, \omega) = x_0(\omega). \tag{2}$$

Here x, y are stochastic processes, $\omega \in \Omega$ (a probability space), $' = d/dt$, $A(\omega)$ is an $n \times n$ matrix whose elements are measurable functions, f is a vector valued function defined on $I \times R^n \times \times R^n \to R^n$, where $I = [0, \infty)$ is an interval, R^n is an Euclidean n space and T is a continuous operator from R^n into R^n.

In a paper published a few years ago, C. C o r d u n e a n u [2] has used the idea of admissibility theory to study the nonstochastic integral equations. C o r d u n e a n u 's results in this direction are the starting point of extensive later developments; some recent ones are found in a series of papers by P a d g e t t and T s o k o s, and brought together in a monograph [6]. The problem considered in this paper is in the general spirit of the investigations in [6]. In particular, if we impose on T various meanings, it is apparent that equation (1) has a great diversity. For example, if $f(t, x, z)$ in (1) is of the form

$$f(t, x, z) = f_1(t, x) + z,$$

and if the operator T is defined by

$$(Tx)(t, \omega) = \int_0^t k(t, \tau, \omega)x(\tau, \omega)d\tau,$$

$$(Tx)(t, \omega) = h(t, \omega) + \int_0^t k(t, \tau, \omega)\, g(\tau, x(\tau, \omega))d\tau,$$

and so on, then the stochastic differential equation (1) will reduce to various types of stochastic integrodifferential equations.

Nonstochastic versions falling into the form (1) (when $Tx = x$) have been the subject of numerous studies in the theory of ordinary differential equations. The main purpose of this paper is to obtain some very general results concerning the existence, uniqueness, boundedness and asymptotic behavior of random solution of (1) under some suitable conditions on f and on the operator T. The tools that will be employed are the well known fixed point theorem of Banach and the integral inequality of the Gronwall-Bellman type recently established in [4].

2. Let R_+ denote the set of nonnegative real numbers. The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix norm. Let $B\subset [0, \infty)$ denote the set of all bounded functions $x: I\times\Omega\to R^n$, and if $x\in B\subset [0, \infty)$, denote its norm by $\|x(t, \omega)\|=\sup\limits_{t\geq 0}|x(t, \omega)|$. By a *random solution* of a stochastic differential equation (1) we shall mean that for each $t\in I$, $x(t, \omega)$ satisfies the equation almost surely.

Let $k(t, \tau, \omega)$ be a nonnegative stochastic kernel defined for t and τ satisfying $0\leq \tau\leq t<\infty$ such that

$$|Y(t, \omega)Y^{-1}(\tau, \omega)|\leq k(t, \tau, \omega), \tag{3}$$

where $Y(t, \omega)$ is the stochastic fundamental matrix solution of the homogenous system (2) such that $Y(0, \omega)$ is the unit matrix. Let $y(t, \omega)$ be a bounded random solution of a linear homogeneous system (2) such that $y(0, \omega)=x_0(\omega)$. It is easy to observe that $y(t, \omega)=Y(t, \omega)x_0(\omega)$.

In order to establish our results in Section 3, we require the following integral inequality proved in [4].

Lemma 1. *Let $u(t)$, $p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on I, for which the inequality*

$$u(t)\leq u_0+\int_0^t p(s)u(s)ds+\int_0^t p(s)\left(\int_0^s q(\tau)u(\tau)d\tau\right)ds, \quad t\in I,$$

holds, where u_0 is a nonnegative constant. Then

$$u(t)\leq u_0\left(1+\int_0^t p(s)\exp\left(\int_0^s (p(\tau)+q(\tau))\,d\tau\right)ds\right), \quad t\in I.$$

3. In this section we state and prove a theorem that gives conditions under which the stochastic differential equation (1) possesses a unique random solution. A well known Banach fixed point theorem will be used in the proof.

Theorem 1. *We consider equation (1) subject to the following conditions:*

(H_1) suppose that there exists a constant $M>0$ such that

$$\sup_{t\geq 0}\int_0^t k(t, \tau, \omega)d\tau\leq M<\infty;$$

(H_2) $f(t, 0, 0)\equiv 0$ for any fixed $t\geq 0$;

(H_3) for each $\alpha>0$, there exists $\eta>0$ such that

$$|f(t, x(t, \omega), (Tx)(t, \omega))-f(t, \bar{x}(t, \omega), (T\bar{x})(t, \omega))|\leq$$

$$\leq \alpha[|x(t, \omega)-\bar{x}(t, \omega)|+|(Tx)(t, \omega)-(T\bar{x})(t, \omega)|]$$

uniformly in $t\geq 0$ whenever $|x(t, \omega)|$, $|\bar{x}(t, \omega)|\leq\eta$;

$$(H_4)\qquad |(Tx)(t, \omega)-(T\bar{x})(t, \omega)|\leq\gamma|x(t, \omega)-\bar{x}(t, \omega)|,$$

where $\gamma>0$ is a constant and $|x(t, \omega)|$, $|\bar{x}(t, \omega)|\leq\eta$.

