$$\iiint_{\Omega}\left\{\left[\beta_{ij}+\frac{\nu}{E}\sigma_{hh}\delta_{ij}-\frac{1+\nu}{E}\sigma_{(ij)}-\frac{1}{2\alpha}\sigma_{[ij]}\right]\delta\sigma_{ij}+\left[\frac{\nu^*}{E^*}\mu_{hh}\sigma_{ij}-\frac{1+\nu^*}{E^*}\mu_{(ij)}-\right.\right.$$

$$\left.\frac{1}{2\varepsilon}\mu_{[ij]}\right]\delta\mu_{ij}-\sigma_{ij,i}\,\delta U_j-(\mu_{ij,i}+\varepsilon_{ijk}\sigma_{ik})\delta\Phi_j\Big\}\,dx_1\,dx_2\,dx_3+\iint_{S}(\sigma_{\lambda j}n_\lambda\,\delta U_j+ \tag{17}$$

$$+\mu_{\lambda j}n_\lambda\delta\Phi_j)dx_3\,ds+2\iint_{D}(\sigma_{3j}\delta U_j+\mu_{3j}\delta\Phi_j)_{x_3=h}dx_1\,dx_2-\iint_{S}(\theta\sigma_0\overline{p}_\alpha\delta U_\alpha+\sigma_0\overline{p}_3\delta U_3+$$

$$\mu_0 q_\alpha\delta\Phi_\alpha+\theta\mu_0\overline{q}_3\delta\Phi_3)dx_3ds+2\iint_{D}\left(\theta\sigma_0\frac{p}{2}\,\delta U_3+\theta\mu_0\,\frac{q_\lambda}{2}\,\delta\Phi_\lambda\right)_{x_3=h}dx_1\,dx_2=0.$$

Taking into account (7) to (11), and using (12), the equation (17) reduces to

$$\iint_{S^*}\left\{\theta^2(P^*_\lambda-\overline{p}^*_\lambda)\delta u_\lambda+(P^*_3-\overline{p}^*_3)\delta u_3+\frac{E}{E}[(Q^*_\lambda-\overline{q}^*_\lambda)\,\delta\varphi_\lambda+\theta^2(Q^*_3-\overline{q}^*_3)\,\delta\varphi_3]\right\}d\zeta\,d\sigma=0. \tag{18}$$

Introduction of the expansions $P^*_\lambda=P^{*(0)}_\lambda+\theta^2P^{*(2)}_\lambda+\ldots$, etc., into (18) yields

$$\iint_{S^*}\left[(P^{*(0)}_3-\overline{p}^{*(0)}_3)\delta u_3+\frac{E}{E}(Q^{*(0)}_\lambda-\overline{q}^*_\lambda)\,\delta\varphi_\lambda\right]d\zeta\,d\sigma=0, \text{ etc.} \tag{19}$$

If we take in (19) $\delta u_3=\delta u^{(0)}_3$, $\delta\varphi_\alpha=\delta\varphi^{(0)}_\alpha$, where $u^{(0)}_3$, $\varphi^{(0)}_\alpha$ are given by (14), we obtain for the first approximation

$$\int_{-1}^{1}P^{*(0)}_3d\zeta=\int_{-1}^{1}\overline{p}^*_3\,d\zeta,\quad\int_{-1}^{1}Q^{*(0)}_\alpha d\zeta=\int_{-1}^{1}\overline{q}^*_\alpha\,d\zeta. \tag{20}$$

or

$$s^{(0)}_{\lambda 3}n_\lambda=\frac{1}{2}\int_{-1}^{1}\overline{p}^*_3d\zeta,\quad m^{(0)}_{\lambda\alpha}n_\lambda=\frac{1}{2}\int_{-1}^{1}\overline{q}^*_\alpha\,d\zeta,\quad(\xi_\alpha=\xi^*_\alpha(\sigma),\ |\zeta|\leq 1). \tag{21}$$

The boundary value problem (15), (21) is similar to that of [3], (for which an existence and uniquenes theorem has been proved).

