C^k. Soit maintenant $F: R\times U_0\times C_0^1(I,\ U_0)\rightarrow C^0(I,\ E)$ définie par $F(a, x, \varphi)(t)=\dot{\varphi}(t) - af(at, x+\varphi(t))$ et montrons que F est de classe C^k. Eidemment F s'écrit comme différence de deux applications $F=F_1-F_2$ où $F_1: R\times U_0\times C_0^1(I,\ U_0)\rightarrow C^0(I,\ E)$ est définie par $F_1(a,\ x,\ \varphi)(t)=\dot{\varphi}(t)$ et $F_2: R\times U_0\times C_0^1(I,\ U_0)\rightarrow C^1(I,\ E)$ est définie par $F_2(a, x, \varphi)(t)=a\ f(at, x+\varphi(t))$. Comme F_1 est une application linéaire et continue, donc de classe C^1, il reste à montrer que F_2 est de classe C^1. Soit alors $\tau: R\rightarrow C^1(I,\ R)$ l'application définie par $[\tau(a)](t)=a.t$ pour tout $a\in R$ et $t\in I$. L'application τ est linéaire et continue. Notons par $\sigma: C(I,\ U_0)\times C_0^1(I,\ U_0)\rightarrow C^1(I,\ U)$ l'application définie par: $\sigma(x,\ \varphi)=x+\varphi$ où par $C(I,\ U_0)$ on a noté les applications $x: I\rightarrow U_0$ définies par $x(t)=x$ pour tout $t\in I$. L'application σ est un plongement de $C(I,\ U_0)\times C_0^1(I,\ U_0)$ dans $C^1(I,\ U)$ et donc elle est de classe C^1. Notons encore par $\mathfrak{M}$ $R\times C^1(I,\ E)\rightarrow C^1(I,\ E)$ l'application définie par $\mathfrak{M}(a,\ g)=ag$ qui, évidemment, est une application linéaire et continue. On déduit alors que:

$$R\times U_0\times C_0^1(I,\ U_0)\xrightarrow{\tau\times\sigma} C^1(I,\ R)\times C^1(I,\ U)\xrightarrow{\Omega_f} R\times C^1(I,\ E)\xrightarrow{\mathfrak{M}} C^1(I,\ E)$$

donc $F_2=\mathfrak{M}\cdot\Omega_f\circ(\tau\times\sigma)$ est une application de classe C^1 comme composition d'applications de classe C^1.

D'une façon analogue, si f est de classe C^k, il résulte que F_2 est de classe C^k et donc que F est de classe C^k.

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Reçu le 18.III.1874

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Analele științifice ale Universității „Al. I. CUZA" din Iași
Tomul XXI, s. I a, 1975

UNSTEADY FLOW OF A SECOND ORDER FLUID BETWEEN TWO POROUS CO-AXIAL CIRCULAR CYLINDERS

BY

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Introduction. Markovitz and Coleman [1] have suggested some practical methods for the determination of the material constants of a Second order fluid. One of those methods makes use of Helical flow. A Helical flow is a combination of two flows, one axial flow through an annulus of two co-axial circular cylinders and the other, a rotational flow due to the rotation of one of the cylinders about their common axis. Several authors have discussed similar flow problems [2—5].

In this paper a generalized Helical flow of a Second order fluid is considered by assuming the cylinders to be porous. Different cases of this problem have been solved. It has been assumed that the rate of suction at one cylinder is equal to the rate of injection at the other cylinder and that the Reynolds number of the flow is small.

Equations of Motion.: The constitutive equation of an incompressible second order fluid as suggested by Coleman and Noll [6] is

$$p_{ij} = -p\delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)ik} A_{(1)kj}$$

where

$$A_{(1)ij} = v_{i,j} + v_{j,i}$$

and

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i} v_{m,j},$$

p_{ij} is the stress tensor, v_i and a_i the velocity and acceleration vectors, μ_1, μ_2, μ_3 the material constants, p an undetermined hydro-static pressure and a comma denotes covariant differentiation. Let us consider the cylindrical polar co-ordinates system $(r,\ \theta,\ z)$ assuming z-axis along the common axis and r along the radii of the cylinders. Let an incompressible second order fluid be contained between two porous co-axial circular cylinders of radii a and b $(a<b)$.

