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ON THE PROLONGATIONS OF THE FOLIATE BANACH MANIFOLDS AND OF THE FOLIATE FIBRE BUNDLES

BY

LILIANA MAXIM

The study of the prolongations of the manifolds and fibre bundles is an important subject for many theories in analysis, differential geometry and topology and, in view of this fact, it is worth to consider the special case of a foliate Banach manifold. This is the question to be investigated in the present Note, whose starting point is the paper of I. Vaisman [15] dedicated to the special case of foliate finite-dimensional vector bundles; generalizations to more general maps than the vector bundles were suggested in the end of the mentioned paper.

1. Foliate Banach manifolds. Let σ be a foliate family of Banach spaces ([2]), namely every Banach space $\mathbf{E}\in\sigma$ is endowed with a standard foliate structure given by a decomposition $\mathbf{E}=\mathbf{E}_1\times\mathbf{E}_2$, where $\mathbf{E}_1$, $\mathbf{E}_2$ are closed complementary subspaces; the leaves of this structure are the subspaces $\mathbf{E}_1\times\{g\}$, $\forall g\in\mathbf{E}_2$. It will be noted by $\sigma=\sigma_1\times\sigma_2$ in the sense that, for every $\mathbf{E}\in\sigma$, we have $\mathbf{E}=\mathbf{E}_1\times\mathbf{E}_2$ with $\mathbf{E}_1\in\sigma_1$, $\mathbf{E}_2\in\sigma_2$.

Now, consider a C^r-differentiable paracompact manifold V of the type σ endowed with the partitions of the unity of C^r-classes, and the topology τ subordinate to the manifold structure of V. A *foliate structure (foliation)* $\mathfrak{F}$ on V is defined in [2] by a class of the C^r-differentiable equivalente foliate atlases; a C^r-differentiable foliate atlas $\mathfrak{U}=\{h_\alpha=(U_\alpha,\varphi_\alpha,\mathbf{E}_\alpha)\}_{\alpha\in I}$ is a collection of foliate maps (f. m) h_α, where $U_\alpha\in\tau$, $\mathbf{E}_\alpha\in\sigma$ and φ_α is a C^r-diffeomorphism of U_α on an open set in $\mathbf{E}_\alpha$; these maps are C^r-compatible in the following sense: $h_\alpha=(U_\alpha,\varphi_\alpha,\mathbf{E}_\alpha)$ is C^r-compatible with $h_\beta=(U_\beta,\varphi_\beta,E_\beta)$ if:

$\varphi_\alpha(U_\alpha\cap U_\beta)$ (resp. $\varphi_\beta(U_\alpha\cap U_\beta)$) is open in $\mathbf{E}_\alpha$ (resp. $\mathbf{E}_\beta$)

$\varphi_\alpha\circ\varphi_\beta^{-1}:\varphi_\beta(U_\alpha\cap U_\beta)\to\varphi_\alpha(U_\alpha\cap U_\beta)$ is a C^r-diffeomorphism given, locally, by:

$$\varphi_\alpha\circ\varphi_\beta^{-1}:(e_\beta^1,e_\beta^2)\to(e_\alpha^1,e_\alpha^2)\qquad e_\alpha^1=\varphi_1^{\alpha\beta}(e_\beta^1,e_\beta^2)\quad e_\alpha^2=\varphi_2^{\alpha\beta}(e_\beta^2)\tag{1.1}$$

The pair $(V,\mathfrak{F})$ is called a *C^r-foliate manifold*; if $\forall\,\mathbf{E}_2\in\sigma_2$, $\dim\,\mathbf{E}_2<+\infty$ the foliation $\mathfrak{F}$ is called a finite codimensional foliation.

We denote by $\tau_{\mathcal{F}}$ the finest topology on V, so that, for every f. m. $h_\alpha=(U_\alpha,\varphi_\alpha,\mathbf{E}_\alpha)$, the map φ_α is an homeomorphism of U_α on the set $\varphi_\alpha(U_\alpha)$ which is open in the topology of $\mathbf{E}_\alpha$ given by the product of the natural topology of $\mathbf{E}_1^\alpha$ with the discrete topology of $\mathbf{E}_2^\alpha$; Obviously, the topology $\tau_{\mathcal{F}}$ is finer that τ; it is called the *topology subordinate to the foliation* $\mathfrak{F}$

The connected components of the topology $\tau_{\mathcal{F}}$ are called (as in [2]) the *leaves* of $\mathfrak{F}$. they are the C^r-differentiable open submanifolds (in relation to the topology $\tau_{\mathcal{F}}$) of V, pure of the type $\mathbf{E}_1\in\sigma_1$. Let V/R be the space of the leaves and $p:V\to V/R$ their corresponding quotient map. The foliation $\mathfrak{F}$ is *regular* if on V/R there exists a structure of C^r-manifold such that the projection p be a submersion of C^r-classes.

The tangent spaces to the leaves of $\mathcal{F}$ define an integrable subvector bundle ([1], 7.5.1.) $T(\mathcal{F})$, of the tangent vector bundle of manifold V, (TV, V, τ_V); it is called the *structural bundle* of $\mathcal{F}$.

The elements which depend only on the leaves, are called *foliate*; such that we can speak about differentiable foliate morphisms $f: V \to W$, W being a C^k-differentiable Banach manifold. Particularly, a foliate function $f: V \to R$ is characterized, for a f. m. $h_\alpha = (U_\alpha, \varphi_\alpha, \mathrm{E}_\alpha)$, by the condition that the function $f \circ \varphi_\alpha^{-1}$: $\varphi_\alpha(U_\alpha) \to R$ is a foliate map. Hence, a vector field X on V is a section of $T(\mathcal{F})$ if and only if $Xf = 0$ for any foliate function f on V. Let

$$\mathrm{Tr}(\mathcal{F}) = TV / T(\mathcal{F}) \tag{1.2}$$

be the corresponding quotient bundle ([1], 7.5.2); it is called the *transversal bundle* of $\mathcal{F}$.

In our hypotheses, from [1], 7.7.1., 7.5.7, it results that

$$TV = T(\mathcal{F}) \oplus \mathrm{Tr}(\mathcal{F}).$$

We shall denote by π_{tr} the canonical projection

$$\pi_{tr} : TV \to \mathrm{Tr}\ (\mathcal{F}). \tag{1.3}$$

2. Foliate fibre bundle. Now, consider a triplet $(M, V, \pi)^k$ which consists of the C^k-differentiable Banach manifolds M, V, and the surjective submersion (surmersion from [7]) $\pi: M \to V$; let Φ be a C^k-differentiable groupoide with (V, i) as base manifold and (a, b): $\Phi \to V \times V$ their source — target map ([3], [4], [5], [8]), which operates on the surmersion $(M, V, \pi)^k$ in the sense given in the mentioned papers, with the law: $\psi: \Phi \oplus_{a,\pi} M \to M$. We shall denote by $\tau: \Phi \oplus_{b,a} \Phi \to \Phi$ the composition map of Φ and by $s: \Phi \to \Phi$ its inversion law.

It is known [4], that the existence of a differentiable locally transitiv ((a, b) surmersion) groupoide Φ which operates on the triplet in connection with the base (V, i) is a sufficient condition that the surmersion $(M, V, \pi)^k$ be a fibre bundle ([1] 6.1.1.). As in [8], we call that fibre bundle a *fibre bundle associated to the groupoide* Φ; it will be noted by $(M, V, \pi)^k_\Phi$.

