Si $(M', N') \in \mathfrak{S}_j'$, alors $N' \in \sigma_j'(M')$, c'est-à-dire $\exists \alpha \in \mathcal{C}l_{\mathfrak{F}}(M')$ telle que $r(\alpha, j) \subset N'$ et par (3.21), compte tenu de la propriété préfiltrale de $\sigma_j(M')$, il résulte $N' \subset \sigma_j(M')$ où $(M', N') \in \mathfrak{S}_j$ Donc $\mathfrak{S}_i = \mathfrak{S}_i' \; \forall i \in I$.

On peut démontrer aisément que dans la famille des relations d'approximation engendrant par (3.8) – (3.14) une structure syntopogène $\mathfrak{F}$ sur X la structure $\mathcal{C}l_{\mathfrak{F}}$ constante par (3.21) est la moins fine.

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STRUCTURI DE APROXIMARE, OPERATORI DE ÎNCHIDERE GENERALIZAŢI ŞI STRUCTURI SINTOPOGENE

Rezumat

În această Notă se introduce noţiunea de structură de aproximare pe o mulţime X, în raport cu dirijanta A, ca o submulţime a produsului cartezian $X \times \mathfrak{K}(A, X)$, în care $\mathfrak{K}(A, X)$ este mulţimea tuturor $\alpha \subset A \times X$.

În primul paragraf sînt studiate structurile iniţiale şi limitele proiective de spaţii cu structură de aproximare.

În al doilea paragraf se stabileşte, prin propoziţia 2.3, legătura dintre structurile de aproximare unipunctual dirijate şi operatorii de închidere generalizaţi.

Se stabilesc, de asemenea, în ultimul paragraf, legăturile între structurile de aproximare şi cele sintopogene.

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OPERATORS AND FUNCTIONALS ON BANACH SPACES THAT ARE TOPOLOGICALLY INVARIANT UNDER THE ACTION OF A σ-COMPACT UNIMODULAR AMENABLE GROUP

BY

JEAN-PAUL PIER

§ 1. **Topologically invariant operators.** Let X be a locally compact space endowed with a bounded positive measure λ and let $E(X)$ be a complex Banach subspace either of the space $\mathcal{C}(X)$ of bounded complex continuous functions defined on X or of $L^p(X) = L^p(X, \lambda)$; $1 \leqslant p \leqslant \infty$. Suppose further that, if $f \in E(X)$, then $\bar{f} \in E(X)$ and $|f| \in E(X)$. By $K(E(X), E(X))$ will be denoted the convex set of all positive linear (continuous) operators on $E(X)$ such hat $|||T||| \leqslant 1$. In case $E(X)$ contains the constant functions, assume that $T(1_X) = 1_X$; hence $|||T||| = 1$. In the same manner, $K(E(X))$ will be the convex set of all positive linear (continuous) functionals L on $E(X)$ such that $|||L||| \leqslant 1$. In case $E(X)$ contains the constant functions, assume again that L is normalized; hence $K(E(X))$ will then be the set of all the means on $E(X)$.

Definition 1.1. *A topological group G is said to act on X with respect to $E(X)$ if there exists a measurable mapping $(x, \xi) \to x\xi$ from $G \times X$ to X such that, if $A_x f(\xi) = f(x\xi)$ for $f \in E(X)$ $x \in G$, $\xi \in X$, then*

(i) *for every $x \in G$, $f \to A_x f$ belongs to $K(E(X), E(X))$.*

(ii) *for every $f \in E(X)$ and all $x, x' \in G$, $A_{x'} (A_x f) = A_{xx'} f$.*

(iii) *λ is invariant by the action of G* i. e. *for every $f \in E(X)$ and every $x \in G$, $\lambda(A_x f) = \lambda(f)$.*

In the sequel, G will always be a *locally compact group* with Haar measure denoted by μ or dx; $L^+(G) = \{\varphi \in L^1(G) : \varphi \geqslant 0, \int_G \varphi(x) dx = 1\}$.

