$\in N$, $\overline{m}\neq 0$, then $\overline{\alpha}_1\, \overline{m}_1 = \ldots = \overline{\alpha}_k\, \overline{m}_k$. Suppose $\overline{m}\notin M$, then at least one of the components, e.g. $\overline{m}_1\notin M_1$. Then there exists an element $\lambda_1\in R$, $\lambda_1\neq 0$, such that $0\neq\lambda_1\overline{m}_1 = m_1'\in M_1$. Then $\lambda_1\overline{m} = (\lambda_1\overline{m}_1, \ldots, \lambda_1\overline{m}_k)$ satisfies

$$\overline{\alpha}_1(\lambda_1\, \overline{m}_1) = \ldots = \overline{\alpha}_k(\lambda_1\, \overline{m}_k).$$

or

$$\alpha_1(m_1') = \overline{\alpha}_2(\lambda_1\, \overline{m}_2) = \ldots = \overline{\alpha}_k(\lambda_1\, \overline{m}_k).$$

If $\lambda_1\overline{m}_2\notin M_2$, then we continue by multiplying with a suitable $\lambda_2\in R$, $\lambda_2\neq 0$, such that

$$0\neq\lambda_2\,\lambda_1\, \overline{m}_2 = m_2'\in M_2,$$

hence

$$\lambda_2\,\lambda_1\overline{m} = (\lambda_2\, m_1',\ m_2',\ \lambda_2\,\lambda_1\overline{m}_3, \ldots, \lambda_2\lambda_1\overline{m}_k).$$

Continuing this process we obtain, after a finite number of steps, an element $m\in M$,

$$m = \lambda_k\ldots\lambda_1\overline{m} = (m_1, m_2, \ldots, m_k)\neq 0,$$

such that

$$\alpha_1\, m_1 = \alpha_2\, m_2 = \ldots = \alpha_k\, m_k.$$

That proves, that

$$N = \hat{M}_1 \underset{\sim}{\times} \ldots \underset{\sim}{\times} \hat{M}_k(\overline{\alpha}_1, \ldots, \alpha_k;\ F)$$

is an injective essential extension of M, therefore N is an injective hull of M.

Theorem 5.6. *If M is an indecomposable injective R-module, then any representation of M as a subdirect product of the finitely many R-modules $M_1, \ldots, M_n$, $M = M_1 \underset{\sim}{\times} \ldots \times M_n$, is trivial.*

Proof. Suppose there existed a non-trivial representation $M = M_1 \underset{\sim}{\times} \ldots \underset{\sim}{\times} M_n$ of M, then M contained a set of non-zero submodules $N_i, \ldots, N_n$ with $\bigcap_i N_i = 0$. We may suppose that the system $\{N_i\}$ is irredundant (no N_i contains the intersection of the remaining N_j). Denoting by $N_1' = \bigcap_{i\neq 1} N_i$, we have $N_1\neq 0$, $N_1'\neq 0$ and $N_1\cap N_1' = 0$. Then $M\cong M/N_1 \underset{\sim}{\times} M/N_1' = M_1 \underset{\sim}{\times} M_1'$, and the kernels N_1, N_1' are non-zero. Using the result of thm. 5.1. we find $M\cong \widehat{M\cap M_1}\oplus\widehat{M\cap M_1'}\oplus K$, contradicting the hypothesis that M is indecomposable. Therefore a non-trivial representation of M as a subdirect product of finitely many modules cannot exist.

Remark. If M is indecomposable and injective there may be a representation of M as a non-trivial subdirect product, but not with a finite number of components. Take e. g. the injective Z-module Q and define for the prime number p: $Q_p = \{m/p^n \mid m, n\in Z\}$, then $Q = \bigoplus_p Q_p$.

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NOTE DESPRE PRODUSE SUBDIRECTE DE MODULE

Rezumat

În lucrare autorul stabilește unele proprietăți ale R-modulelor subdirect ireductibile și dă o reprezentare iredundantă a unui modul M ca produs subdirect de module subdirect ireductibile, dacă M este noetherian. În § 3 se dă o condiție necesară și suficientă pentru ca un produs subdirect să fie un produs subdirect special. Paragraful 4 este consacrat unei aplicații a produselor subdirecte speciale iar paragraful 5 studiului relației dintre produsul subdirect și injectivitate.