The fibre structure on $(M, V, \pi)^k_\Phi$ is defined as follows: the map $(a, b): \Phi \to V \times V$ being a submersion, for every $x_0 \in V$, there exists a local cross-section $\sigma_U: U \times U \to \Phi$ of this mapping where U is a neighbourhood of the point x_0; it defines a morphism

$$\sigma_U(x_0): U \to \Phi, \quad x \to \sigma_U(x, x_0). \tag{2.1}$$

which is a local cross-section of the submersion $a: \Phi \to V$. This section defines the bundle map $(U, \varphi_U, \pi^{-1}(x_0))$, where the trivialization diffeomorphism φ_U is given by:

$$\varphi_U : \pi^{-1}(U) \to U \times \pi^{-1}(x_0), \quad z \to (\pi(z),\ \psi(\sigma_U\ (x_0),\ z)).$$

The bundle maps $h_U = (U, \varphi_U, \pi^{-1}(x_0))$, $h_{U'} = (U', \varphi_{U'}, \pi^{-1}(y_0))$ are C^k-compatible in the sense that the C^k-morphism:

$$\lambda_{U,U'} : U \cap U' \to \Phi \tag{2.2}$$

exists. So, denoting by $\varphi_{U,x} = \varphi_U \mid \pi^{-1}(x)$, $\varphi_{U',x} = \varphi_{U'} \mid \pi^{-1}(x) \forall x \in U \cap U'$, we have: $\varphi_{U',x} \circ \varphi_{U,x}^{-1} = \lambda_{U,U'}(x)$. The mapping $\lambda_{U,U'}$ is given by:

$$\lambda_{U,U'}(x) = \tau(\sigma_{U'}(y_0)\ (x),\ s(\sigma_U(x_0)\ (x))).$$

Definition 2.1. *A foliate fibre bundle associated to the groupoide Φ (f.f.b.) is a fibre bundle, $(M, V, \pi)^k_\Phi$ with the foliate manifold V as base space, so that, for every bundle maps h_U, $h_{U'}$, $U \cap U' \neq \emptyset$, the C^k-morphism $\lambda_{U,U'}$ (2.2) is foliate.*

Remark 2.1 If the cross-sections (2.1) are foliate maps $\forall U \subset V$ then, obviously, the maps (2.2) are foliate. Conversely, if $\lambda_{U,U'}$ are foliate maps $\forall U, U'$, V is a connected manifold and σ_U is foliate for an open set $U \subset V$ then, for every $U' \subset V$, $\sigma_{U'}$ is foliate too.

Remark 2.2 If the considered fibre bundle is a vector bundle then the previous definition lead to the notion of foliate vector bundle (f. v. b.) which is introduced, in the finite-dimensional case, by I. Vaisman ([14], [15]), P. Molino ([10]) and F. W. Kamber Ph. Tondeur ([6]); then a Banach vector bundle $(E, V, \pi)^k$ with the foliate Banach manifold V as base space is called foliate vector bundle if, for every two vector maps ([1] 7.1.2.), $t = (U, \varphi, \mathrm{F})$, $t' = (U', \varphi', \mathrm{F}')$ of E, the C^k-map $\lambda: U \cap U' \to L(\mathrm{F}, \mathrm{F}')$ is a foliate one.

Examples: 1. The transversal bundle $\mathrm{Tr}(\mathcal{F})$ of $\mathcal{F}$ is, obviously, a foliate vector bundle.

2. Let $\mathcal{E} = \{(E^i, V, \pi_i)\}_{i \in I}$ be a collection of f. v. b. and ω a vectorial functor ([1] 7.6.1.). It easy to verify that the vector bundle obtained from the collection $\mathcal{E}$ by means of the functor ω ([1] 7.6.2.) is a f.v.b. Particularly, if E and $\bar{E}$ are the f.v.b. then E' (the dual of E) ([1] 7.7.3.), $L(E, \bar{E})$ ([1] 7.7.3.), $T^p_q(E)$ ([1] 7.9.3), $\Lambda^q E$([1] 7.9.7.) are f.v.b. too.

3. Let (E, V, π) be a pure f.v.b. of type F ([1] 7.10.1). Then the principal fibre bundle of the linear frames of E is a f.f.b. associated to the groupoide $V \times Gl(\mathrm{F})$.

4. If $\lambda = (P, G, V, \pi)$ is a foliate principal fibre bundle associated to the groupoide $V \times G$ and F is a Banach manifold on which G operates on the right, then the fibre bundle associated to λ with F as type fibre ([1] 6.5.1.) is a f.f.b. associated to the same groupoide.

Now, consider two f.f.b. $(M, V, \pi)^k_\Phi$ and $(M', V, \pi')^{k'}_{\Phi'}$ and $h: M \to M'$ a V-morphism of this fibrations ([1] 6.1.3.). The morphism h will be called a *foliate V-morphism* if his local expression $\varphi' \circ h \circ \varphi^{-1}$ for two trivialization maps $\varphi: \pi^{-1}(U) \to U \times F$, $\varphi': \pi'^{-1}(U) \to U \times F'$ is a foliate map. Hence, if U is the domain of a f. m. of V, the local expression of the mapping $\varphi' \circ h \circ \varphi^{-1}$ for the maps $(U \times W, \psi \times \chi)$ and $(U \times W', \psi \times \chi')$ have the following form:

$$(\psi \times \chi') \circ \varphi' \circ h \circ \varphi^{-1} \circ (\psi^{-1} \times \chi^{-1}): \psi(U) \times \chi(W) \to \psi(u) \times \chi'(W')\ (e^1, e^2, f) \to (e^1, e^2, f'(e^2, f)).$$

Now, it is easy to see that the f.f.b. on V and their foliate V-morphisms make up a category.

A cross-section of a f.f.b. $(M, V, \pi)^k_\Phi$, is called *foliate* if it is constant on the leaves of $\mathcal{F}$. It is easy to see that every foliate V-morphism h induces an homomorphism of the sheafs of germs of differentiable foliate sections of two considered f.f.b.

3. Prolongations of the foliate Banach manifolds and of the foliate fibre bundles. 3.1. We consider first the prolongation of a foliate Banach manifold $(V, \mathcal{F})$ in the sense of Morimoto [11]; hence, we study the prolongation of the foliation $\mathcal{F}$ to the tangent bundle TV. The results obtained are summarized in:

Proposition 3.1. *Let $(V, \mathcal{F})$ be a C^k-differentiable foliate Banach manifold of type σ and $(TV, V, \tau_V)^{k-1}$ his tangent vector bundle.*

a) *The foliation $\mathcal{F}$ induces a foliation $T\mathcal{F}$ on the manifold TV.*

b) *If $\mathcal{F}$ is of finite codimension, then $T\mathcal{F}$ is of finite codimension;*

c) *If $\mathcal{F}$ is regular, then $T\mathcal{F}$ is regular too.*

d) *If $M \subset V$ is saturated relative to $\mathcal{F}$, then $\tau_V^{-1}(M)$ is a saturated set relative to the foliation $T\mathcal{F}$;*

e) *Every subfoliation $\mathcal{G}$ of $\mathcal{F}$ defines a subfoliation $T\mathcal{G}$ of $T\mathcal{F}$ on the manifold TV.*

Proof a) Let $\mathfrak{A} = \{h_\alpha = (U_\alpha, \varphi_\alpha, \mathrm{E}_\alpha), \mathrm{E}_\alpha \in \sigma\}_{\alpha \in I}$ be the maximal foliate atlas which defines the foliation $\mathcal{F}$ on the manifold V and $\{\tau_\alpha : \tau_V^{-1}(U_\alpha) \to U_\alpha \times \mathrm{E}_\alpha\}_{\alpha \in I}$ be the trivialization maps on TV induced by the f.m. h_α. $T\mathfrak{A} = \{H_\alpha = (\tau_V^{-1} f(U_\alpha), \Phi_\alpha, \mathrm{E}_\alpha \times \mathrm{E}_\alpha)\}_{\alpha \in I}$ is the maximal atlas on TV determined by the trivializations $\{\tau_\alpha\}_{\alpha \in I}$. It is a C^{k-1}-differentiable foliate atlas on the manifold TV. In fact, the C^{k-1} diffeomorphism $\Phi_\alpha \circ \Phi_\beta^{-1}: \Phi_\beta(\tau_V^{-1}(U_\alpha \cap U_\beta)) \to \Phi_\alpha(\tau_V^{-1}(U_\alpha \cap U_\beta))$ have the following local expression:

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (e_\beta^1, e_\beta^2, e_\beta^{-1}, e_\beta^{-2}) \to (e_\alpha^1, e_\alpha^2, e_\alpha^{-1}, e_\alpha^{-2})$$

$$(3.1) \quad \left\{ e_\alpha^1 = \Phi_1^{\alpha\beta}(e_\beta^1, e_\beta^2), \quad e_\alpha^2 = \Phi_2^{\alpha\beta}(e_\beta^2), \quad e_\alpha^{-1} = \Phi_2^{\alpha\beta}(e_\beta^1, e_\beta^2, e_\beta^{-1}, e_\beta^{-2}), \quad e_\alpha^{-2} = \Phi_2^{\alpha\beta}(e_\beta^2, e_\beta^{-2}). \right.$$

Hence, TV is a paracompact manifold of type $T\sigma = \{T\mathbf{E} = \mathbf{E}_1 \times \mathbf{E}_1 \times \mathbf{E}_2 \times \mathbf{E}_2\}$. The foliation defined by the atlas $T\mathcal{U}$ will be denoted by $T\mathcal{F}$. Now, it immediately follows the proof of b).

Obviously, $\tau_{T\mathcal{F}} = \tau_V^{-1}(\tau_{\mathcal{F}})$ and the projection τ_V is a continuous and open map relative to the topologies $\tau_{\mathcal{F}}$ and $\tau_{T\mathcal{F}}$. Since V with the topology τ is a local connected space ([1] 5.1.6.) it follows that $\tau_{\mathcal{F}}$ is locally connected and a leaf F of $\mathcal{F}$ is open and closed relative to the topology $\tau_{\mathcal{F}}$. Consequently, the set $\tau_V^{-1}(F)$ is open and closed relative to the topology $\tau_{T\mathcal{F}}$; hence it is an union of the connected components of this topology. We denote by ${}^TR$ the equivalence relation on TV defined by the foliation $T\mathcal{F}$, by TV/T_R the quotient space and by ${}^Tp : TV \to TV/T_R$ the corresponding projection (continuous and open).

Now we prove that the equivalence relation ${}^TR$ is regular, and, moreover, there exists the submersion $\tau_V/R : TV/{}^TR \to V/R$ such that the following diagram of manifolds commuts:

$$\begin{tikzcd}
TV \arrow[rr, "{}^Tp"] \arrow[d, "\tau_V"'] & & TV/{}^TR \arrow[d, dotted, "\tau_V/R"] \\
V \arrow[rr] & & V/R
\end{tikzcd}$$

For this purpose, according to [1] 5.9.5., we shall prove that:

1) graph ${}^TR$ is a submanifold in $TV \times TV$,

2) $\overline{\mathrm{pr}}_1 : \text{graph } {}^TR \to TV$ is a submersion.

In fact, from the previous remarks, it follows that graph ${}^TR = (\tau_V \times \tau_V)^{-1}$ (graph R). Since R is a regular relation, graph R is a submanifold in $V \times V$; $\tau_V \times \tau_V$ is a submersion because τ_V is submersion ([1] 5.9.3.) and 1) follows from [1] 5.11.6. Denoting by pr_1 the submersion $\mathrm{pr}_1 : \text{graph } R \to V$, we have:

$$\overline{\mathrm{pr}}_1 \circ \tau_V = \tau_V \times \tau_V \circ \mathrm{pr}_1 :$$

then 2) results from [1] 5.9.2. Moreover, from the previous considerations, it immediately follows that, if V/R is a separate space, $TV/{}^TR$ is separate too and the submersion τ_V/R exists.

d) Let $M \subset V$ be a saturated set relative to $\mathcal{F}$; hence $M = \bigcup_{\gamma \in \Gamma} F_\gamma$ where F_γ are the leaves of $\mathcal{F}$. Then $\tau_V^{-1}(M) = \bigcup_{\gamma \in \Gamma} \tau_V^{-1}(F_\gamma)$ and, for each F_γ, $\tau_V^{-1}(F_\gamma)$ is a union of the leaves of the foliation $T\mathcal{F}$.

e) it follows immediatelly from the manner of the definition of the foliation $T\mathcal{F}$.

Remark: For the finite-dimensional manifolds, a similar result (a), e)) is proved in [13].

3.2. Now, consider the prolongations of the f.f.b. in Ehresmann's sense. For this, the usual notations for the well-known notions in the prolongation theory should be used. ([3], [4], [7], [8]); we recall them: $J^n(M, N)$ is the manifold of the n-jets of local maps of C^k Banach manifold M in C^k Banach manifold $N (n \leq k)$, $J^n(M) = J^n(M, M)$, $j_x^n 1_M = j_x^n$, (α_n, β_n): $J^n(M, N) \to M \times N$ the corresponding source-target submersion, $\Pi^n(M)$ the open submanifold of the inversible n-jets in $J^n(M)$, $Y.X$ the composition of the jets $X \in J^n(M, N)$, $Y \in J^n(N, L)$, $fX = j_y^n(f).X$ where $X \in J^n(M, N)$, $y = \beta_n(X)$ and $f : N \to L$, ${}^nf : J^n(M, N) \to J^n(M, L)$ the holonomical prolongation of n-th order of the mapping f given by $X \to fX$, $\forall X \in J^n(M, N)$.

Firstly we prove the following result:

Proposition 3.2. *Let V be a C^k-differentiable foliate Banach manifold of type σ and M be a C^k-differentiable Banach manifold. The set $J_f^n(V, M)$ of the n-jets of local foliate maps of V in M is a submanifold in $J^n(V, M)$; particularly, the set $\Pi_f^n(V)$ of the inversible n-jets of the foliate maps on V, so that their inverses are foliate too, is a submanifold in $\Pi^n(V)$.*

Proof. Let $\mathcal{U} = \{h_\alpha = (U_\alpha, \varphi_\alpha, \mathbf{E}_\alpha)\}_{\alpha \in I}$ be the atlas of the foliation $\mathcal{F}$ on V and $\mathcal{V} = \{(V_\beta, \psi_\beta, \mathbf{F}_\beta)\}_{\beta \in J}$ a maximal atlas of the differential structure on M. It is known from [3] that $\{(J^n(U_\alpha, V_\beta), [\varphi_\alpha, \psi_\beta]_n, L^n(\mathbf{E}_\alpha, \mathbf{F}_\beta))\}_{(\alpha,\beta) \in I \times J}$ is an atlas of the differential structure on $J^n(V, M)$, where $L^n(\mathbf{E}_\alpha, \mathbf{F}_\beta) = \mathbf{E}_\alpha \times \mathbf{F}_\beta \times \prod_{p=1}^{n} L_{\mathrm{sym}}^p(\mathbf{E}_\alpha, \mathbf{F}_\beta)$ and $[\varphi_\alpha, \psi_\beta]_n : J^n(U_\alpha, V_\beta) \to L^n(\mathbf{E}_\alpha, \mathbf{F}_\beta)$ carry $X \in J^n \times (U_\alpha, V_\beta)$ in $\psi_\beta . X . j_{\alpha_n(X)}^n(\varphi_\alpha^{-1})$. According to [1] 5.8.3., the first part of the proposition follows from the remark that $\{J_f^n(U_\alpha, V_\beta), [\varphi_\alpha, \psi_\beta]_n J_f^n(U_\alpha, V_\beta), L^n(\mathbf{E}_2^\alpha, \mathbf{F}_\beta)$ is an atlas of a differential structure on the set $J_f^n(V, M)$ and $L^n(\mathbf{E}_2^\alpha, \mathbf{F}_\beta)$ is a closed split vector subspace in $L^n(\mathbf{E}_\alpha, \mathbf{F}_\beta)$ $\forall(\alpha, \beta) \in I \times J$. The second part of the proposition results from the fact that $L^n(\mathbf{E}_2^\alpha, \mathbf{E}_2^{\alpha'})$ is a closed split vector subspace in $L^n(\mathbf{E}_\alpha, \mathbf{E}_{\alpha'})$, $\forall(\alpha, \alpha') \in I \times I$.