The following lemmas can be readily proved:

Lemma 1.2. *Let G act on X with respect to $E(X)$. If $f \in E(X)$, $\varphi \in L^1(G)$, then $\int_G \varphi(x) f(x\xi) dx$ exists for almost all $\xi \in X$. Put $\varphi * f(\xi) = \int_G \varphi(x) f(x\xi) dx$ for almost all $\xi \in X$. If $f \in L^p(X)$, $1 \leqslant p \leqslant \infty$, one has $\varphi * f \in L^p(X)$ and $\|\varphi * f\|_{X,p} \leqslant \|\varphi\|_{G,1} \|f\|_{X,p}$.*

Lemma 1.3. *If, for some $f \in L^p(X)$, $1 \leqslant p \leqslant \infty$, one has $A_x f = f$, for every $x \in G$, then also $\varphi * f = f$, for every $\varphi \in L^+(G)$.*

Definition 1.4. *$E(X)$ is said to be topologically invariant (for the action of G) if, for every $f \in E(X)$ and every $\varphi \in L^+(G)$, $\varphi * f \in E(X)$. We then say that an operator $T \in K(E(X), E(X))$ is topologically invariant (for the action of G), if, for every $f \in E(X)$ and every $\varphi \in L^+(G)$, $T(\varphi * f) = T(f)$; a functional $L \in K(E(X))$ is said to be topologically invariant for the action of G if, for every $f \in E(X)$ and every $\varphi \in L^+(G)$, $L(\varphi * f) = L(f)$.*

By $K_G(E(X), E(X))$ will be denoted the set of all topologically invariant operators in $K(E(X), E(X))$; $K_G(E(X))$ will be the set of all topologically invariant functionals in $K(E(X))$.

In § 2. we consider some particular examples of topologically invariant operators for the action of a σ-compact amenable unimodular group. In § 3. we mention an application.

First we state some more immediate lemmas for the case where G acts with respect to a subspace of $L^p(X)$; $1 \leqslant p \leqslant \infty$.

Lemma 1.5. *If $\varphi \in L^1(G)$, $f \in L^p(X)$, $1 \leqslant p \leqslant \infty$, and $a \in G$, one has*

$$A_a(\varphi * f) = ({}_{a^{-1}}\varphi) * f = \varphi * A_a f.$$

(Here ${}_x\varphi(y) = \varphi(xy)$ for a function φ defined on G and $x, y \in G$).

Lemma 1.6. *If $\varphi, \psi \in L^1(G)$ and $f \in L^p(X)$, $1 \leqslant p \leqslant \infty$, one has*

$$\varphi * (\psi * f) = (\varphi * \psi) * f,$$

$*$ being the convolution product for G.

Lemma 1.7. *If G acts on X with respect to $E(X)$, then every $T \in K_G(E(X), E(X))$ is invariant for the action of G i. e. for every $f \in E(X)$ and every $a \in G$, $T(A_a f) = T(f)$. The analogous result holds for every element of $K_G(E(X))$.*

§ 2. Examples. Recall that if G is a σ-*compact amenable group*, there exists a sequence $\{H_n\}$ of measurable subsets in G such that $0 < \mu(H_n) < \infty$ and $\lim\limits_{n\to\infty} \mu(xH_n \Delta H_n)/\mu(H_n) = 0$, uniformly for every $x \in G$; if moreover G is *unimodular*, then also $\lim\limits_{n\to\infty} \mu(H_n x \Delta H_n)/\mu(H_n) = 0$ uniformly for every $x \in G$ ([4]). Let $\theta_n = 1/\mu(H_n)\, 1_{H_n}$; $n \in N^*$.

Lemma 2.1. *If G is a σ-compact amenable group, then $\lim\limits_{n\to\infty} \| \theta_n - \varphi * \theta_n \|_1 = 0$, uniformly for every $\varphi \in L^+(G)$; if moreover G is unimodular, then $\lim\limits_{n\to\infty} \| \theta_n - \theta_n * \varphi \|_1 = 0$, uniformly for every $\varphi \in L^+(G)$.*

Proof. Given $\varepsilon > 0$, let $n \in N^*$ be such that, for every $x \in G$, $\mu(xH_n \Delta H_n)/\mu(H_n) < \varepsilon$ and, in case G is unimodular, also $\mu(H_n x \Delta H_n)/\mu(H_n) < \varepsilon$.