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REMARKS ON TWO GEOMETRIC CONJECTURES OF L. FEJES TÓTH

BY

I. J. SCHOENBERG

Recently Professor L. F e j e s T ó t h discussed in a Seminar on Intuitive Geometry at the University of Wisconsin-Madison the following two conjectures. I quote in full his note [1]:

„Starting with an arbitrary convex polygon p_0, we successively construct the polygons $p_1, p_2, \ldots$ so that the vertices of p_n are the sidemidpoints of p_{n-1}. Again, starting with an arbitrary convex polygon P_0, we successively construct the polygons $P_1, P_2, \ldots$ in such a way that P_n arises by drawing a line through each vertex of P_{n-1} parallel to the line joining the two adjacent vertices of P_{n-1}. Some years ago I made two conjectures [2] which may be considered as dual counterparts of each other.

Conjecture 1. *There are affine images of $p_1, p_2, \ldots$ which converge to a regular polygon.*

Conjecture 2. *There are affine images of $P_1, P_2, \ldots$ which converge to a regular polygon.*

In my paper [2], I proved the correctness of Conjecture 1 for pentagons and hexagons. This paper started some further papers (see [1] for nine further references) giving, among others, various complete proofs for the correctness of Conjecture 1. On the other hand, the question whether Conjecture 2 is correct or not is not solved as yet".

The main results of the present note are as follows.

1. Twenty four years ago the author showed in [3] how the finite Fourier Series can solve a number of (mostly extremum) problems of elementary geometry, in particular that the polygons p_n, of Conjecture 1, shrink to a point. In section 1 it is shown how this approach yields in a few lines the results of H. R e i c h a r d t and G. L ü k ö (see [1] for references) concerning Conjecture 1.

2. It is shown in section 2 by a counter-example that Conjecture 2 is false for hexagons. The counter-example generalizes to polygons of an *even* number k of vertices (third paragraph of section 4). However, *the question whether Conjecture 2 is correct or not remains undecided if the number k of vertices is odd* ($k = 5, 7, 9, \ldots$). Computations done by my colleague C. de B o o r ($n = 1, 2, \ldots, 20$) show beyond a shadow of a doubt that Conjecture 2 is correct for pentagons having an axis of symmetry.

3. Starting from a convex polygon $P_0 = A_1 A_2 \ldots A_k$ we obtain a new convex polygon $\overline{P}_1$ by removing from P_0 all k corner triangles $A_{i-1} A_i A_{i+1}$ ($A_1 = A_{k+1}$, $A_0 = A_k$). Iterating this „peeling" process we obtain the polygons $\overline{P}_2, \overline{P}_3, \ldots$. I conjecture that the intersection

$$\bigcap_{n=1}^{\infty} \overline{P}_n$$

reduces to a point. Computer evidence, again obtained by d e B o o r, strongly supports this conjecture for pentagons having an axis of symmetry. I am much indebted to C. de B o o r for his help.

1. **On Conjecture 1.** The sequence of polygons (p_n) of Conjecture 1 was the subject of Problem No. 3547, American Mathematical Monthly, Vol. 39 (1932), p. 239, proposed by Martin R o s e n m a n. The objective was to show that the polygon p_n shrinks to

the centroid of the vertices of p_0 as $n \to \infty$. For a discussion of this problem, and others, by means of the finite Fourier Series (f. F. S.) see [3].

The main tool used in [3] is the lemma stated below. Let $p_0 = (z_0, z_1, \dots, z_{k-1})$ be the vertices of the polygon p_0 in the complex plane. Writing $\omega_\nu = \exp(2\pi i\nu/k)$, we expend the sequence (z_ν) in its f. F. S.

$$z_\nu = \zeta_0 + \zeta_1 \omega_\nu + \zeta_2 \omega_\nu^2 + \dots + \zeta_{k-1} \omega_\nu^{k-1} \quad (\nu = 0, \dots, k-1), \tag{1}$$

the f. F. coefficients (ζ_ν) being given by the inverse transformation

$$\zeta_\nu = \frac{1}{k}(z_0 + z_1 \overline{\omega}_\nu + z_2 \overline{\omega}_\nu^2 + \dots + z_{k-1} \overline{\omega}_\nu^{k-1}), \quad (\nu = 0, \dots, k-1). \tag{2}$$