Since the rate of suction at one cylinder is equal to the rate of injection at the other, the velocity profile will be independent of θ and z. For the present case the equations of motion and continuity, in cylindrical co-ordinates $(r,\ \theta,\ z)$, are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = \frac{\partial p_{rr}}{\partial r} + \frac{p_{rr} - p_{\theta\theta}}{r}, \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\right) = \frac{\partial p_{\theta r}}{\partial r} + \frac{2p_{r\theta}}{r},$$

(1)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r}\right) = \frac{\partial p_{zr}}{\partial r} + \frac{p_{zr}}{r},$$

(2)
$$\frac{\partial}{\partial r}(ur) = 0,$$

where ρ is the density of the fluid and u, v and w the velocity components along r, θ and z respectively. The equation of continuity (2) suggests that the form of radial velocity should be

(3)
$$u = \frac{H(t)}{r}.$$

We assume that the other two velocity components are of the form

$$v = rG(r,\ t)\ ;\ w = F(r,\ t). \tag{4}$$

For an arbitrary generalized Helical flow, for the present case, the dynamical equations in the absence of the body forces reduce to

$$\frac{1}{r}\frac{\partial H}{\partial t} - \frac{H^2}{r^4} - G^2 r = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu_2\left\{-\frac{24\,H^2}{r^5} - 4\overline{G}\overline{G}' + 2r\overline{G}'^2 - 2r\overline{G}\overline{G}'' + \right.$$

$$\left. + 2r^2\overline{G}'\overline{G}'' + 4\overline{F}'\overline{F}'' + \frac{2}{r}\overline{F}'^2\right\} + \nu_3\left\{-\frac{16\,H^2}{r^5} + \frac{\overline{F}'^2}{r} + 2\overline{F}'\overline{F}'' + 2r\overline{G}'^2 + 2r\overline{G}'\overline{G}''\right\},$$

(5)

$$r\frac{\partial G}{\partial r} + 2\frac{\overline{H}\overline{G}}{r} + \overline{H}\overline{G}' = \nu_1(3\overline{G}' + r\overline{G}'') + \nu_2\left\{3\frac{\partial G'}{\partial t} + r\frac{\partial G''}{\partial t} + \frac{5\overline{H}\overline{G}''}{r} + \frac{3\overline{H}\overline{G}'}{r^2} + \overline{H}\overline{G}'''\right\},$$

$$\frac{\partial\overline{F}}{\partial t} + \frac{\overline{H}\overline{F}'}{r} = \nu_1\left(\overline{F}'' + \frac{\overline{F}'}{r}\right) + \nu_2\left\{\frac{\partial\overline{F}''}{\partial t} + \frac{\partial\overline{F}'}{r\partial t} + \frac{\overline{H}\overline{F}'''}{r} - \frac{\overline{H}\overline{F}''}{r^2} + \frac{\overline{H}\overline{F}'}{r^3}\right\} + \nu_3\left\{2\frac{\overline{F}'\overline{H}}{r^3} - \frac{2\overline{H}\overline{F}''}{r^2}\right\}$$

where a prime denotes differentiation with respect to r.

From the above equations it is clear that the three differential equations contain four unknown functions i. e. p, $\overline{H}$, $\overline{G}$ and $\overline{F}$. Hence in order that one could get a solution of these equations under suitable boundary conditions, one has to assume one of these functions to be known. Here if we assume the suction, injection to be steady i. e. $\overline{H}$ a constant, p, $\overline{F}$ and $\overline{G}$ can be determined by equations (5). In order to make the functions $\overline{F}$, $\overline{G}$, $\overline{H}$ and p, r and t non-dimensional we put

$$\overline{H}(t) = a^2\Omega H(T)\ ;\ \overline{G}(r,\ t) = \Omega G(\eta,\ T), \qquad p = -\nu_1\Omega\rho P\ ;\ R = \frac{\Omega a^2}{\nu_1}$$