Then there exists a number $\varepsilon_0>0$ such that to any $\varepsilon\in(0, \varepsilon_0]$, if $\|y(t, \omega)\|\leq\lambda\varepsilon$ for some fixed $\lambda\in(0, 1)$, then there exists a unique random solution $x(t, \omega)$ of the equation (1) for all $t\geq 0$, satisfying $\|x(t, \omega)\|\leq\varepsilon$.

Proof. Fix $\alpha>0$, $\gamma>0$ such that $\alpha M(1+\gamma)<1$. Fix a number $\beta>0$ such that $\beta M(1+\gamma)\leq 1-\lambda$. Using (H_2), (H_3) and (H_4) pick $\delta\geq 0$ such that

$$|f(t, x(t, \omega), (Tx)(t, \omega))|\leq\beta(1+\gamma)|x(t, \omega)|, \tag{4}$$

uniformly in $t\geq 0$ whenever $|x(t, \omega)|\leq\delta$. Define $\varepsilon_0=\min(\delta, \eta)$. For any ε, $0<\varepsilon\leq\varepsilon_0$ we define

$$S(\varepsilon)=\{x(t, \omega): x(t, \omega)\in B\subset[0, \infty), \|x(t, \omega)\|\leq\varepsilon\}.$$

Define the operator U by the relation

$$(Ux)(t, \omega)=y(t, \omega)+\int_0^t Y(t, \omega)Y^{-1}(\tau, \omega)f(\tau, x(\tau, \omega)(Tx), (\tau, \omega))d\tau \tag{5}$$

for $x(t, \omega)\in S(\varepsilon)$, whose fixed point corresponds to the solution of the equation (1). Using (3), (H_1), (4) and (5) we obtain

$$|(Ux)(t, \omega)|\leq\|y(t, \omega)\|+\int_0^t |Y(t, \omega)Y^{-1}(\tau, \omega)f(\tau, x(\tau, \omega), (Tx)(\tau, \omega))|d\tau\leq$$

$$\leq\lambda\varepsilon+\int_0^t k(t, \tau, \omega)\beta(1+\gamma)\|x(\tau, \omega)\|d\tau\leq\lambda\varepsilon+\beta M(1+\gamma)\varepsilon\leq\lambda\varepsilon+(1-\lambda)\varepsilon=\varepsilon.$$

Hence U maps $S(\varepsilon)$ into itself. On the other hand using (3), (H_1), (H_2), (H_3) and (5) we have

$$\|(Ux)(t, \omega)-(U\bar{x})(t, \omega)\|\leq\int_0^t k(t, \tau, \omega)\alpha(1+\gamma)\|x(\tau, \omega)-\bar{x}(\tau, \omega)\|d\tau$$

and

$$\|(Ux)(t, \omega)-(U\bar{x})(t, \omega)\|\leq\alpha M(1+\gamma)\|x(t, \omega)-\bar{x}(t, \omega)\|$$

for any $x(t, \omega)$, $\bar{x}(t, \omega)\in S(\varepsilon)$. Since $\alpha M(1+\gamma)<1$, U is a contraction on $S(\varepsilon)$. Hence by the well known contraction mapping principle, the equation (1) has a unique solution $x(t, \omega)\in S(\varepsilon)$ with $\|x(t, \omega)\|\leq\varepsilon$. This completes the proof of the theorem.

4. In this section we shall study the boundedness and asymptotic behavior of equation (1) as a perturbation of the system (2).

Theorem 2 below establishes that corresponding to every bounded random solution of (2) there corresponds a bounded random solution of (1) under some suitable conditions on f and on the operator T.

Theorem 2. *Suppose that*

$$|Y(t, \omega)Y^{-1}(\tau, \omega)f(\tau, x(\tau, \omega), (Tx)(\tau, \omega))|\leq p(\tau, \omega)(|x(\tau, \omega)|+|(Tx)(\tau, \omega)|), \tag{6}$$

where $p: I\times\Omega\to R_+$ and $\int_0^\infty p(\tau, \omega)d\tau<\infty$. Further, suppose that the operator T satisfies the inequality

$$|(Tx)(t, \omega)|\leq\int_0^t q(\tau, \omega)|x(\tau, \omega)|d\tau, \tag{7}$$

where $q : I \times \Omega \to R_+$ and $\int_0^{\infty} q(\tau, \omega) d\tau < \infty$. *Then to every bounded random solution* $y(t, \omega)$ *of* (2) *on* I, *the corresponding random solution* $x(t, \omega)$ *of* (1) *is bounded on* I.