Lemma 1. *If we subject the sequence (z_ν) to the cyclic transformation*

$$\begin{aligned} z_0' &= a_0 z_0 + a_1 z_1 + \dots + a_{k-1} z_{k-1} \\ z_1' &= a_{k-1} z_0 + a_0 z_1 + \dots + a_{k-2} z_{k-1} \\ &\vdots \\ z_{k-1}' &= a_1 z_0 + a_2 z_1 + \dots + a_0 z_{k-1}, \end{aligned} \tag{3}$$

and write

$$f(z) = a_0 + a_1 z + \dots + a_{k-1} z^{k-1}, \tag{4}$$

then the f. F. coefficients (ζ_ν'), of the transformed sequence (z_ν'), are given by

$$\zeta_\nu' = \zeta_\nu f(\omega_\nu), \quad (\nu = 0, \dots, k-1). \tag{5}$$

The derivation of the vertices of the polygon p_1, from those of p_0, is seen to be expressed by a cyclic transformation (3) such that (4) takes on the simple form $f(z) = (1+z)/2$. Iterating n times we find the f. F. S. of the vertices of $p_n = (z_0^{(n)}, \dots, z_{k-1}^{(n)})$ to be given by

$$z_\nu^{(n)} = \sum_{\alpha=0}^{k-1} \zeta_\alpha \left(\frac{1+\omega_\alpha}{2}\right)^n \omega_\nu^\alpha. \tag{6}$$

We also assume that $\zeta_0 = k^{-1} \sum z_\alpha = 0$, hence that the centroid of the vertices of p_0, and therefore also of p_n, is at the origin 0. Observe, furthermore that

$$\frac{1+\omega_\alpha}{2} = \cos\frac{\alpha\pi}{k} \cdot \exp\left(\frac{\alpha\pi i}{k}\right), \quad \cos\frac{\alpha\pi}{k} = \left|\frac{1+\omega_\alpha}{2}\right| < \left|\frac{1+\omega_1}{2}\right| = \cos\frac{\pi}{k}, \quad (\alpha = 2, 3, \dots, k-2). \tag{7}$$

Using (6) and $\zeta_0 = 0$ we easily obtain that

$$z_\nu^{(n)} \Big/ \left(\cos\frac{\pi}{k}\right)^n = (\zeta_1 + \zeta_{k-1}) \cos\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + i(\zeta_1 - \zeta_{k-1}) \sin\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + O(\rho_k^n), \tag{8}$$

where

$$\rho_k = \cos\frac{2\pi}{k} \Big/ \cos\frac{\pi}{k} = 1 - \frac{3\pi^2}{k^2} + \dots, \quad (0 < \rho_k < 1). \tag{9}$$

Writing

$$\zeta_1 + \zeta_{k-1} = a + ic, \quad i(\zeta_1 - \zeta_{k-1}) = b + id, \quad z_\nu^{(n)} = x_\nu^{(n)} + i y_\nu^{(n)} \tag{10}$$

where $a, c, \dots, y_\nu^{(n)}$ are reals, we may rewrite (8) as

$$\begin{aligned} x_\nu^{(n)} \left(\sec\frac{\pi}{k}\right)^n &= a \cos\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + b \sin\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + O(\rho_k^n), \\ y_\nu^{(n)} \left(\sec\frac{\pi}{k}\right)^n &= c \cos\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + d \sin\left(\frac{2\pi\nu}{k} + \frac{n\pi}{k}\right) + O(\rho_k^n). \end{aligned} \tag{11}$$

In terms of the affine transformation

$$A : x' = ax + by, \quad y' = cx + dy, \tag{12}$$

and the regular polygon $\Pi_0 = (\omega_\nu)$, we may restate (11) as

Theorem 1. 1. *The Conjecture 1 is true without the assumption that p_0 is convex.*

2. *As $n \to \infty$ we have*

$$\left(\sec\frac{\pi}{k}\right)^n p_n = A(e^{\frac{n\pi i}{k}} \Pi_0) + O(\rho_k^n), \tag{13}$$

where A is defined by (12), (10), *and ρ_k by* (9).