Now, let $(M, V, \pi)^k$ be a surmersion with foliate manifold V as base space and $(J^n(\pi), V, \pi_n)^{k-n}$ with $J^n(\pi) = \{X / X \in J^n(V, M), \pi X = j_{\alpha_n(X)}^n$ in $J^n(V)\}$ $(J_v^n(\pi), V, \pi_n^v)^{k-n}$ with $J_v^n(\pi) = \{X / X \in J^n(V, M), \pi X = j_{\alpha_n^\wedge(X)}^n \in J^n(V)\}$ (here $\alpha_n^\wedge(X)$ denote the constant map defined on V having their values in $\alpha_n(X)$) be their holonomical prolongations of n-th order $(n \leq k)$ (transversal and vertical respectively) which are introduced in [4] and [8].

The intersections:

$$J_f^n(\pi) = J^n(\pi) \cap J_f^n(V, M), \qquad J_{v,f}^n(\pi) = J_v^n(\pi) \cap J_f^n(V, M)$$

will be important. We call them the *foliate prolongations of n^{th} order of the surmersion* $(M, V, \pi)^k$ (transversal and vertical respectively).

We get the following result:

Proposition 3.3. *The sets $J_f^n(\pi)$ and $J_{v,f}^n(\pi)$ are submanifolds in $J^n(\pi)$ and $J_v^n(\pi)$ (respectively) (hence they are the submanifolds in $J^n(V, M)$). The restrictions of the natural epimorphism $\pi_{n-1}^n : J^n(V, M) \to J^{n-1}(V, M)$ ([3]) have the properties that:*

$$\pi_{n-1,f}^n = \pi_{n-1}^n \big|_{J_f^n(\pi)} : J_f^n(\pi) \to J_f^{n-1}(\pi).$$

Proof. Consider the submersions: ${}^n\pi : J^n(V, M) \to J^n(V, V)$, $\beta_n : J^n(V, M) \to M$, $({}^n\pi, \beta_n)$: $J^n(V, M) \to J^n(V, V) \oplus_{\beta_n, \pi} M$ and $({}^n\pi, \beta_n) \mid J_f^n(V, M)$. The pull-back from the last submersion of the embedding: $M \to J^n(V, V) \oplus_{\beta_{n,\pi}} M$, $\quad z \to (j_{\pi(z)}^n, z)$ is just the submanifold $J_f^n(\pi)$.

In the same manner, the pull-back from $({}^n\pi, \beta_n)$ $J_f^n(V, M)$ of the embedding:

$$M \to J^n(V, V) \oplus_{\beta_{n,\pi}} M, \qquad z \to (j_{\pi(z)}^{n\wedge}, z)$$

is the submanifold $J_{v,f}^n(\pi)$. The last part of the proposition is obvious.

We denote by $i_n : J_f^n(\pi) \to J^n(\pi)$ the natural embedding.

In order to approach the problem of the prolongations of f.f.b we shall consider, firstly, the foliate prolongations of the differential groupoides with the foliate manifolds as base spaces.

Let Φ be a differential local transitive groupoide with the foliate manifold V as base space and the submersion $(a, b) : \Phi \to V \times V$ as source-target map.

The transversal holonomic prolongation of nth order of this groupoide [4], is the submanifold $\Phi^n(V)$ in $J^n(V, \Phi)$:

$$\Phi^n(V) = \{X / X \in J^n(V, \Phi) ;\ aX = j_{\alpha_n(X)} ;\ bX \in \Pi^n(V)\},$$

such that the composition law $\divideontimes$ defined by:

$$\divideontimes : (X, Y) \to Y \divideontimes X = {}^n\tau(YbX, X),$$

determins on this submanifold a structure of C^{k-n}– differentiable groupoide with V as base space and $(a^n = \alpha_n|_{\Phi^n(V)}, b^n = \beta_n \circ b_{\Phi^n(V)})$ as source-target map.

The vertical holonomical prolongation of nth order of the groupoide Φ [4], [8] is the fibre product $\Phi^n_v(V) \oplus \Pi^n(V)$ of the maps $(a, b) \circ \beta_n|_{\Phi^n_v(V)}$ and $(\alpha_n, \beta_n) : \Pi^n(V) \to V \times V$, where

$$\Phi^n_v(V) = \{X / X \in J^n(V, \Phi) ;\ aX = j^{nV}_{\alpha_n(X)} ;\ bX = j^n_{\alpha_n(X)}(b(\beta^{\wedge}_n(X))\},$$

with the composition law given by:

$$\otimes : (X, Y), (X_1, Y_1) \to ({}^n\tau(X_1.Y, X), Y_1.Y)$$

and the source-target map: $(a^n_v = (a, b) \circ \beta_n|_{\Phi^n_v(V)}, b^n_v = (\alpha_n, \beta_n))$.

Finally, the bilateral holonomic prolongation of nth order of Φ [4], [8] is the fibre product $\Phi^n_v(V) \oplus \Phi^n(V)$ with the composition law $\odot$ given by:

$$\odot : ((X, Y), (X_1, Y_1)) \to ({}^n\tau(X_1.bY, X), {}^n\tau(Y_1.bY, Y)).$$

Now, consider the following sets:

$$\Phi^n_f(V) = \{X / X \in J^n_f(V, \Phi) ;\ aX = j_{\alpha_n(X)} ;\ bX \in \Pi^n_f(V)\},$$

$$\Phi^n_{v,f}(V) = \Phi^n_v(V) \cap J^n_f(V, \Phi), \quad \Phi^n_{v,f}(V) \oplus \Pi^n_f(V), \quad \Phi^n_{v,f}(V) \oplus \Phi^n_f(V),$$

and prove:

Proposition 3.4. *The sets $\Phi^n_f(V)$, $\Phi^n_{v,f}(V) \oplus \Pi^n_f(V)$ and $\Phi^n_{v,f}(V) \oplus \Phi^n_f(V)$ are the C^{k-n}– differential subgroupoides in $\Phi^n(V)$, $\Phi^n_v(V) \oplus \Pi^n(V)$ and $\Phi^n_v(V) \oplus \Phi^n(V)$, respectively.*

Proof. According to [4], [7], [8], the proposition will be proved if we shall prove that these sets are submanifolds in the respective groupoides which are stable at the composition laws $\divideontimes$, $\otimes$, $\odot$ and at their inversion laws.

For this purpose, consider the submersion ${}^nb|_{J^n_f(a)} : J^n_f(V, \Phi) \to J^n(V, V)$; it is obvious that:

$$\Phi^n_f(V) = ({}^nb|_{J^n_f(a)})^{-1}(\Pi^n_f(V)) = ({}^nb|_{J^n(a)})^{-1}(\Pi^n_f(V)) = ({}^nb_{\Phi^n(V)})^{-1}(\Pi^n_f(V));$$

hence $\Phi^n_f(V)$ is a submanifold in $\Phi^n(V)$. The stability of $\Phi^n_f(V)$ relative to $\divideontimes$ follows immediately because $Y.bX \in J^n_f(V, \Phi)$, $\forall X, Y \in \Phi^n_f(V)$ and ${}^n\tau(J^n_f(V, \Phi) \times J^n_f(V, \Phi)) \subset J^n_f(V, \Phi)$.

The restriction to $\Phi^n_f(V)$ of the inversion law from $\Phi^n(V)$ given by $X \to sX(bX)^{-1}$, is obviously a morphism on $\Phi^n_f(V)$. Hence this is a subgroupoide in $\Phi^n(V)$.