5. Validity of iterative procedure. Assuming ν and ν^* of order unity, the conditions for the validity of the proposed iterative procedure are

$$(\beta,\ \gamma,\ \varepsilon)=O(L^2(\lambda,\ \mu,\ \alpha)). \tag{22}$$

REFERENCES

1. Green A. E. and Naghdy P. M. — *Micropolar and Director Theories of Plates*. Quart. J. Mech. Appl. Math., 20 (1967), p. 183—199.
2. Idelovici A. — *On a Micropolar Theory for Bending of Plates*. An. şt. Univ. Iaşi, 19 (1973), p. 191—207.
3. Ieşan D. — *Torsion of Micropolar Elastic Beams*. Int. J. Eng. Sci., 9 (1971), p. 1047—1060.
4. Reissner E. — *On a Variational Theorem in Elasticity*, J. Math. Phys., 29 (1950), p. 90—95.
5. Reissner E. — *On the Derivation of Boundary Conditions for Plate Theory*, Proc. Roy. Soc., A 276 (1963), p. 178—186.
6. Şandru N. — *On some Problems of the Linear Theory of the Asymmetric Elasticity*, Int. J. Eng. Sci., 4 (1966), p. 81—94.

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GENERALIZED LINEAR AUTOMATA

BY

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1. In this paper we have studied a notion of generalized linear automaton (G. L. A.), whose particular case is the notion of linear automaton. We have investigated the properties of the G. L. A. s regarding the equivalence of automata, minimal automata, a. s. o. Then we have obtained an application of the G. L. A. s to the generation of the numerical pseudo-random sequences.

1.1. Definition. *A G. L. A. is a 5-uple (I, S, O, f, g), where :*

a) I, S, O are nonempty sets called respectively the set of inputs, the set of states and the set of outputs, $f: S\times I\to S$ and $g: S\times I\to S$ are the next state function, and respectively the output fonction ;

b) S and O are organized as semigroups with the internal operation denoted by $\divideontimes$ and the unit elements O_s, respectively O_0 ;

c) there exists the semigroup homomorphisms $A: S\to S$, $C: S\to O$ and the mappings $B: I\to S$, $D: I\to O$ so that $f(s, i)=As\divideontimes Bi$ and $g(s, i)=Cs\divideontimes Di$, for every $(s, i)\in S\times I$;

d) there exists the element $O_I\in I$, so that $BO_I=O_s$ and $DO_I=O_0$. When $|I|=1$, the G.L.A. is called autonomous.

1.2. If I, S, O are linear spaces and A, B, C, D are linear mappings, we obtain the linear automata ([2], p. 26).

1.3. The mappings f and g can be extended to $S\times I^*$ by taking

$$f(s,\ px)=f(f(s,\ p),\ x),$$

$$g(s,\ px)=g(f(s,\ p),\ x),$$

where $s\in S$, $x\in I$ and $p\in I^*$.

We get by induction the formulae :

$$f(s,\ i_1\ldots i_n)=A^n s\divideontimes\sum_{j=1}^{n}A^{n-j}\,Bi_j\ \text{(the general next state equation)},$$

$$g(s,\ i_1\ldots i_n)=CA^{n-1}s\divideontimes\sum_{j=1}^{n-1}CA^{n-j-1}\,Bi_j\divideontimes Di_n\ \text{(the general output equation)},$$

where $i_j\in I$ for $j=1, 2, \ldots, n$ and $\sum$ is referring to the operation $\divideontimes$.

1.4. Since the mappings A, B, C and D determine the next state and output functions, we shall also denote a G. L. A. by $\mathfrak{a}=(I, S, O\,;\ A, B, C, D)$.

1.5. Definition. *Let $\mathfrak{a}=(I, S, O\,;\ A, B, C, D)$ and $\mathfrak{a}'=(I', S', O'\,;\ A', B', C', D')$; a G.L.A. epimorphism is a triple $h=(\alpha, \beta, \gamma)$ of surjections $\alpha: I\to I'$, $\beta: S\to S'$, $\gamma: O\to O'$ so that*

$$\beta(As\divideontimes Bi)=A'\beta(s)\divideontimes B'\alpha(i),\qquad\gamma(Cs\divideontimes Di)=C'\beta(s)\divideontimes D'\alpha(i),\qquad\forall(s,\ i)\in S\times I.$$

If α, β and γ are bijections, then h will be called G. L. A. isomorphism.