$$\text{Then, } \|\theta_n - \varphi * \theta_n\|_1 = \int_G |\theta_n(y) - \varphi * \theta_n(y)|\, dy = \int_G \left| \int_G \theta_n(y)\varphi(x)\, dx - \right.$$

$$\left. \int_G \varphi(x)\, \theta_n(x^{-1}y)\, dx \right| dy \leqslant \int_G\int_G |\theta_n(y) - \theta_n(x^{-1}y)|\, \varphi(x) dx\, dy \leqslant$$

$$\int_G \frac{\mu(H_n \Delta x H_n)}{\mu(H_n)} \varphi(x) dx < \varepsilon \int_G \varphi(x)\, dx = \varepsilon;$$

n case G is unimodular,

$$\|\theta_n - \theta_n * \psi\|_1 = \int_G |\theta_n(y) - \theta_n * \varphi(y)|\, dy = \int_G \left| \int_G \theta_n(y)\varphi(x)\, dx - \int_G \theta_n(yx^{-1})\varphi(x)\, dx \right| dy \leqslant$$

$$\leqslant \int_G\int_G |\theta_n(y) - \theta_n(yx^{-1})|\, \varphi(x)\, dx\, dy \leqslant \int_G \frac{\mu(H_n \Delta H_n x)}{\mu(H_n)} \varphi(x)\, dx < \varepsilon \int_G \varphi(x)\, dx = \varepsilon.$$

Greenleaf ([6]) Corollary 3.4.) proved the following

Proposition 2.2. *Let the σ-compact amenable group G act on X with respect to $L^p(X)$ for $1 < p < \infty$ or for $p = 1$ in case G is compact. Then to every $f \in L^p(X)$ one may associate $\tilde{f} \in L^p(X)$ such that*

(i) *for every $a \in G$, $A_a \tilde{f} = \tilde{f}$,*

(ii) $\lim\limits_{n\to\infty} \| \theta_n * f - \tilde{f} \|_p = 0$.

Next we consider a general version of a definition stated by Chatard ([1]) in case $p = 1$ and G is σ-compact abelian.

Definition 2.3. *Let G be a locally compact group acting on X with respect to $L^p(X)$, $1 \leqslant p \leqslant \infty$. Then a function $f \in L^p(X)$ is said to be L^p—almost periodic if the orbit $o(f) = \{A_x f : x \in G\}$ of f is relatively compact in $L^p(X)$.*

We will denote by $P^p(X)$ the set of all $f \in L^p(X)$ that are L^p-almost periodic. If $f \in P^p(X)$, then $\bar{f} \in P^p(X)$ and $|f| \in P^p(X)$; the latter result is a consequence of the inequality

$$||f(\xi)| - |f(\eta)|| \leqslant |f(\xi) - f(\eta)|,$$

for $f \in P^p(X)$ and $\xi, \eta \in X$. Obviously, $P^p(X)$ is a vector space. One can easily prove that $P^p(X)$ is closed in $L^p(X)$, $1 \leqslant p \leqslant \infty$. Hence, $P^p(X)$ is a *Banach subspace* of $L^p(X)$.

Proposition 2.4. *$P^p(X)$ is topologically invariant in $L^p(X)$; $1 \leqslant p \leqslant \infty$.*

Proof. Notice that, for every $\varepsilon > 0$, every $f \in P^p(X)$ and every $\varphi \in L^1(G)$, there exist $a_1, \ldots, a_m \in G$ such that to each $x \in G$ there corresponds $k = 1, \ldots, m$ for which $\| A_x f - A_{a_k} f \|_p < \varepsilon / \|\varphi\|_1$. It then follows from 1.5. and 1.2. that

$$\| A_x(\varphi * f) - A_{a_k}(\varphi * f) \|_p = \| \varphi * (A_x f - A_{a_k} f) \|_p \leqslant \|\varphi\|_1 \| A_x f - A_{a_k} f \|_p < \varepsilon.$$

Hence, if $f \in P^p(X)$, $\varphi * f \in P^p(X)$.

In proposition 2.5., Chatard's proof is carried over.

Proposition 2.5. *Let the σ-compact amenable unimodular group G act on X with respect to $P^p(X)$, $1 \leqslant p \leqslant \infty$. Then to every $f \in P^p(X)$ one may associate an element $\tilde{f} \in P^p(X)$ such that*

(i) *for every $a \in G$, $A_a \tilde{f} = \tilde{f}$,*

(ii) $\lim\limits_{n\to\infty} \| \theta_n * f - \tilde{f} \|_p = 0$.