(6)

$$\overline{F}(r,\ t) = a\Omega F(\eta, T);\ \ T = nt\ ;\ \ \eta = \frac{r}{a}\ ; \qquad \alpha = \frac{\nu_2}{a^2}\ ;\ \beta = \frac{\nu_3}{a^2}\ ;\ \ \nu_n = \frac{\mu_n}{\rho}\ ;\ \ (n = 1, 2, 3).$$

$$\omega = \frac{n}{\Omega} \tag{7}$$

The equation of motion now takes the form

$$R\left\{\frac{\omega}{\eta}\frac{\partial H}{\partial T} - \frac{H^2}{\eta^3} - G^2\eta\right\} = \frac{\partial P}{\partial\eta} + \alpha R\left\{-\frac{24H^2}{\eta^5} - 4GG' + 2\eta G'^2 - 2\eta GG'' + 2\eta^2 G'G'' + \right.$$

(8 a)

$$\left. + 4F'F'' + \frac{2F'^2}{\eta}\right\} + \beta R\left\{-\frac{16\,H^2}{\eta^5} + \frac{F'^2}{\eta} + 2F'F'' + 2\eta G'^2 + 2\eta^2 G'G''\right\},$$

$$R\left\{\omega\eta\frac{\partial G}{\partial T} + \frac{2HG}{\eta} + HG'\right\} = 3G' + \eta G'' + \alpha R\left\{3\omega\frac{\partial G'}{\partial T} + \eta\omega\frac{\partial G''}{\partial T} + \frac{5HG''}{\eta} + \frac{3HG'}{\eta^2} + HG'''\right\}, \tag{8 b}$$

$$R\left\{\omega\frac{\partial F}{\partial T} + \frac{HF'}{\eta}\right\} = F'' + \frac{F'}{\eta} + \alpha R\left\{\omega\frac{\partial F''}{\partial T} + \frac{\omega}{\eta}\frac{\partial F'}{\partial T} + \frac{HF'''}{\eta} - \frac{HF''}{\eta^2} + \frac{HF'}{\eta^3}\right\} +$$

(8 c)

$$+\beta R\left\{\frac{2F'H}{\eta^3} - \frac{2F''H}{\eta^2}\right\}.$$

From here and onwards, a prime will denote differentiation with respect to η. We should mention here that the following analysis is valid only for small ω. If we assume H to be prescribed, G and F can be calculated by equations (8 b) and (8 c) and p can be determined by equation (8 a).

Solution: Let us put

$$G(\eta,\ T) = e^{ik_1T}\sum_{m=0}^{\infty} R^m\, g_m(\eta, w), \qquad F(\eta,\ T) = e^{ik_2T}\sum_{m=0}^{\infty} R^m\, f_m(\eta,\ w) \tag{9}$$

in equations (8 b), (8 c) respectively and assume that only real parts of functions have any physical meanings. Equating terms in like powers of R in (8 b, 8 c), we get a set of ordinary differential equations. Integrating them we have

$$e^{-ik_1T}\, G(\eta,\ T) = A_0 + B_0\eta^{-2} + RH\{A_1 + B_1\eta^{-2} +$$

$$+ g_1(\eta,\ \omega_1)\} + (RH)^2\{A_2 + B_2\eta^{-2} + g_2(\eta,\ \omega_1)\},$$

(10)

$$e^{-ik_2T}\, F(\eta,\ T) = \bar{A}_0 + \bar{B}_0\log\eta + RH\{\bar{A}_1 + \bar{B}_1\log\eta +$$

$$+ f_1(\eta,\ \omega_2)\} + (RH)^2\{\bar{A}_2 + \bar{B}_2\log\eta + f_2(\eta,\ \omega_2)\}$$

where

$$g_1(\eta,\ \omega_1) = \frac{(2A_0 + i\omega_1 B_0)(2\log\eta - 1)}{4} + \frac{i\omega_1 A_0\eta^2}{8}, \tag{11}$$

$$g_2(\eta,\ \omega_1) = \frac{(2A_1 + i\omega_1 B_1)(2\log\eta - 1)}{8} + \frac{i\omega_1 A_1\eta^2}{8} -$$

$$-\frac{\omega_1^2 A_0\eta^4}{192} + \frac{i\omega_1(2A_0 + i\omega_1 B_0)}{64}\eta^2(4\log\eta - 5) +$$