2. A connection between states which give the same answer when the same sequence is introduced in the automaton is given by :

2.1. Definition. *Let* $(I_j, S_j, O_j; A_j, B_j, C_j, D_j)$ *with* $j=1, 2$, *be two G.L.A.s. The states* $s_1 \in S_1$ *and* $s_2 \in S_2$ *are called equivalent (and denoted* $s_1 \sim s_2$*) if*

$$g_1(s_1, p) = g_2(s_2, p) \quad \text{for every} \quad p \in I^*.$$

Two G.L.A.s a_1 *and* a_2 *are equivalent if for every state* s_1 *of* a_1 *there exists at least one state* s_2 *of* a_2 *such that* $s_1 \sim s_2$, *and for every state* s_2' *of* a_2 *there exists at least one state* s_1' *of* a_1 *so that* $s_2' \sim s_1'$.

2.2. Theorem. *Given the G.L.A.* $a=(I, S, O; A, B, C, D)$, *a necessary and sufficient condition that* $s_1 \sim s_2$, *with* $s_1, s_2 \in S$, *is:*

$$CA^{n-1}s_1 = CA^{n-1}s_2 \quad \text{for every} \quad n \in N.$$

The proof is trivial by using 1.4 and 2.1.

2.3. The property 2.2 is generally undecidable for G.L.A.s; for the special case of linear automata this becomes a decidable one ([2], p. 34).

In the sequel we shall study a class of G.L.A.s for which the property 2.2 is decidable

2.4. Definition. *A G.L.A.* $a=(I, S, O; A, B, C, D)$ *has the power* t, *if* t *is the smallest natural number* h *so that for the semigroup homomorphism* A *we have:*

$$A^k \in \{a_0 I_s + a_1 A^1 + a_2 A^2 + \dots + a_{h-1} A^{h-1} \mid a_i \in \{0,1\}, i=\overline{0, h-1}\},$$

for every $k \in N$ *(if such* h *exist).*

The set

$$\{a_0 1_S + \dots + a_{t-1} A^{t-1} \mid a_i \in \{0, 1\}, i=\overline{0,t-1}\}$$

can be organized as a semigroup with unit element by means of the operation of S, taking $(A'+A'')s = A's \divideontimes A''s$.

Here $OA = O_S$ (the null mapings on S) and $1A = A$.

2.5. Theorem. *If a G.L.A* $a=(I, S, O; A, B, C, D)$ *has the finite power* t, *then a necessary and sufficient condition that* $s_1 \sim s_2$ *is* $CA^{h-1}s_1 = CA^{h-1}s_2$ *for every* $h \leq t$, *where,* $s_1, s_2 \in S$.

2.6. The following example shows that there exists G.L.A.s which have not a finite power. Let $a=(I, S, O; A, B, C, D)$ with $I=N$, $S=(N^*, +)$, $O=(N^*, +)$, $A: S \to S$ by $As = 2s$, $C: S \to O$ by $Cs=s$, and B and D the null mappings. This G.L.A. has not a finite power because $\forall k \in N$, 2^k can not be expressed as a sum of $2^0, 2^1, 2^2, \dots, 2^{k-1}$.

2.7. The notion from 2.5 can be introduced analogously for linear automata by: $A^k \in [1_s, A, A^2, \dots, A^{t-1}]$ (i. e. $\forall k \in N$, A^k belongs to the linear space generated by $\{1_s, A, A^2, \dots, A^{t-1}\}$).

According to Caley-Hamilton's theorem the power defined above cannot exceed the dimension of the space S.

2.8. Theorem. *Let* h *be a G.L.A. epimorphism* $h: a_1 \to a_2$ *given by* $h=(1_I, \beta, 1_0)$ *with* $a_1=(I_1, S_1, O_1, f_1, g_1)$ *and* $a_2=(I_2, S_2, O_2, f_2, g_2)$; *then if* $s_1 \in S_1$ *we have* $s_1 \sim \beta(s_1)$ *and* a_1, a_2 *are equivalent.*

3. In what follows we shall study a class of G.L.A.s for which the states giving the same answers, if any input sequence is introduced, are equal.