Proof. It is kuwon [6], the random solutions of (1) and (2) with the same initial values are related by

$$x(t, \omega) = y(t, \omega) + \int_0^t Y(t, \omega) Y^{-1}(\tau, \omega) f(\tau, x(\tau, \omega), (Tx)(\tau, \omega)) d\tau. \tag{8}$$

From (6), (7) and (8), we obtain

$$|x(t, \omega)| \leqslant c + \int_0^t p(\tau, \omega) \, | x(\tau, \omega) | \, d\tau + \int_0^t p(\tau, \omega) \left(\int_0^\tau q(s, \omega) |x(s, \omega)| ds \right) d\tau,$$

where c is the upper bound for $| y(t, \omega) |$. Now an application of Lemma 1 with $u(t) = | x(t, \omega)|$ yields

$$|x(t, \omega)| \leqslant c \left(1 + \int_0^t p(\tau, \omega) \exp \left(\int_0^\tau (p(s, \omega) + q(s, \omega)) \, ds \right) d\tau \right).$$

The above estimation implies the boundedness of $x(t, \omega)$ on I, and the theorem is proved.

Our next theorem shows that under some suitable conditions on the fundamental matrix of (2) and the perturbation term in (1), all the random solutions of (1) approach zero as $t \to \infty$.

Theorem 3. *Let the fundamental matrix* $Y(t, \omega)$ *of* (2) *satisfy an inequality*

$$|Y(t, \omega) Y^{-1}(\tau, \omega)| \leqslant k e^{-\alpha(t-\tau)}, \tag{9}$$

where $k > 0$, $\alpha > 0$ *are constants. Suppose that the perturbation term* $f(t, x(t, \omega), (Tx)(t, \omega))$ *in* (1) *satisfies*

$$| f(t, x(t, \omega), (Tx)(t, \omega)) | \leqslant p(t, \omega) \, (|x(t, \omega)| + |(Tx)(t, \omega)|), \tag{10}$$

where $p : I \times \Omega \to R_+$ *and* $\int_0^{\infty} p(\tau, \omega) d\tau < \infty$. *Further, suppose that the operator* T *satisfies the inequality*

$$|(Tx)(t, \omega)| \leqslant e^{-\alpha t} \int_0^t q(\tau, \omega) \, | x(\tau, \omega)| d\tau, \tag{11}$$

where $q : I \times \Omega \to R_+$ *and* $\int_0^{\infty} q(\tau, \omega) d\tau < \infty$. *Then all random solutions of* (1) *approach zero as* $t \to \infty$.

Proof. The random solutions of (1) and (2) with the same initial values are related by

$$x(t, \omega) = Y(t, \omega) x_0(\omega) + \int_0^t Y(t, \omega) Y^{-1}(\tau, \omega) f(\tau, x(\tau, \omega), (Tx)(\tau, \omega)) \, d\tau.$$

Using conditions (9), (10) and (11), we obtain

$$| x(t, \omega) | \leqslant k \, | x_0(\omega) | \, e^{-\alpha t} + e^{-\alpha t} \left[\int_0^t kp(\tau, \omega) \, | x(\tau, \omega) | \, e^{\alpha \tau} d\tau + \right.$$

$$\left. + \int_0^t kp(\tau, \omega) \left(\int_0^\tau q(s, \omega) e^{-\alpha s} |x(s, \omega)| e^{\alpha s} \, ds \right) d\tau \right].$$

Multiplying both sides of the above inequality by $e^{\alpha t}$, applying Lemma 1 with $u(t) = | x(t, \omega) | e^{\alpha t}$, then multiplying by $e^{-\alpha t}$, we obtain

$$|x(t, \omega)| \leqslant e^{-\alpha t} k \, | x_0(\omega)| \left(1 + \int_0^t kp(\tau, \omega) \exp \left(\int_0^\tau (kp(s, \omega) + q(s, \omega) e^{-\alpha s}) ds \right) d\tau \right).$$

The above estimate yields the desired result if we choose k and $| x_0(\omega) |$ small enough, and the proof of the theorem is complete.

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ASUPRA ECUAŢIILOR DIFERENŢIALE STOCASTICE PERTURBATE

Rezumat

Lucrarea este consacrată studiului existenţei, unicităţii, mărginirii şi comportării asimptotice a soluţiilor aleatoare ale unor ecuaţii diferenţiale stocastice perturbate.