(12)

$$+ \frac{i\omega_1 A_0\eta^2}{16} + \frac{(2A_0 + i\omega_1 B_0)}{4}\{(\log^2\eta - \log\eta)\} -$$

$$-\alpha\left\{\frac{i\omega_1(4A_0 + i\omega_1 B_0)(2\log\eta - 1)}{4} - \frac{\omega_1^2 A_0\eta^2}{8}\right\}.$$

$$f_1(\eta,\ \omega_2) = \frac{i}{4}\{\bar{A}_0\eta^2 + \bar{B}_0\eta^2(\log\eta - 1)\}\omega_2 + \frac{\bar{B}_0}{2}(\log\eta)^2 - \frac{\bar{B}_0}{\eta^2}(\alpha + \beta). \tag{13}$$

$$f_2(\eta,\ \omega_2) = \frac{i\omega_2}{4}\{\bar{A}_1\eta^2 + \bar{B}_1\eta^2(\log\eta - 1)\} + \frac{\bar{B}_1}{2}(\log\eta)^2 - \frac{\bar{B}_1}{\eta^2}(\alpha + \beta) - \omega_2^2\left\{\frac{\bar{A}_0\eta^4}{64} + \right.$$

$$\left. + \frac{\bar{B}_0\eta^4}{64}(\log\eta - 1/2) - \frac{\bar{B}_0\eta^4}{64}\right\} + i\omega_2\left\{\frac{\bar{B}_0\eta^2}{8}(\log^2\eta - \log\eta + 3/4) - \frac{\bar{B}_0(\alpha + \beta)\log^2\eta}{2}\right\} +$$

$$+ \frac{i\omega_2}{8}\left\{\bar{A}_0\eta^2 + \bar{B}_0\eta^2(\log\eta - 1) - \frac{\bar{B}_0\eta^2}{2}\right\} +$$

$$+ \frac{\bar{B}_0}{6}(\log\eta)^3 - \frac{\bar{B}_0}{2\eta^2}(\alpha + \beta) - \alpha\left\{-\omega_2^2\left(\frac{\bar{A}_0}{2}\log^2\eta + \right.\right.$$

(14)

$$\left. + \frac{\bar{B}_0}{6}\log^3\eta\right) - \frac{i\omega_2\bar{B}_0}{4}\log^2\eta - \frac{i\omega_2}{\eta^2}\bar{B}_0(\alpha + \beta) +$$

$$\left. + \frac{\bar{B}_0}{2}\log\eta + \frac{2\bar{B}_0}{\eta^4}(\alpha + \beta)\right\} -$$

$$-2\beta\left\{\frac{\bar{B}_0}{2\eta^2}(\log\eta + 1) - \frac{\bar{B}_0}{4\eta^2} - \frac{i\omega\bar{B}_0}{2}\log^2\eta + \frac{\bar{B}_0(\alpha + \beta)}{2\eta^4}\right\}$$

and

$$\omega_1 = \frac{k_1 n}{H\Omega}; \qquad \omega_2 = \frac{k_2 n}{H\Omega} \tag{15}$$

and A_0, B_0, A_1, B_1, A_2, B_2, $\bar{A}_0$, $\bar{B}_0$, $\bar{A}_1$, $\bar{B}_1$, $\bar{A}_2$, and $\bar{B}_2$ are constants of integration which will determine the pattern of flow.

We should point out that by the principle of superposition, a more general form of time dependence of F and G can be discussed.

In order to know the nature of flow in more details, the following six cases have been discussed:

(1) The inner cylinder is oscillating about the common axis of the cylinders and the outer rotates with a constant angular velocity.

(2) The inner cylinder oscillates about the common axis of the cylinders whereas the outer oscillates parallel to its length.

(3) Both the inner and the outer cylinders oscillate in 0 direction about common axes.

(4) The inner cylinder is rotating with a constant angular velocity when the outer oscillates parallel to its length.

(5) Both the inner and the outer cylinders oscillate parallel to their lengths.

(6) The inner cylinder moves parallel to its length with a constant velocity and the outer oscillates parallel to its length.