3.1. Definition. *A G.L.A. is called minimal if* $g(s_1, p) = g(s_2, p)$ *for every* $p \in I^*$ *implies* $s_1 = s_2$.

3.2. Theorem. *A G.L.A. is minimal if and only if the relation introduced in 2.1 (which when* $a_1 = a_2$ *is an equivalence on the set* $S_1 = S_2$*) is the unit relation (i. e. every class contains only one element).*

3.3. Theorem. *Given the G.L.A.s* $a=(I, S, O, f, g)$ *and* $a_1=(I_1, S_1, O_1, f_1, g_1)$, *if* a *is minimal and* $h: a \to a_1$ *is a G.L.A. epimorphism given by* $h=(1_I, \beta, 1_0)$ *then* h *is a G.L.A. isomorphism and* a_1 *is also minimal.*

3.4. Theorem. *If* a_1 *and* a_2 *are equivalent and minimal G.L.As. then they are isomorphic.*

Proof. We define $\beta: S_1 \to S_2$ by $\beta(s_1) = s_2$, where s_2 is one of the states of a_2 for which $s_1 \sim s_2$.

By using the minimality of a_1 and a_2 one can prove that this mapping is well defined, bijective, and finally, h is indeed a G.L.A. isomorphism (one uses also 3.2).

4. We intend to attach to every G.L.A. another G.L.A., with the same „external behaviour", (i. e. which gives the same answer when the same input sequence is introduced) and, in addition, „simpler" than the first one.

4.1 Definition. *A minimal G.L.A.* a', *equivalent to a G.L.A.* a, *is called a minimal form of* a.

4.2. Theorem. *For every G.L.A.* $a=(I, S, O; A, B, C, D)$ *there exists a minimal G.L.A.* $a'=(I, S', O; A', B', C', D')$ *and G.L.A. epimorphism* $h: a \to a'$ *given by* $h=(1, \beta_I, 1_0)$, *such that* $\beta: S \to S'$ *is a monoid epimorphism.*

Proof. We shall consider the quotient set $S/\sim$, where „$\sim$" is the equivalence considered at 2.1. The G.L.A. $a'=(I, S/\sim, O; A', B', C', D')$, where

$A': S/\sim \to S/\sim$	by $A'([s])=[As]$,
$B': I \to S/\sim$	by $B'(i)=[Bi]$,
$C': S/\sim \to O$	by $C'([s])=Cs$,
$D': I \to O$	by $D'i=Di$

will be the minimal automaton.

By using 2.2 it follows that the operators A', B', C', D' are well defined. Then, we shall organize $S/\sim$ as a monoid using the operation $[s_1] \divideontimes [s_2] = [s_1 \divideontimes s_2]$, and one can prove that A and C are homomorphisms. The condition d) from 1.1 is fulfiled for a' as a consequence of the corresponding property of a. Therefore a' is a G. L.A. Taking into account that $(S/\sim)/\sim = S/\sim$ and the result 3.2, it follows that a' is minimal. The epimorphism h will be constructed by means of the mapping $\beta: S \to S/\sim$ given by $\beta(s)=[s]$ which also is a monoid epimorphism.

4.3. Theorem. *Every G.L.A. admits a minimal form.*

The proof is immediate, by using 4.2 and 2.8.

4.4. We note that 4.2 points the existence of minimal G.L.A.s and from 4.3 it follows the existence of a minimal form for every G.L.A.

4.5. Theorem. *A minimal form of a G.L.A. is unique up to a G.L.A. isomorphism. The sets of states of the minimal forms are isomorphic as semigroups.*

Proof. If a' and a'' are two minimal forms for the G.L.A. a and $h_1: a \to a'$ and $h_2: a \to a''$ are the epimorphisms from 4.2, given by $h_1=(1_I, \beta_1, 1_0)$, respectively $h_2=(1_I, \beta_2, 1_0)$, then $h: a' \to a''$ with $h=(1_I, \beta, 1_0)$, where $\beta: S' \to S''$ is given by $\beta\beta_1 = \beta_2$, is the required isomorphism.

5. The simulation techniques in computers use numerical pseudorandom sequences. Such a sequence is given by the linear congruential method:

$$x_{n+1} \equiv ax_n + b \pmod{m} \text{ with } a, b, x_0, m \text{ from } N \text{ and } x_0 < m.$$

For $b=0$ we obtain the multiplying linear congruential method.