The relation (13) shows that the left side of (13) has $2k$ distinct limit positions, one for each residue class of n, mod. $2k$. For the polygon p_n *as a whole* there are only two limit positions: $A(\Pi_0)$ and $A(e^{\pi i/k}\Pi_0)$.

2. On Conjecture 2. My reaction on hearing of Conjecture 2 was to try to establish it by the f. F. S. which was so effective for Conjecture 1. My attempt failed, naturally, because now the construction of P_1, from P_0, can not be expressed by a cyclic transformation of the form (3). For if it could, then the assumption of convexity of P_0 would clearly be unnecessary, while we know that P_1 need not exist if P_0 fails to be strictly convex. A bit vaguely we may say that the construction of P_1 is „cyclic" but not linear (in the Algebra of C). See [3, Theorem 3 on p. 398] for the conditions on (3) to lead, on iteration, to a polygon p_n having an affinity for affine images of regular polygons as $n \to \infty$.

This remark made me doubt the correctness of Conjecture 2. The following is a

Counter-example. Figure 1 shows five expanding convex hexagons. Just now we are only concerned with the three outer hexagons.

Let $P_0 = (\nu)$ (here and below ν runs from 1 to 6) be a convex hexagon (Figure 1) which is a modification of a regular hexagon: The two triangles $T_1 = (1, 3, 5)$ and $T_2 = (2, 4, 6)$ are still concentric regular triangles with sides parallel, the only difference being that the triangle T_2 is smaller than the triangle T_1.

We now derive from P_0 by the prescribed construction the hexagon $P_1 = (\nu')$. Notice the following

Property of P_1. *The six chords joining two nearly adjacent vertices of P_1 (like 6′2′), are parallel to the sides of $P_0 = (\nu)$ (like 12).*

This follows from the fact that our figure is invariant with respect to all rigid motions including reflexions, that leave invariant the triangle $T_1 = (1, 3, 5)$. In particular this implies that the vertices of P_0 are the midpoints of the sides of P_1, and this implies the above property of P_1.

It follows that $P_2 = (\nu'')$ is obtained by drawing through the vertices of P_1 parallel to the sides of P_0. Thus 1″2″ is parallel to 12, etc. However, the invariance mentioned above implies that P_2 is a hexagon having the same structure as P_0: The triangles ($1''$, $3''$, $5''$) and ($2''$, $4''$, $6''$) are concentric equilateral triangles having parallel sides. However, the parallelism of the sides of P_0 and P_2 now implies that

(14) *P_0 and P_2 are homothetic with respect to their common center 0.*

We conclude that if we continue the process obtaining P_0, P_1, P_2 ,..., then the hexagons P_0, P_2, P_4 ,..., P_{2m} ,... are all homothetic to P_0. Likewise P_1, P_3 ,..., P_{2m+1} ,... are all homothetic to P_1. However, *neither P_0, nor P_2, are affine images of a regular hexagon.* Indeed, P_0 is not, because opposite sides are not parallel even though they are equal in lengths. Also P_1 is not, because opposite sides are not of equal length even though they are parallel. This disproves Conjecture 2 for hexagons.

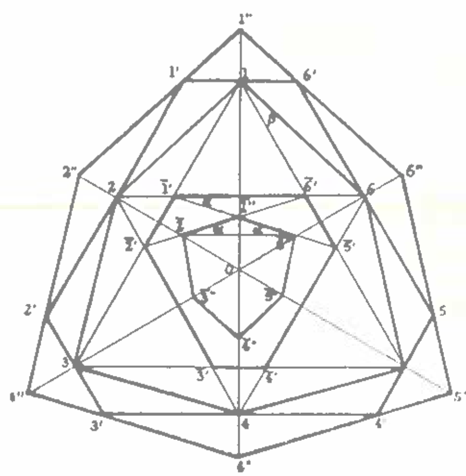

Figure 1

3. **A modification of Conjecture 2: The peeling process.** Let $\overline{P}_0=(\overline{\nu})$ ($\nu=1$,..., k) be a convex polygon. From $\overline{P}_0$ we derive the convex polygon $\overline{P}_1=(\overline{\nu}')$ whose sides are formed by the chords $\overline{k\,2}$, $\overline{1\,3}$,..., $\overline{k-1\,1}$ of $\overline{P}_0$. Clearly $\overline{P}_1$ is obtained from $\overline{P}_0$ by „peeling off" its „bark" formed by the union of triangles