Now, it is known from [4] that $\Phi^n_v(V)$ is the pull-back from the submersion $(({}^na, {}^nb), \beta_n) : J^n(V, \Phi) \to (J^n(V, V) \oplus J^n(V, V)) \oplus \Phi$ of the embedding $\chi : \Phi \to (J^n(V, V) \times J^n(V, V)) \times \Phi$ given by: $f \to (j^n_{a(f)}, j^n_{a(f)}(b(f)), f)$.

The set $\Phi^n_{v,f}(V)$ is the pull-back from the submersion $(({}^na, {}^nb), \beta_n)_{J^n_f(V,\Phi)}$ of the same embedding; hence this set is a submanifold in $J^n_f(V, \Phi)$; moreover $(({}^na, {}^nb), \beta_n)^{-1}(\chi(\Phi)) = \Phi^n_v(V) \supset (({}^na, {}^nb), \beta_n)_{J^n_f(V,\Phi)})^{-1}(\chi(\Phi)) = \Phi^n_{v,f}(V)$, such that $\Phi^n_{v,f}(V)$ and $\Phi^n_v(V)$ are the submanifolds in $J^n(V, \Phi)$ and $\Phi^n_{v,f}(V) \subset \Phi^n_v(V)$. Hence $\Phi^n_{v,f}(V)$ is a submanifold in $\Phi^n_v(V)$ and $\Phi^n_{v,f}(V) \oplus \Pi^n_f(V)$ is a submanifold in $\Phi^n_v(V) \oplus \Pi^n(V)$.

It is easy to see that this last submanifold is stable from the composition law $\otimes$ and from the inversion law in $\Phi^n_v(V) \oplus \Pi^n(V)$ given by [4]: $(X, Y) \to (sX.Y^{-1}, Y^{-1})$.

The submanifold $\Phi^n_{v,f}(V) \oplus \Phi^n_f(V)$ is also stable at the law $\odot$ and the proposition is proved.

We shall denote by (a^n_f, b^n_f), $(a^n_{v,f} b^n_{v,f})$ respectively, the source-target maps of the first two groupoides from the previous proposition.

Now, we can state the principal result of this section:

Theorem 3.5. *Let $(M, V, \pi)^k_\Phi$ be a f.f.b. associated to the groupoide Φ with the foliate connected manifold V as base space.*

Then:

a) *The surmersion $(J^n_f(\pi), V, \pi_n|_{J^n_f(\pi)})^{k-n}$ is a f.f.b. associated to the groupoide $\Phi^n_f(V)$. We call this the foliate transversale holonomic prolongation of n^{th} order of π.*

b) *The surmersion $(J^n_{v,f}(\pi), V, \pi^v_n|_{J^n_{v,f}(\pi)})^{k-n}$ is a fibre bundle associated to the groupoide $\Phi^n_{v,f}(V) \oplus \Pi^n_f(V)$.*

Proof. a) The groupoide $\Phi^n_f(V)$ operates on the surmersion $(J^n_f(\pi), V, \pi_n|_{J^n_f(\pi)})^{k-n}$ with the following law:

$$\psi^n : \Phi^n_f(V) \oplus J^n_f(\pi) \to J^n_f(\pi), \quad (A, X) \to ({}^n\psi(A, X)).(bA)^{-1}$$

where ${}^n\psi$ is the prolongation of the law $\psi : \Phi \oplus_{a,\pi} M \to M$. Hence this surmersion is a fibre bundle associated to the groupoide $\Phi^n_f(V)$. Moreover it is a f.f.b. In fact, consider the set of local foliate cross-sections $\{\sigma_U(x_0)\}$ (Remark 2.1) and define a set of local foliate cross-sections $\{\sigma^n_U(x_0)\}$ of the submersion $a^n_f : \Phi^f_n(V) \to V$ which defines the previous fibre structure. For this purpose we remark that the map:

$$j^n\sigma_U(x_0) : U \to J^n(V, \Phi), \quad x \to j^n_x\sigma_U$$

is a local foliate cross-section of the submersion $\alpha_n : J^n(V, \Phi) \to V$, $\forall \sigma_U(x_0)$ and $j^n\sigma_U(x_0)(U) \subset J^n_f(V, \Phi)$. Moreover, $j^n\sigma_U(x_0)(U) \subset \Phi^n_f(V)$. We shall put $\{\sigma^n_U(x_0) = j^n\sigma_U(x_0)\}$; consequently the corresponding set $\{\lambda^n_{U,U'}\}$ (def. 2.1) is a set of the C^{k-n}– differentiable foliate morphisms.

b) Using the same notations, the groupoide $\Phi^n_{v,f}(V) \oplus \Pi^n_f(V)$ operates on the surmersion $(J^n_{v,f}(\pi), V, \pi^v_n|_{J^n_{v,f}(\pi)})^{k-n}$ by the following law:

$$\psi^n_v : [\Phi^n_{v,f}(V) \oplus \Pi^n_f(V)] \oplus J^n_{v,f}(\pi) \to J^n_{v,f}(\pi) \quad ((A, Z), X) \to ({}^n\psi(A, X))Z^{-1}.$$

Hence this surmersion is a fibre bundle associated to this groupoide. Later, we shall prove that this fibre bundle is a f.f.b.

If the f.f.b. from the Theorem 3.5 is the f.v.b. $(M, V, \pi)^k \in VB^k(V, \mathbf{F})$, then the previous considerations permit to define their holonomic transversale prolongations $J^n(\pi)$ (vector bundles $\forall n \leqslant k$) and their foliate transversale holonomic prolongations $J^n_f(\pi)$ (f.v.b). Moreover $J^n_f(\pi) \in VB^{k-n}(V, \mathbf{F} \times L(\mathbf{E}_2, \mathbf{F}) \times \ldots \times L^n_{\text{sym}}(\mathbf{E}_2, \mathbf{F}))$ and is a sub-vector bundle in $J^n(\pi)$ ([1] 7.5.1.).

Denoting:

$$Q^n(\pi) = J^n(\pi)/J^n_f(\pi),$$

we find again the following local direct exact sequence ([1] 7.5.7) from [15]:

$$0 \longrightarrow J^n_f(\pi) \xrightarrow{i_n} J^n(\pi) \xrightarrow{p^n} Q^n(\pi) \longrightarrow 0,$$

where p^n is the canonical projection.

It is known [3], [7], [8] that a natural epimorphism of jet bundles $\pi^n_{n-1}: J^n(\pi) \to J^{n-1}(\pi)$ whose kernel is $L^n_{sym}(\tau_V, \pi)$, exists. In the same manner one get a foliate epimorphism $\pi^n_{n-1,f}$: $J^n_f(\pi) \to J^{n-1}_f(\pi)$ whose kernel is the sub-vector bundle $L^n_{sym}(\tau_V|_{Tr(\tilde{\mathcal{F}})}, \pi)$. Finally we find again the diagram $\Delta^n(\pi)$ from [15] which has commutative squares, and exact rows and columns; we shall denote it $\Delta^n_{n-1}(\pi)$:

$$(\nabla^n_{n-1}(\pi)) \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L^n_{sym}(\tau_V|_{Tr(\tilde{\mathcal{F}})}, \pi) \arrow[r, "\varepsilon^n_{n-1,f}"] \arrow[d, "\pi^*_{Tr}"'] & J^n_f(\pi) \arrow[r, "\pi^n_{n-1,f}"] \arrow[d, "i_n"'] & J^{n-1}_f(\pi) \arrow[r] \arrow[d, "i_{n-1}"] & 0 \\ 0 \arrow[r] & L^n_{sym}(\tau_V, \pi) \arrow[r, "\varepsilon^n_{n-1}"] \arrow[d] & J^n(\pi) \arrow[r, "\pi^n_{n-1}"] \arrow[d, "p^n"'] & J^{n-1}(\pi) \arrow[r] \arrow[d, "p^{n-1}"] & 0 \\ 0 \arrow[r] & L^n_{sym}(\tau_V, \pi)/L^n_{sym}(\tau_V|_{Tr(\tilde{\mathcal{F}})}, \pi) \arrow[r, "\bar{\varepsilon}^n_{n-1}"] \arrow[d] & Q^n(\pi) \arrow[r, "\bar{\pi}^n_{n-1}"] \arrow[d] & Q^{n-1}(\pi) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where π^*_{Tr} is the map induced by π_{Tr}.