5.1. Next we shall construct a G.L.A. which produces the same numerical pseudorandom sequences as linear congruential method determined by the numbers a, b, x_0 and m.

Denote by „$+_m$" the operation $a +_m b \equiv a+b \pmod{m}$ which determines on M a monoid structure and let be $a=(I, S, O; A, B, C, D)$ where $I=N^*$, $S=(M, +_m)$, $O=(M, +_m)$ $M=\{0, 1, 2, \dots, m-1\}$

$$A: S \to S \text{ by } As = as = s +_m s +_m \dots +_m s$$

$$B: I \to S \text{ by } Bi = \begin{cases} 0 & \text{for } i=0 \\ b & \text{for an element } x \\ \text{anyhow} & \text{for the remainder.} \end{cases}$$

$C = 1_M$ and D the zero mapping.

One can easily see that this is a G.L.A We shall impose the supplementary condition that the initial state of automaton, i. e. the state in which it starts to work, to be x_0.

5.2. The automaton works in the following way: by introducing at the input wire the symbol x, the automaton being in the state x_k, gives at the output wire the symbol $Cx_k +_m Dx$, hence x_k, and turns into the state $Ax_k +_m Bx$, hence $ax_k +_m b$.

5.3. If we wish to obtain a multiplying linear congruential method determined by x_0, a and m, we shall use the same construction as in 5.1, by taking as B the zero mapping

5.4. One can see that the G.L.A. constructed in 5.1 is minimal. Concerning the G.L.A from 5.3 one can prove, by using 2.8 and 3.4, that given the multiplying linear congruential method, the autonomous G.L.A. (B and D being zero mappings) which one can attach to it by 5.3 is unique up to an isomorphism.

5.5. Taking into account the results of the theory of linear congruential sequences, one can construct G.L.A.s which supply numerical pseudo-random sequences of maximum period.

REFERENCES

1. Creangă, I., Reischer, C., Simovici, D. — *Teoria automatelor*, Ed. Junimea Iaşi, 1973.
2. Gill, A. — *Linear Sequential Circuits*. McGraw Hill Book Co., New York, 1967.
3. Naylor, T. H., Balintfy, J. L., Burdick, D. S., Chu, K. — *Computer Simulation Techniques*. John Wiley and Sons, New York, London, Sidney, 1966.

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AUTOMATE LINIARE GENERALIZATE

Rezumat

În această Notă se studiază noţiunea de automat liniar generalizat (A.L.G.) care conţine drept caz particular noţiunea de automat liniar. Sint cercetate proprietăţile A.L.G.-urilor legate de echivalenţă, automate minimale, etc.

Se găseşte apoi o aplicaţie a A.L.G.-urilor la generarea şirurilor de numere pseudo-aleatoare.

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXI, S. I a, 1975

RECENZII

HELMUT HASSE: *Mathematische Abhandlungen. Herausgegeben von Heinrich Wolfgang Leopoldt und Peter Roquette*. Band 1, 2, 3. Walter de Gruyter, Berlin, New York, 1975. IX + 535 S; DM 471.

Culegerea de faţă a fost alcătuită cu scopul de a uşura accesul cititorilor interesaţi la opera lui Helmut Hasse, operă care a influenţat profund dezvoltarea matematicii în secolul nostru, în special a teoriei numerelor. Au fost incluse numai lucrările apărute pînă în 1961; acestea sînt grupate pe capitole, corespunzînd domeniilor matematice cărora le aparţin, după cum urmează. Volumul I: I. Forme pătratice; II. Resturi de norme şi corpuri locale de clase; III. Legi de reciprocitate; IV. Teoria corpurilor de clase; V. Algebre. Volumul II: VI. Multiplicare complexă; VII. Corpuri de funcţii de congruenţe; VIII. Corpuri de funcţii aritmetice. Volumul III: IX. Sume Gauss; X. Problema de scufundare; XI. Calculul numărului de clase şi al unităţilor; XII. Generalităţi şi bazele teoriei numerelor.

Jack Weinstein

CHARLES P. ENZ and JAGDISH MEHRA (edited by): *Physical reality and mathematical description*. Dordrecht, D. Reidel Publ. Comp. 1974. XXIV, 552 p. $, 39,50.