$$(\overline{k}, \overline{1}, \overline{2}) \cap (\overline{1}, \overline{2}, \overline{3}) \cap \dots \cap (\overline{k-1}, \overline{k}, \overline{1}).$$

We iterate the construction n times obtaining the convex polygon $\overline{P}_n$.

***Question**: Does an appropriate affine image of $\overline{P}_n$ converge, as $n\to\infty$, to a regular polygon?*

The answer is again negative. For a counter-example we refer again to Figure 1, where $\overline{P}_0=P_0=(\nu)$ ($\nu=1$,..., 6) is our old hexagon. Figure 1 shows the peeled polygons $\overline{P}_1=(\overline{\nu}')$ and $\overline{P}_2(\overline{\nu}'')$.

Lemma 2. *The hexagon $\overline{P}_2=(\overline{\nu}'')$ is similar to $P_0=(\nu)$.*

Proof: The new figure preserves its old invariance property. It suffices therefore to show that

(15) $$\alpha=\sphericalangle\,\overline{6}''\overline{2}''\overline{1}'' \text{ is equal to } \beta=\sphericalangle\,516.$$

Evidently $\alpha=\sphericalangle\,\overline{6}''\overline{2}''\overline{1}''=\sphericalangle\,\overline{2}''\overline{6}''\overline{1}''=\sphericalangle\,\overline{5}'\overline{1}'6$. However, this last angle is equal to β as we immediately see if we observe that the four points 1, $\overline{1}'$, $\overline{5}'$ 6 are on a circle, while $\beta=\sphericalangle\,\overline{5}'16$.

A proof that no affine image of $\overline{P}_n$ can possibly converge to a regular hexagon is now completed as in the case of Conjecture 2.

4. **Concluding remarks.** The above remarks raise some questions and a few will now be stated.

1. At the end of section 1 we found for p_n, as $n\to\infty$, a single limit shape in *two* distinct positions. In Figure 1 we found for P_n *two* distinct shapes, hence also limit shapes. We found the same situation for $\overline{P}_n$.

The construction of Figure 1 generalizes readily to polygons P_0 with an even number $k=2m$ of vertices. We start from a regular $2m$-gon $P_0=(1, 2, 3$,...), when $T_1=(1, 3, 5$,...) and $T_2=(2, 4, 6$,...) are regular m-gons. We now shrink T_2 slightly making it smaller, but still concentric and parallel to the original T_2. Again we find only *two*, distinct shapes, one for P_0, P_2 ,..., the other for P_1, P_3 ,..., and neither of these are affine regular.

Problem 1. *Does the appearance of just two limit shapes in all these cases have any general significance? Are there convex polygons P_0, with $k=2m$ vertices, such that there are three, or more, limit shapes for P_m, or for $\overline{P}_n$? Or none at all?*

The remark in the paragraph before the last disproves Conjecture 2 for all *even* values of k.

Problem 2. *Does Conjecture 2 hold for odd k, e. g. pentagons?*

Concerning the „peeled" polygon $\overline{P}_n$ I wish to state

Conjecture 3. The set

(16) $$\overline{P}_\infty=\lim_{n\to\infty}\overline{P}_n=\bigcap_{n=1}^{\infty}\overline{P}_n$$

reduces to a point.

If $\overline{P}_0$ is a pentagon obtained from a triangle by cutting off two of its vertices by two small triangles, then graphic experimentation seems to indicate that the limit set (16) might be a segment. Again Carl de Boor kindly performed the „constructions" on the computer obtaining $\overline{P}_n$ for $n=1, 2$,..., 15, and these show beyond a shadow of a doubt that $\overline{P}_\infty$ reduces to a point in the case of pentagons having an axis of symmetry.

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