Proof. As for every $f \in P^p(X)$, the operator $T_f : \varphi \to T_f(\varphi) = \varphi * f$ from $L^1(G)$ into $L^p(X)$ is compact, there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\{\theta_{n_k} * f\}$ converges in the L^p-norm to an element $\tilde{f}$ of $L^p(X)$.

By 1.5. and 1.1 (i), for every $a \in G$,

$$\| A_a \tilde{f} - \theta_{n_k} * A_a f \|_p = \| A_a \tilde{f} - A_a(\theta_{n_k} * f) \|_p = \| A_a(\tilde{f} - \theta_{n_k} * f) \|_p \leqslant \| \tilde{f} - \theta_{n_k} * f \|_p;$$

hence

$$\lim_{k\to\infty} \| A_a \tilde{f} - \theta_{n_k} * A_a f \|_p = 0. \tag{1}$$

By 1.5 and 1.2,

$$\lim_{k\to\infty} \| \theta_{n_k} * (A_a f - f) \|_p = \lim_{k\to\infty} \| ({}_{a^{-1}}\theta_{n_k} - \theta_{n_k}) * f \|_p \leqslant$$

$$\leqslant \lim_{k\to\infty} \| {}_{a^{-1}}\theta_{n_k} - \theta_{n_k} \|_1 \| f \|_p = \lim_{k\to\infty} \frac{\mu(aH_{n_k} \Delta H_{n_k})}{\mu(H_{n_k})} \| f \|_p.$$

Thus, because G is σ-compact amenable,

$$\lim_{k\to\infty} \| \theta_{n_k} * (A_a f - f) \|_p = 0. \tag{2}$$

From (1) and (2) one deduces that $\lim\limits_{k\to\infty} \| A_a \tilde{f} - \theta_{n_k} * f \| = 0$, i. e. $A_a \tilde{f} = \tilde{f}$. Hence $\tilde{f} \in P^p(X)$ and, by 1.3, for every $\varphi \in L^+(G)$, $\varphi * \tilde{f} = \tilde{f}$.

If $f=0$, $\tilde{f}=0$ and (ii) is trivially satisfied. Suppose then that $f\neq 0$.

Given $\varepsilon>0$, there exists $k_0\in N^*$ such that $\|\theta_{n_{k_0}} * f-\tilde{f}\|_p<\frac{\varepsilon}{2}$. By 2.1, there exists $n_0\in N^*$ such that, for $n\geq n_0$,

$$\|\theta_n-\theta_n * \theta_{n_{k_0}}\|_p<\frac{\varepsilon}{2\|f\|_p}.$$ Then, for $n\geq n_0$, by 1.2 and 1.6,

$$\|\theta_n * f-\tilde{f}\|_p=\|\theta_n * f-\theta_n * \tilde{f}\|_p\leq$$

$$\leq\|\theta_n * f-(\theta_n * \theta_{n_{k_0}}) * f\|_p+\|\theta_n * (\theta_{n_{k_0}} * f)-\theta_n * \tilde{f}\|_p\leq$$

$$\leq\|\theta_n-\theta_n * \theta_{n_{k_0}}\|_1\|f\|_p+\|\theta_n\|_1\|\theta_{n_{k_0}} * f-\tilde{f}\|_p<\frac{\varepsilon}{2\|f\|_p}\|f\|_p+\frac{\varepsilon}{2}=\varepsilon.$$

Theorem 2.6. *Let G be a σ-compact amenable unimodular group acting on X with respect to $L^p(X)$, $1\leq p<\infty$ (in case $p=1$, X is supposed to be compact) [resp. with respect to $P^p(X)$; $1\leq p\leq\infty$]. Then there exists $T\in K_G(L^p(X), L^p(X))$ [resp. $T\in K_G(P^p(X), P^p(X))$] such that, for every $f\in L^p(X)$ [resp. $f\in P^p(X)$],*

(i) *for every $a\in G$, $A_a(T(f))=T(f)$.*

(ii) $\lim\limits_{n\to\infty}\|\theta_n * f-T(f)\|_p=0.$

Proof. Define $T:f\to T(f)=\tilde{f}$ according to 2.2 and 2.5. It suffices to establish the topological invariance of T.