Moreover, one can obviously construct the following diagram, $\Delta^n_0(\pi)$, with exact rows and columns and commutative squares:

$$(\Delta^n_0(\pi)) \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & J^n_{0,f}(\pi) \arrow[r, "\varepsilon^n_{0,f}"] \arrow[d] & J^n_f(\pi) \arrow[r, "\pi^n_{0,f}"] \arrow[d] & \pi \arrow[d, equal] & \\ 0 \arrow[r] & J^n_0(\pi) \arrow[r, "\varepsilon^n_0"] \arrow[d] & J^n(\pi) \arrow[r, "\pi^n_0"] \arrow[d] & \pi \arrow[r] & 0 \\ 0 \arrow[r] & J^n_0(\pi)/J^n_{0,f}(\pi) \arrow[r, "\bar{\varepsilon}^n_0"] \arrow[d] & Q^n(\pi) \arrow[r, "\bar{\pi}^n_0"'] \arrow[d] & \pi \arrow[u, equal] & \\ & 0 & 0 & & \end{tikzcd}$$

where $J^n_0(\pi) = \operatorname{Ker} \pi^n_0 \in VB^{k-n}(V, \prod_{p=1}^{n} L^p_{sym}(E, F))$, $J^n_{0,f}(\pi) = \operatorname{Ker} \pi^n_{0,f} \in VB^{k-n}(V, \prod_{p=1}^{n} L^p_{sym}(E_2, F))$. At the end of this section it is important to mention the following remarks:

1) If the foliate manifold V is C^∞-differentiable of type σ, where σ is a foliate family of Banach C-spaces [5] we can use the theory of infinite order prolongations of the surmersions, fibre bundles, groupoides and fibre bundles associated to the groupoides developed in [8], and we obtain the foliate prolongations of infinite order of the previous configurations. Particularly, the foliate transversale holonomic prolongations of infinite order of the f.v.b. $(M, V, \pi)^\infty \in VB^\infty(V, F)$ is the f.v.b. $J^\infty_f(\pi) \in VB^\infty(V, F \times \prod_{n=1}^{\infty} L^n_{sym}(E_2, F))$ with the local convex space indicated in parentheses as type fibre.

2) The results of this section admit a generalization to the case of finite and infinite order semiholonomic foliate prolongations. For this, one constructs, by recurrence, the semiholonomic jet manifolds $\bar{J}^n_f(V, M)$ of foliate morphisms; they are the submanifolds in $\bar{J}^n(V, M)$ (the semiholonomic n-jet manifold) and contains $J^n_f(V, M)$ as embedded submanifolds ($\forall n \leqslant k$). Using the embeddings $\omega_{n,f}: J^n_f(V, M) \to \bar{J}^n_f(V, M)$ we can define the semiholonomic foliate finite-dimensional prolongations of the surmersions, groupoides and f.f.b. In the particular case of the f.v.b. one obtains the semiholonomic transversale prolongations $\bar{J}^n_f(\pi) \in VB^{k-n}(V, F \times \prod_{p=1}^{n} L^p(E_2, F))$, $\bar{J}^\infty_f(\pi) \in VB^\infty(V, F \times \prod_{p=1}^{\infty} L^p(E_2, F))$ (if the model space for the manifold V is a Banach C-space) and the diagrams $\bar{\Delta}^n_{n-1}(\pi)$, $\bar{\Delta}^n_0(\pi)$ which are similar to $\Delta^n_{n-1}(\pi)$ and $\Delta^n_0(\pi)$.

3) Using some results from [7] we can construct the diagrams similar to $\Delta^n_{n-1}(\pi)$, $\bar{\Delta}^n_{n-1}(\pi)$ for every surmersion $(M, V, \pi)^k$ or f.f.b. $(M, V, \pi)^k_\Phi$. Such that, from the propositions 5.1, 6.1, 7.3 from [7], it results the following local direct exact sequences of vector bundles with $J^n(\pi)$ and $\bar{J}^n(\pi)$ as base spaces respectively:

$$(*) \quad 0 \to \pi^{n*}_0 L^n_{sym}(\pi^*\tau_V, \nu_r\tau_M) \to TJ^n(\pi) \to \pi^{n}_{n-1}{}^* TJ^{n-1}(\pi) \to 0$$

$$(**) \quad 0 \to \bar{\pi}^{n*}_0 L^n(\pi^*\tau_V, \nu_r\tau_M) \to T\bar{J}^n(\pi) \to \bar{\pi}^{n}_{n-1}{}^* T\bar{J}^{n-1}(\pi) \to 0,$$

where „$*$“ denotes the pull-back of a vector bundle from a map ([1] 7.2.4.) and $\nu_r\tau_M$ is the vector bundle of the „vertical“ vectors, namely the tangent vectors of the fibres of π.

According to the results from Proposition 3.3, the vector bundle $(TJ^n_f(\pi), J^n_f(\pi), \tau_{J^n_f(\pi)})$ is a sub-vector bundle in $(TJ^n(\pi)|_{J^n_f(\pi)}, J^n_f(\pi), \tau_{J^n(\pi)})$ and we obtain the following diagram of vector bundles with exact rows and columns and commutative squares and the manifold $J^n_f(\pi)$ as base space:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \pi^{n*}_{0,f} L^n_{sym}(\pi^*\tau_{V|Tr(F)}, \nu_V\tau_M) \arrow[r] \arrow[d] & TJ^n_f(\pi) \arrow[r, "T\pi^n_{n-1,f}!"] \arrow[d] & \pi^{n}_{n-1,f}{}^* TJ^{n-1}_f(\pi) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi^{n}_{0,f} L^n_{sym}(\pi^*\tau_V, \nu_V\tau_M) \arrow[r] \arrow[d] & TJ^n(\pi)|_{J^n_f(\pi)} \arrow[r, "T\pi^n_{n-1}!"] \arrow[d] & \pi^{n*}_{n-1,f} TJ^{n-1}(\pi)|_{J^{n-1}_f(\pi)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi^{n}_{0,f}{}^*\left(L^n_{sym}(\pi^*\tau_V, \nu_r\tau_M)/L^n_{sym}(\pi^*\tau_{V|Tr(F)}, \nu_r\tau_M)\right) \arrow[r] \arrow[d] & TJ^n(\pi)|_{J^n_f(\pi)}/TJ^n_f(\pi) \arrow[r] \arrow[d] & \pi^{n}_{n-1}{}^*\left(J^{n-1}(\pi)|_{J^{n-1}_f(\pi)}/TJ^{n-1}_f(\pi)\right) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where the maps $T\pi^n_{n-1}!$ and $T\pi^n_{n-1,f}!$ are defined in [12].

From remark 2 and the sequence $(**)$, it follows a diagram $\tilde{\bar{\Delta}}^n_{n-1}(\pi)$ similar to $\tilde{\Delta}^n_{n-1}(\pi)$. If the f.f.b. $(M, V, \pi)^k_\Phi$ is a f.v.b., then $\nu_r\tau_M = \pi^* M$ and from the diagram $\tilde{\Delta}^n_{n-1}(\pi)$ we can obtain the diagram $\Delta^n_{n-1}(\pi)$ (and similar in the semiholonomic case).