Case 1. When the inner cylinder oscillates as $r\Omega_1 \cos k_1 nt$ and the outer rotates with a constant angular velocity Ω_2, the boundary conditions are

$$v = r\Omega_1 \cos k_1 nt; \qquad w = 0 \quad \text{at} \quad r = a$$

$$v = r\Omega_2; \qquad w = 0 \quad \text{at} \quad r = b.$$

In the light of the transformations (3) and (6), the above boundary conditions become

$$G = \frac{\Omega_1}{\Omega}\cos k_1 T; \quad F = 0 \text{ at } \eta = 1, \qquad G = \frac{\Omega_2}{\Omega}; \; F = 0 \quad \text{at} \quad \eta = K \tag{16}$$

where $K = \dfrac{b}{a} > 1$.

The substitution (9) changes (16) into

$$g_0 = \frac{\Omega_1}{\Omega}; \; g_{m+1} = 0; \; f_m = 0 \text{ for } m \geq 0 \text{ at } \eta = 1$$

(17)

$$(g_0)_{\omega_1 = 0} = \frac{\Omega_2}{\Omega}; \; g_{m+1\,|\,\omega_1 = 0} = 0; \; f_m = 0 \text{ for } m \geq 0 \text{ at } \eta = K.$$

Evaluating the constants, we get

$$A_0 = \frac{K^2\Omega_2 - \Omega_1}{\Omega(K^2-1)}; \qquad B_0 = \frac{K^2}{K^2-1} \cdot \frac{\Omega_1 - \Omega_2}{\Omega}$$

$$A_1 = \frac{(2A_0 + i\omega_1 B_0)}{4} - \frac{i\omega_1 A_0}{8} -$$

$$- \frac{K^2}{K^2-1}\left\{\frac{(2A_0 + i\omega_1 B_0)}{2}\log K + \frac{i\omega_1 A_0(K^2-1)}{8}\right\}. \tag{18}$$

$$B_1 = \frac{K^2}{K^2-1}\left\{\frac{(2A_0 + i\omega_1 B_0)}{2}\log K + \frac{i\omega_1 A_0(K^2-1)}{8}\right\}$$

$$A_2 = -g_2(1, \omega_1) - \frac{K^2}{K^2-1}\{g_2(K, 0) - g_2(1, \omega_1)\} \qquad B_2 = \frac{K^2}{K^2-1}\{g_2(K, 0) - g_2(1, \omega_1)\}$$

$$\bar{A}_0 = \bar{B}_0 = \bar{A}_1 = \bar{B}_1 = \bar{A}_2 = \bar{B}_2 = 0. \tag{19}$$

Case 2. When the inner cylinder oscillates as $r\Omega_1 \cos k_1 nt$ about their common axes and the outer oscillates as $V_0 \cos k_2 nt$ parallel to its length the boundary conditions are

$$v = r\Omega_1 \cos k_1 nt; \quad w = 0 \quad \text{at} \quad r = a$$

$$v = 0; \qquad w = V_0 \cos k_2 nt \quad \text{at} \quad r = b.$$

In the light of the transformations (3) and (6), the above boundary conditions change into

$$G = \frac{\Omega_1}{\Omega}\cos k_1 T; \; F_{\omega_2} = 0 \text{ at } \eta = 1$$

(20)

$$G = 0 \qquad F = M\cos k_2 T \text{ at } \eta = K \qquad M = (V_0/a\Omega).$$

The substitution of (9) in (20) changes the boundary conditions into

$$g_0 = \frac{\Omega_1}{\Omega}; \quad g_{m+1} = 0; \; f_{m|\omega_2 = 0} = 0 \text{ at } \eta = 1 \text{ for } m \geq 0$$

(21)

$$g_{m|\omega_1 = 0} = 0; \; f_0 = M; \; f_{m+1} = 0 \text{ for } m \geq 0 \text{ at } \eta = K.$$

Evaluating the constants we have

$$A_0 = -\frac{\Omega_1}{\Omega(K^2-1)}; \qquad B_0 = \frac{\Omega_1}{\Omega}\,\frac{K^2}{(K^2-1)}. \tag{22}$$

A_1, B_1, A_2, B_2 are the same as in case 1.

$$\bar{A}_0 = 0; \; \bar{B}_0 = \frac{M}{\log K}$$

$$\bar{A}_1 = B_0(\alpha+\beta); \; \bar{B}_1 = \frac{1}{\log K}\left\{\frac{\bar{B}_0}{K^2}(\alpha+\beta) - \right.$$