Let $f\in L^p(X)$ [resp. $f\in P^p(X)$] and $\varphi\in L^1(G)$. By 2.2 [resp. 2.5.], 1.6, 1.2, 2.1, one obtains

$$\|T(\varphi * f)-T(f)\|_p=\|T(\varphi * f-f)\|_p=$$

$$=\lim_{n\to\infty}\|\theta_n * (\varphi * f-f)\|_p=\lim_{n\to\infty}\|(\theta_n * \varphi-\theta_n) * f\|_p\leq$$

$$\leq\lim_{n\to\infty}\|\theta_n * \varphi-\theta_n\|_1\|f\|_p=0.$$

Remark 2.7. *In case $p=1$ and λ is a probability measure, if $K_G(E(X), E(X))\neq\emptyset$, then also $K_G(E(X))\neq\emptyset$.*

It suffices to associate $M:f\to\lambda(T(f))$ to $T\in K_G(E(X), E(X))$.

§ 3. Application. The following theorem is modelled after C o n v e r s e - N a m i o k a - P h e l p s ([3] Theorem 2.1).

Theorem 3.1. *Let X be a locally compact space endowed with a measure λ and let $E(X)$ be, for the ordinary operations of functions on X, a Banach subalgebra either of $\mathcal{C}(X)$ or of $L^\infty(X, \lambda)$. Suppose that $E(X)$ contains the constant functions and that, if $f\in E(X)$, then $|f|\in E(X)$ and $\bar{f}\in E(X)$. Let G be a σ-compact unimodular amenable group acting on X with respect to $E(X)$. Suppose that $E(X)$ is topologically invariant and that $K_G(E(X))\neq\emptyset$. Then, for every $M\in K_G(E(X))$, the following assertions are equivalent:*

(i) *M is an extreme point in $K_G(E(X))$.*

(ii) *for every $f\in E(X)$ and every $g\in E(X)$,*

$$\lim_{n\to\infty}M(\theta_n * fg)=M(f)\,M(g).$$

(iii) *for every $f\in E(X)$, there exists a sequence $\{\psi_n\}$ in $L^1(G)$ such that, for every $g\in E(X)$,*

$$\lim_{n\to\infty}M(\psi_n * f\psi_n * g)=M(f)\,M(g).$$

Proof. $\|\cdot\|$ will denote the norm of the Banach space $E(X)$.

(i) $\to$ (ii).

If $\|g\|=0$, the result is trivial. Suppose then that $\|g\|\neq 0$.

Let $U_n(f)=\frac{1}{\|g\|}[M(\theta_n * fg)-M(f)\,M(g)]$ for $f\in E(X)$, $n\in N^*$.

For every $n\in N^*$, U_n is a linear functional on $E(X)$.

Suppose for a moment that $g\geq 0$. Then, for every $f\in E(X)$, $f\geq 0$, and every $n\in N^*$, one has

$$(M+U_n)(f)=M(f)\left[1-\frac{M(g)}{\|g\|}\right]+\frac{1}{\|g\|}M(\theta_n * fg)\geq 0$$

and

$$(M-U_n)(f)=\frac{1}{\|g\|}M(f)\,M(g)+M\left(f-\theta_n * f\frac{g}{\|g\|}\right)\geq$$

$$\geq\frac{1}{\|g\|}M(f)\,M(g)+M(f-\theta_n * f)=\frac{1}{\|g\|}M(f)\,M(g)\geq 0;$$

hence, if $g\geq 0$, $(M\pm U_n)(f)\geq 0$ for every $f\geq 0$. Moreover, for every $n\in N^*$, $(M\pm U_n)(1_X)=1$. Thus, for every $n\in N^*$, $(M\pm U_n)\in K(E(X))$.

We will show that the sequence $\{U_n(f)\}$ is convergent in $\mathbf{C}$, for $f\in E(X)$.

If $f=0$, the sequence converges trivially to $U(f)=0$.