4. Adapted and foliate connections of higher order on f. f. b. Let $(M, V, \pi)^k_\Phi$ be a f.f.b. associated to the groupoide Φ. In [4] P. Ver Eecke introduces the submanifold of n^{th} order Φ-connection elements $(n \leq k)$ given by:

$$Q^n(V, \Phi) = \{ X / X \in J^n(V, \Phi) ;\ \beta_n(X) = i(\alpha_n(X)),\ aX = j^n_{\alpha_n(X)},\ bX = j^n_{\alpha_n(X)} \}.$$

Denoting by q the map $q = \alpha_n\, Q^n(V, \Phi)$, $(Q^n(V, \Phi), V, q)^{k-n}$ is a surmersion; moreover, the groupoide $\Phi^n_r(V) \oplus \Phi^n(V)$ operates on this surmersion by:

$$(4.1)\quad \odot : (\Phi^n_r(V) \oplus \Phi^n_r(V)) \oplus Q^n(V, \Phi) \to Q^n(V, \Phi),\quad ((A, B), X) \to {}^n\pi({}^n\pi(A, X),\ sB), (bB)^{-1},$$

so that the previous surmersion is a fibre bundle associated to this groupoide with a Banach space as the type fibre [4].

The *Φ-connections of n^{th} order* on the manifold V are the C^{k-n}-differentiable cross-sections of this fibre bundle.

Now, for every n, $n \leq k$, we can consider the set $Q^n_f(V, \Phi) = Q^n(V, \Phi) \cap J^n_f(V, \Phi)$ and $q_f = q\,|\,Q^n_f(V, \Phi)$; we prove:

Proposition 4.1. *1) $Q^n_f(V, \Phi)$ is a submanifold in $Q^n(V, \Phi)$, 2) The groupoide $\Phi^n_{r,f} \oplus \Phi^n_f(V)$ operates on the manifold $Q^n_f(V, \Phi)$ so that $(Q^n_f(V, \Phi), V, q_f)^{k-n}$ is a fibre bundle associated to this groupoide.*

Proof. 1) The submersion $(a, b) : \Phi \to V \times V$ induces the submersion $({}^n(a, b), \beta_n) : J^n(V, \Phi) \to J^n(V, V \times V) \oplus \beta_{n}, (a, b)\, \Phi$; we can identify this last submersion with: $(({}^na, {}^nb), \beta_n) : J^n(V, \Phi) \to (J^n(V, V) \oplus J^n(V, V)) \oplus \Phi$. Then $Q^n(V, \Phi)$ is the pull-back of the embedding $V \to (J^n(V, V) \oplus J^n(V, V)) \oplus \Phi$ $(x \to (j^n_x, j^n_x X, i(x)))$ from the previous submersion. The pull-back of the same embedding from the submersion $(({}^na, {}^nb), \beta_n)|J^n_f(V, \Phi)$, is the set $Q^n_f(V, \Phi)$; so that it is a submanifold in $J^n_f(V, \Phi)$, hence in $J^n(V, \Phi)$ and obviously it is included in $Q^n(V, \Phi)$. From these considerations it results that $Q^n_f(V, \Phi)$ is a submanifold in $Q^n(V, \Phi)$

2) it immediately follows from the remark that $\odot(\Phi^n_{r,f}(V) \oplus \Phi^n_f(V)) \oplus Q^n_f(V, \Phi)) \subset Q^n_f(V, \Phi)$.

Definition 4.1. *A cross-section of the fibre bundle $(Q^n_f(V, \Phi), V, q_f)^{k-n}$ is called an adapted holonomic Φ-connection of n^{th} order on the foliate manifold V. His value in $x \in V$ is called an adapted holonomic Φ-connection element of order n in $x \in V$.*

According to [4] (III 5.6) the following existence theorem results:

Theorem 4.2. *Let V be a paracompact foliate manifold of type σ, where σ is a foliate family of C^k-normal separable Banach spaces. Then on V there exists adapted holonomic Φ-connections of order n $(\forall n \leq k)$.*

The morphism

$$Q^n_f(V, \Phi) \oplus_{qf,\, \pi \circ \beta n} J^n_f(\pi) \to J^n_{r,f}(\pi),\qquad (X, Y) \to \nabla_X Y = {}^n\psi(X, Y)$$

permits to define the *absolute differentiation* of a cross-section $\mathfrak{Y} : x \to Y_X$ of $J^n_f(\pi)$ relative to adapted Φ-connection $\mathfrak{X} : x \to X_x$ on V; this is a cross-section $\nabla_{\mathfrak{X}} \mathfrak{Y}$ of $J^n_{r,f}(\pi)$ given by: $\nabla_{\mathfrak{X}} \mathfrak{Y} : x \to \nabla_{X_x} Y_x$. Now, we can prove the following result:

Proposition 4.3. *Let $(M, V, \pi)^k_\Phi$ be a f.f.b. associated to the groupoide Φ with the foliate connected manifold V as base space; then the fibre bundles $(J^n_{r,f}(\pi), V, \pi^r_n\,|\,J^n_{r,f}(\pi))^{k-n}$ (Theorem 3.5) and $(Q^n_f(V, \Phi), V, q_n)^{k-n}$ (Proposition 4.1) are foliate fibre bundles.*

Proof. We consider a family $\{\sigma_U(x_0)\}$ of foliate cross-sections of submersion $a : \Phi \to V$ (Remark 2.1) so that $\forall\, U \subset V$ there exists a f. m. $h_\alpha = (U_\alpha, \varphi_\alpha, E_\alpha)$ with $U \subset U_\alpha$ and we associate to this family a family $\{\sigma^n_{U,r}(x_0)\}$ of foliate local cross-sections of the submersion $a \circ \beta_n | \varphi^n_{r,f}(V)$. For this purpose, let $\sigma_U(x_0)$ be a fixed cross-section from the given

family and $\varkappa_U$ a foliate cross-section over U of the fibre bundle $Q^n_f(V, \Phi)$. Using the morphism $\theta : Q^n_f(V, \Phi) \oplus_{q_f, {}^n a \circ \beta_n | \Phi^n_f(V)} \Phi^n_f(V) \to \Phi^n_{r,f}(V)$, $(X, Y) \to {}^n\pi(X, Y)$ we shall define the foliate local cross-section $\sigma^n_{U,r}(x_0) : U \to \Phi^n_{r,f}(V)$, $x \to (\theta(\varkappa_n(x), j^n_U \sigma_U(x_0))$.
Now, it is easy to obtain the families of the foliate local cross-sections of the groupoides $\Phi^n_{r,f}(V) \oplus \Pi^n_f(V)$ and $\Phi^n_{r,f}(V) \oplus \Phi^n_f(V)$ and the proposition is proved.

If $(M, V, \pi)^k_\Phi \in VB^k(V, F)$ then, according to the principal result from [9], every holonomic Φ-connection of order n on V, $\varkappa$, determines an isomorphism $\mathcal{J}$ of the fibre bundle $J^n_r(\pi)$ and $J^n(\pi)$: $\mathcal{J} : J^n_r(\pi) \to j^n(\pi)\ j^n_x(g) \to j^n_x(fg)$ where $\varkappa(x) = j^n_x f$. This isomorphism defines an *n-connection* on the considered vector bundle (i. e. a differentiable splitting of the second row of $\Delta^n_0(\pi)$).

It is easy to see that, if $\varkappa$ is an adapted Φ-connection on V, it determines an isomorphism $\mathcal{J}_f$ of the fibre bundles $J^n_{r,f}(\pi)$ and $J^n_f(\pi)$. We shall call an *adapted n-connection* on the vector bundle $(M, V, \pi)^k$ a differentiable splitting of the second row of $\Delta^n_0(\pi)$ with their values in the sub-vector bundle $J^n_f(\pi)$. Summing up the previous results we obtain:

Proposition 4.4. *Every adapted holonomic Φ-connection of n^{th} order on the foliate manifold V determines an adapted n-connection on the f.v.b. $(M, V, \pi)^k_\Phi$.*

The theory of adapted holonomic n-connections on f.v.b. may be developed in a similar way as the theory of holonomic n-connections on vector bundles [8].

If V is a C^∞-differentiable foliate manifold of type σ with σ a foliate family of Banach C-spaces, the results of this section admits a generalization for the infinite order case. We can also consider a similar theory for the semiholonomic case.