(23)

$$\left. - \bar{B}_0(\alpha+\beta) - \frac{i\omega_2}{4}\bar{B}_0 K^2(\log K - 1) - \frac{\bar{B}_0}{2}(\log K)^2\right\}$$

$$\bar{A}_2 = -f_2(1, 0);$$

$$\bar{B}_2 = \frac{1}{\log K}\left\{-f_2(K, \omega_2) + f_2(1, 0)\right\}.$$

Case 3. Let the two cylinders oscillate with a phase difference of H. In the present case the boundary conditions are

$$v = r\Omega_1 \cos k_1 nt; \qquad w = 0 \text{ at } r = a$$

$$v = r\Omega_2 \cos(k_1 nt + H), \quad w = 0 \text{ at } r = b.$$

This in view of (3), (6) and (9) becomes

$$g_0 = \frac{\Omega_1}{\Omega}; \; g_{m+1} = 0; \; f_m = 0 \text{ at } \eta = 1 \text{ for } m \geq 0$$

(24)

$$g_0 = \frac{\Omega_2}{\Omega}; \; g_{m+1} = 0; \; f_m = 0 \text{ at } \eta = K \text{ for } m \geq 0.$$

Applying the above boundary conditions we get

$$A_0=-\frac{K^2\Omega_2+\Omega_1}{\Omega(K^2-1)}\,;\ B_0=\frac{K^2}{K^2-1}\cdot\frac{\Omega_1+\Omega_2}{\Omega}$$

(25)
$$A_1=\frac{2A_0+i\omega_1 B_0}{4}-\frac{i\omega_1 A_0}{8}-$$
$$-\frac{K^2}{K^2-1}\left\{\frac{(2A_0+i\omega_1 B_0)}{2}\log K+\frac{i\omega_1 A_0K^2}{8}-\frac{i\omega_1 A_0}{8}\right\}$$
$$B_1=\frac{K^2}{K^2-1}\left\{\frac{(2A_0+i\omega_1 B_0)}{2}\log K+\frac{i\omega_1 A_0K^2}{8}-\frac{i\omega_1 A_0}{8}\right\}$$

$$A_2=-B_2-g_2(1,\ \omega_1)$$

(26)
$$B_2=\frac{K^2}{K^2-1}\{g_2(K,\ \omega_1)-g_2(1,\ \omega_1)\}$$

$$\bar{A}_0=\bar{B}_0=\bar{A}_1=\bar{B}_1=\bar{A}_2=\bar{B}_2=0.$$

Case 4. In the present case, the boundary conditions are

$$v=r\Omega_1\,;\quad w=0\qquad\text{at } r=a$$
$$v=0\,;\qquad w=V_0\cos k_2nt\ \text{ at } r=b.$$

In view of (3), (6) and (9) the above boundary conditions change into

(27)
$$g_0\big|_{\omega_1=0}=\frac{\Omega_1}{\Omega}\,;\ g_{m+1}\big|_{\omega_1=0}=0\,;\ f_m\big|_{\omega_2=0}=0\quad\text{for } m\geqslant 0 \text{ at } \eta=1,$$
$$g_m=0\,;\quad f_0=M\,;\quad f_{m+1}=0\ \text{ for } m\geqslant 0\ \text{ at } \eta=K$$
$$M=\frac{V_0}{\Omega a}.$$

Solving for constants we get

(28)
$$A_0=-\frac{\Omega_1}{\Omega(K^2-1)}\,;\quad B_0=\frac{\Omega_1}{\Omega}\frac{K^2}{K^2-1}\qquad A_1=\frac{A_0}{2}-\frac{K^2}{K^2-1}\cdot A_0\log K$$
$$B_1=\frac{K^2}{K^2-1}\cdot A_0\log K$$
$$A_2=-g_2(1,\ 0)-\frac{K^2}{K^2-1}\{g_2(K,\ 0)-g_2(1,\ 0)\}$$
$$B_2=\frac{K^2}{K^2-1}\{g_2(K,\ 0)-g_1(1,\ 0)\}.$$

The values of $\bar{A}_0$, $\bar{B}_0$, $\bar{A}_1$, $\bar{B}_1$, $\bar{A}_2$, $\bar{B}_2$ are the same as in case 2.