Cartea este un omagiu adus — la cea de a 60-a aniversare a naşterii — fizicianului Maria Jauch care prin opera sa ştiinţifică a obţinut un nume de onoare în comunitatea fizicienilor teoreticieni, prin opera sa didactică şi numărul mare de colaboratori distinşi şi-a pus amprenta pe fizica matematică modernă. Un număr de colaboratori, prieteni şi admiratori ai prof. Jauch au contribuit la această colecţie şi lucrările lor reflectă interesele largi ale lui Jauch în vastul domeniu al fizicii teoretice. După o introducere care cuprinde date asupra vieţii şi operei prof. Jauch cartea conţine contribuţii originale asupra unui larg cerc de probleme ale fizicii matematice moderne, fiind împărţită în patru părţi: I. Artă, istorie şi filosofie; II. Fizică matematică; III. Teoria dispersiei, teoria cîmpului; IV. Teoria cuantică, mecanica statistică. În timp ce accentul principal în aceste lucrări se pune pe bazele matematice ale fizicii moderne este scoasă în evidenţă şi îmbinarea matematicii cu probleme de importanţă fizică, filosofică şi istorică.

Gh. Procopiuc

HÜTTE: *Taschenbücher der Technik. Mathematik*. 2 Auflage von I. Szabo, K. Wellnitz, W. Zander, Springer Verlag, Berlin, Heidelberg, New York, 1974, 373 S, 153 Abb., DM 48.

Cartea se adresează inginerilor şi studenţilor de la facultăţile tehnice, punîndu-le la dispoziţie mijloacele matematice, clasice şi moderne, necesare studierii diferitelor discipline tehnice. Materialul prezentat în carte este împărţit în 13 capitole. În raport cu prima ediţie aceasta este îmbunătăţită şi completată conţinînd în plus, printre altele, o tratare amănunţită a funcţiei lui Dirac, ecuaţiile integrale ale polinoamelor Cebîşev, Hermite şi Laguerre, metodele de integrare Cebîşev şi Romberg şi un capitol complet nou, dedicat calculului pe maşinile de calcul. De asemenea se remarcă o tratare mai aprofundată a calculului operaţional, a transformatei Laplace, a ecuaţiilor integrale, a calculului cu diferenţe finite şi a calculului probabilităţilor şi statisticii.

N. Zitron

J. HAINZL: *Mathematik für Naturwissenschaftler*. Stuttgart, B. G. Teubner, 1971, 311 p.

Cartea conţine prelegerile ţinute de autor la Universitatea din Freiburg şi este destinată, în principal, biologilor, chimiştilor şi mineralogilor. În cele şase capitole (clase de numere şi funcţii, calcul diferenţial şi integral, funcţii elementare şi serii Fourier, geometrie analitică şi algebră liniară, funcţii de mai multe variabile, ecuaţii diferenţiale) se parcurge un material vast, într-o manieră accesibilă. Matematica nu apare ca un scop în sine, ci ca un instrument pentru însuşirea disciplinelor de profil. Demonstraţiile se fac, uneori, pe cazuri particulare, alteori, sînt omise; totdeauna autorul are însă grija de a face plauzibil rezultatul prin exemple judicios alese. Fiecare capitol se încheie cu un număr de exerciţii, multe avînd indicaţii privind rezolvarea. Cartea se încheie cu o bibliografie şi un indice de noţiuni.

D. Rimer

RICHARD SWINBURNE (editor): *The Justification of Induction*. Oxford University Press. 1974. 179 p.

Este o colecţie de articole privitoare la inducţie, unele mai vechi — ca cel al lui Bertrand Russel, datînd din 1912 — celelalte, apărute după 1919. Autorii, Paul Edwards, Wesley C. Salmon, Stephen F. Barker, Henry E. Kyburg Jr., J. O. Urmson, John W. Lenz, R. B. Braithwaite, Max Black. Peter Achinstein, Keith Campbell, Nicolas Maxwell, profesori la diferite universităţi americane, sînt cunoscuţi şi din alte lucrări în acest domeniu. Din problematica vastă şi complexă abordată, menţionăm, între altele, evidenţa inductivă, validitatea inducţiei, justificarea inducţiei din diferite puncte de vedere. Cartea conţine şi o bibliografie precum şi un indice de noţiuni şi nume.

D. Rimer