Now, we shall give one more definition: an adapted holonomic Φ-connection of n^{th} order on the foliate manifold V is *foliate* if the cross-section which defines it is a foliate section; an adapted n-connection on a f.v.b. $(M, V, \pi)^k$ is called *foliate n-connection* if the morphism which defines it is foliate.

From the previous considerations it follows:

Proposition 4.5. *Every foliate Φ-connection of n^{th} order on the foliate manifold V defines a foliate n-connection on the f.v.b. $(M, V, \pi)^k_\Phi$, i. e. a foliate splitting of the first row of $\Delta^n_0(\pi)$.*

Using the holonomic and semiholonomic generalized connections of order, n introduced by P. Libermann in [7], for the surmersions or fibre bundles, we can extend the results from [15] concerning the adapted holonomic and semiholonomic connections of order n on f.v.b.

A holonomic (semiholonomic) generalized connection of order n on the surmersion $(M, V, \pi)^k$ is a globale cross-section $C_n : J^{n-1}(\pi) \to J^n(\pi)$ $(C_n : J^{n-1}(\pi) \to \bar{J}^n(\pi))$ of the affine fibre bundle $\pi^n_{n-1} : J^n(\pi) \to J^{n-1}(\pi)$ $(\bar{\pi}^n_{n-1} : \bar{J}^n(\pi) \to J^{n-1}(\pi))$. If $(M, V, \pi)^k \in V\overline{B}^k(V, F)$ then a holonomic (semiholonomic) connection of order n of this fibre bundle is a holonomic (semiholonomic) generalized connection of order n such that C_n (C_n) is a morphism of vector bundles.

We limit here our considerations, to the holonomic case and we remark that every holonomic generalized connection of n^{th} order on the surmersion $(M, V, \pi)^k$ defines a splitting of the second row of $\tilde{\Delta}^n_{n-1}(\pi)$; it is given by the composition of the canonical homomorphism $\pi^{n*}_{n-1}\ TJ^{n-1}(\pi) \to TJ^{n-1}(\pi)$ with the vector bundle morphism $TC_n : TJ^{n-1}(\pi) \to TJ^n(\pi)$. As in [15], *a generalized connection of order n* on the given surmersion will be called *adapted* if $C_n(J^{n-1}_f(\pi)) \subset J^n_f(\pi)$. Now, noting by $C_{n,f} = C_n | J^{n-1}_f(\pi)$ we have $C_n \circ i_{n-1} = i_n \circ C_{n,f}$ hence $TC_n \circ Ti_{n-1} = Ti_n \circ TC_{n,f}$ and by the composition of the canonical morphism π^n_{n-1} $TJ^{n-1}_f(\pi) \to TJ^{n-1}_f(\pi)$ with $TC_{n,f}$ we obtain a differentiable splitting of the first row of $\tilde{\Delta}^n_{n-1}(\pi)$. Applying the results from [15], it follows that each splitting of the first row of $\tilde{\Delta}^n_{n-1}(\pi)$ is the induced splitting of the set of some splittings of the second row of $\tilde{\Delta}^n_{n-1}(\pi)$; but, obviously, we cannot say that these splittings are the splittings defined by some adapted generalized connections of order n.

If $(M, V, \pi)^k$ is a fibre bundle associated to the differential groupoide Φ, then a ge neralized connection of order n on it is called *foliate* if the morphism $C_{n,f}$ is a foliate mor phism. Since the first row of $\widetilde{\Delta}^n_{n-1}(\pi)$ is not a sequence of f.v.b., we say, as in [15], abou the foliate splittings of this sequence that they are induced splittings of the foliate gene ralized connections.

Finally, we remark that the adapted and foliate generalized semiholonomic connec tions and their relation with the splittings of certain sequences of $\widetilde{\Delta}^n_{n-1}(\pi)$ can be develope in a manner similar to the one indicated in the holonomic case.

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Faculty of Mathematics
University of Iaşi
Romania

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UN THÉORÈME D'EXISTENCE POUR LES ÉQUATIONS DIFFÉRENTIELLES ORDINAIRES DANS LES ESPACES LOCALEMENT CONVEXES

PAR

NECULAI PAPAGHIUC

0. Ces dernières années ont paru nombre de travaux de la théorie de la différentiabilité dans les espaces localement convexes (G. Marinescu [8], [9], [10], J. P. Penot [12], I. V. Averbuh et O. G. Smolianov [2], H. H. Keller [6], J. Kijowski et W. Szczyrba [7], R. A. Graff [5], M. Simonnet [14]. L'un des résultats remarquables obtenus le dernier temps est représenté par le théorème d'inversion locale des applications de classe C^k. Ce résultat a été démontré par G. Marinescu et R. A. Graff. Un autre problème fondamental d'analyse locale à implication dans l'analyse globale est constitué par celui de l'intégration des champs de vecteurs. Le but du présent travail est de résoudre ce problème.

1. Soit E un espace localement convexe et $\mathcal{Q}$ l'ensemble de toutes les semi-normes continues sur E. Soit A_0 un sous-ensemble de l'ensemble $\mathcal{Q}$ qui satisfait à la condition:

(1) *pour tout* $x \in E$, $x \neq 0$, *il existe un indice* $\alpha_0 \in A_0$ *tel que* $|x|_{\alpha_0} \neq 0$ (c'est-à-dire A_0 est un ensemble suffisant de semi-normes).

Nous notons par A l'ensemble des semi-normes ayant la forme $|x| = \max\{|x|_1, \ldots, |x|_n\}$, où $\{|\cdot|_1, \ldots, |\cdot|_n\}$ est un sous-ensemble fini quelconque de l'ensemble A_0. Il est évident que l'ensemble A satisfait à la propriété:

(2) *pour tout couple d'indices* $\alpha_1, \alpha_2 \in A$, *il existe un indice* $\alpha \in A$ *tel que*: $|x|_{\alpha_1} < |x|_\alpha$, $|x|_{\alpha_2} < |x|_\alpha$, $\forall x \in E$ (l'ensemble A est dirigé).

Nous supposons que l'ensemble A est équivalent à $\mathcal{Q}$ (les deux définissent dans E la topologie d'espace localement convexe).

Dans [10] (p. 19) on introduit une classe plus large d'ensembles, qui sont des ensembles ouverts dans une topologie plus fine que la topologie localement convexe.

Considérons R_+ l'ensemble des nombres réels positifs et $\alpha \to r_\alpha$ une application de l'ensemble A_0 dans R_+.

On appelle *intersphère ouverte* de centre x_0 et de rayons $\{r_\alpha\}$, $(\alpha \in A_0)$ l'ensemble $S(x_0; \{r_\alpha\}) = \bigcap_{\alpha \in A_0} \{x \in E;\ |x - x_0|_\alpha < r_\alpha\}$. C'est avec l'aide des intersphères ouvertes qu'on définit dans E une nouvelle topologie appellée la *topologie raffinée* de E, dans laquelle un sous-ensemble $U \subset E$ est défini comme ouvert, si, pour tout $x_0 \in U$, il existe une intersphère ouverte des centre x_0 incluse dans U.

Les ouverts dans la topologie raffinée seront appelés des ensembles raffinés ouverts. La topologie raffinée est, évidemment, plus fine que la topologie d'espace localement convexe de E, et, grâce à la condition (1), elle est séparée.

Soit I un ensemble dirigé. On dit qu'une suite généralisée $\{x_i\}_{i \in I}$ (une application de l'ensemble I dans E) est convergente dans la topologie raffinée vers un élément $x \in E$ si pour tout système de nombres réels positifs $\{r_\alpha\}$, $(\alpha \in A_0)$ il existe un $i_0 \in I$ tel que $i > i_0$ implique $|x_i - x|_\alpha < r_\alpha$ (c'est-à-dire $x_i \in S(x; \{r_\alpha\})$).