Case 5. Assuming that both the cylinders oscillate parallel to their length (with a phase difference of π), the motion of the cylinders is described as

(29)
$$v=0\,;\quad w=v_1\cos k_2nt\quad\text{at}\quad r=a$$
$$v=0\,;\quad w=v_2\cos(k_2nt+\pi)\quad\text{at}\quad r=b.$$

In view of (3), (6) and (9), the above boundary conditions will become

$$g_m=0\,;\ f_0=M_1\,,\ f_{m+1}=0\ \text{for } m\geqslant 0 \text{ at } \eta=1$$
$$g_m=0\,;\ f_0=-M_2\,;\ f_{m+1}=0\ \text{for } m\geqslant 0 \text{ at } \eta=K$$
$$M_1=\frac{V_1}{a\Omega}\,;\qquad M_2=\frac{V_2}{a\Omega}.$$

Solving for constants, we have

(30)
$$A_0=B_0=A_1=B_1=A_2=B_2=0\qquad \bar{A}_0=M_1\,;\qquad \bar{B}_0=\frac{1}{\log K}(M_2-M_1)$$

(31)
$$\bar{A}_1=-\frac{i\omega_2}{4}(\bar{A}_0-\bar{B}_0)+\bar{B}_0(\alpha+\beta),\qquad \bar{B}_1=\frac{1}{\log K}\Big[-\frac{\omega_2 iK^2}{4}\{\bar{A}_0+\bar{B}_0(\log K-1)\}$$
$$-\frac{\bar{B}_0}{2}(\log K)^2-\frac{\bar{B}_0(\alpha+\beta)}{K^2}-\bar{A}_1\Big]$$
$$\bar{A}_2=-f_2(1,\ \omega_2),\qquad \bar{B}_2=-\frac{1}{\log K}\{\bar{A}_2+f_2(K,\omega_2)\}.$$

Case 6. In the last case, let the flow of the fluid be given as

$$v=0\,;\qquad w=V_1\qquad\text{at } r=a,$$
$$v=0\,;\ w=V_2\cos k_2nt\ \text{ at } r=b.$$

In view of (3), (6) and (9) the boundary conditions will become

(32)
$$g_m=0\,;\quad f_0\big|_{\omega_2=0}=M_1\,;\ f_{m+1}\big|_{\omega_2=0}=0\ \text{ at } \eta=1$$
$$g_m=0\,;\quad f_0=M_2\,;\qquad f_{m+1}=0\qquad\text{at } \eta=K\qquad\text{for } m\geqslant 0.$$

Solving for constants, we get

(33)
$$A_0=B_0=A_1=B_1=A_2=B_2=0$$
$$\bar{A}_0=M_1\,;\quad \bar{B}_0=-\frac{1}{\log K}(M_1+M_2)$$

(34)
$$\bar{A}_1=\bar{B}_0(\alpha+\beta)$$
$$\bar{B}_1=\frac{1}{\log K}\Big[-\bar{B}_0(\alpha+\beta)-\omega_2 i\Big\{\Big(\frac{\bar{A}_0K^2}{4}+$$
$$+\frac{\bar{B}_0K^2}{4}(\log K-1)\Big)\Big\}-\frac{\bar{B}_0}{2}(\log K)^2\Big]$$
$$\bar{A}_2=-f_2(1,\ 0)$$
$$\bar{B}_2=-\frac{1}{\log K}\{\bar{A}_2+f_2(K,\omega_2)\}.$$

In the same way many other similar problems can be discussed. We should again point out that equations (8 a, b) are linear and therefore if we have to solve a problem whose boundary conditions are the linear combination of the problems we discussed above, the solution of that problem can be obtained by taking the corresponding linear combination of their solutions. For example the solution of the following problem

(35)
$$v=r\Omega_1\cos k_1nt+r\Omega_1\,;\qquad w=0\qquad\text{at}\quad r=a$$
$$v=r\Omega_2\qquad\qquad ;\qquad w=V_0\cos k_2nt\quad\text{at}\quad r=b$$

is the sum of the solutions of cases 1 and 4.

Acknowledgement: This work was partly completed while the author was at Lucknow University, Lucknow, U. P. INDIA and the author is very grateful to Prof. Ram Ballabh, the chairman of the department of Mathematics and Astronomy, for his advice in the preparation of this note.

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Received 1.IV.1974

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CURGEREA NESTAȚIONARĂ A UNUI FLUID DE ORDINUL AL DOILEA ÎNTRE DOI CILINDRI COAXIALI POROȘI

Rezumat

Se studiază curgerea elicoidală nestaționară a unui fluid de ordinul al doilea între doi cilindri coaxiali poroși. Presupunind că numărul lui Reynold, *R*, este mic, se determină viteza în forma unei serii după puterile lui *R*. Citeva cazuri speciale sint considerate.

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXI, s. I a, 1975

ON BENDING OF MICROPOLAR PLATES

BY

AUREL IDELOVICI

1. Statement of the problem. Consider a thin plate of constant thickness $2h$ with its middle surface in the x_1Ox_2 — plane. Let Ω be the region occupied by the plate, Σ — its boundary surface, S — the edge surface, C — its boundary curve given by $x_\alpha = x^*_\alpha(s)$ ($\alpha = 1, 2$), where s is arc length. The plate is assumed to consist of isotropic, homogeneous elastic material.

In what follows, we use the notation of [2], [6]. Greek indices take the values 1, 2, and Latin indices take the values 1, 2, 3. Subscripts following commas denote partial differentiation.

The basic equations of linear micropolar elasticity with independent local rotations, in the absence of body forces and couples, are:

— the equilibrium equations

$$\sigma_{ji,j} = 0, \quad \mu_{ji,j} + \varepsilon_{ikl}\sigma_{kl} = 0; \tag{1}$$

— the kinematical relations

$$\beta_{ij} = U_{j,i} - \varepsilon_{ijk}\Phi_k, \quad \varphi_{ij} = \Phi_{j,i}; \tag{2}$$

— the constitutive equations

$$\sigma_{ij} = \lambda\beta_{hh}\delta_{ij} + 2\mu\beta_{(ij)} + 2\alpha\beta_{[ij]}, \qquad \mu_{ij} = \beta\varphi_{hh}\delta_{ij} + 2\gamma\varphi_{(ij)} + 2\varepsilon\varphi_{[ij]}, \tag{3}$$

where σ_{ij} are stresses, μ_{ij} — couple stresses, β_{ij} and φ_{ij} — strain tensors, U_i — displacements Φ_i — microrotations, $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$ — constants, ε_{ikl} — the permutation symbol, δ_{ij} — Kronecker's delta; the symbols ( ) and [ ] denote the symmetric, respectively the skew-symmetric part of a tensor.

The boundary conditions are

$$\sigma_{ji}n_j = \bar{P}_i, \quad \mu_{ji}n_j = \bar{Q}_i, \tag{4}$$

where $\bar{P}_i$, $\bar{Q}_i$ are exterior loads (surface tractions and couples, respectively), and n_i are the components of the outward unit normal to Σ.

In what follows we consider only the bending of the plate, assuming that U_3, Φ_α, $\sigma_{\alpha\beta}$, $\sigma_{3\alpha}$, μ_{33} are even functions of x_3, and U_α, Φ_3, $\sigma_{\alpha\beta}$, $\mu_{\alpha\beta}$, $\mu_{3\alpha}$, are odd functions of x_3. Accordingly we impose the following boundary conditions:

$$\sigma_{3\alpha} = 0, \quad \sigma_{33} = \mp 0\sigma_0\frac{p}{2}, \quad \mu_{3\alpha} = \mp 0\mu_0\frac{q_\alpha}{2}, \quad \mu_{33} = 0 \quad \text{— on faces } (x_3 = \pm h). \tag{5}$$

$$\begin{aligned} P_\alpha &\equiv \sigma_{\lambda\alpha}n_\lambda = 0\sigma_0\bar{p}_\alpha, \quad P_3 \equiv \sigma_{\lambda 3}n_\lambda = \sigma_0\bar{p}_3 \\ Q_\alpha &\equiv \mu_{\lambda\alpha}n_\lambda = \mu_0\bar{q}_\alpha, \quad Q_3 \equiv \mu_{\lambda 3}n_\lambda = 0\mu_0\bar{q}_3 \end{aligned} \quad \text{— on } S\ (x_\alpha = x^*_\alpha(s), |x_3| \leqslant h), \tag{6}$$