Consider $f\in E(X)$, $\|f\|\neq 0$. As, for every $n\in N^*$, $|U_n(f)|\leq 2\|f\|$, there exists a subsequence $\{U_{n_k}(f)\}$ converging to a complex number $U(f)$. Given $\varepsilon>0$, there exists $k_0\in N^*$ such that, for any $k\geq k_0$, $|U(f)-U_{n_k}(f)|<\varepsilon/3$. By lemma 2.1, there exists $k_1\geq k_0$ such that for any $n\geq n_{k_1}$ and any $m\in N^*$, $\|\theta_n * \theta_m-\theta_n\|_1<\varepsilon/3\|f\|$ and $\|\theta_m * \theta_n-\theta_n\|_1<\varepsilon/3\|f\|$. Hence, for $n\geq n_{k_1}$,

$$|U(f)-U_n(f)|\leq|U(f)-U_{n_{k_1}}(f)|+|U_{n_{k_1}}(f)-U_n(f)|<$$

$$<\frac{\varepsilon}{3}+\frac{1}{\|g\|}|M((\theta_{n_{k_1}}-\theta_n) * fg)|\leq\frac{\varepsilon}{3}+\|\theta_{n_{k_1}}-\theta_n\|_1\|f\|\leq$$

$$\leq\frac{\varepsilon}{3}+\|\theta_{n_{k_1}} * \theta_n-\theta_{n_{k_1}}\|_1\|f\|+\|\theta_{n_{k_1}} * \theta_n-\theta_n\|_1\|f\|<$$

$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Thus, $\lim\limits_{n\to\infty}U_n(f)=U(f)$ and $(M\pm U)\in K(E(X))$.

For every $\varphi\in L^+(G)$ and every $n\in N^*$, by 1.6 and 1.2,

$$|U_n(\varphi * f)-U_n(f)|=\frac{1}{\|g\|}|M(\theta_n * (\varphi * f-f)\,g)|=$$

$$=\frac{1}{\|g\|}|M((\theta_n * \varphi-\theta_n) * fg)|\leq\|\theta_n * \varphi-\theta_n\|_1\|f\|.$$

Then, $|U(\varphi * f)-U(f)| \leqslant \lim_{n\to\infty} |U_n(\varphi * f)-U_n(f)| \leqslant \lim_{n\to\infty} \|\theta_n * \varphi - \theta_n\|_1 \|f\|$; from 2.1 one deduces that $U(\varphi * f)=U(f)$.

We have shown that in case $g \geqslant 0$, $(M \pm U) \in K_G(E(X))$. As M is an extreme point in $K_G(E(X))$, necessarily $U=0$. Thus, for every $g \in E(X)$, $\|g\| \neq 0$, $g \geqslant 0$, one has

$$\lim_{n\to\infty} M(\theta_n * fg)=M(f)\, M(g).$$

for every $f \in E(X)$.

For every $g \in E(X)$ with real values, there exists $a>0$ such that $g+a1_X \geqslant 0$; then owing to the foregoing result, for every $f \in E(X)$,

$$M(f)\, M(g)=M(f)\, M(g+a1_X)-aM(f)=$$

$$=\lim_{n\to\infty} M(\theta_n * f(g+a1_X))-\lim_{n\to\infty} M(\theta_n * fa1_X)=\lim_{n\to\infty} M(\theta_n * fg).$$

Thus the desired equality holds for every real-valued $g \in E(X)$: it is then valid for any $g \in E(X)$.

(ii)$\Longrightarrow$(iii).

Consider in $E(X)$ the ideal $I=\{u \in E(X) : M(u^2)=0\}$ and the non-degenerate bilinear hermitian form $(.,.)$ defined on $E(X)/I$ by $M(u\bar{v})=(\dot{u}, \dot{v})$. Let H be the completion of $E(X)/I$ as a Hilbert space. Let $z=M(f)\dot{1}_X$, for $f \in E(X)$. By hypothesis, given $f \in E(X)$, $g \in E(X)$ and $\varepsilon>0$, there exists $n_0 \in N^*$ such that

$$|(z, \dot{g})-((\theta_{n_0} * f)^{\cdot}, \dot{g})|=|M(f)\, M(g)-M(\theta_{n_0} * fg)|<\varepsilon.$$

Thus, z belongs to the weak closure of $\{(\varphi * f)^{\cdot} : \varphi \in L^1(G)\}$ in H; this set being convex, z belongs to its strong closure. Therefore, there exists a sequence $\{\varphi_n\}$ in $L^1(G)$ such that $M(\varphi_n * fg)$ converges to $M(f)\, M(g)$ uniformly for all $g \in E(X)$ such that $\|g\| \leqslant$ constant c. For every $m \in N^*$, $\|\varphi_m * g\| \leqslant \|g\|$ by 1.2. Thus, if $\|g\| \leqslant c$,

$$\lim_{n\to\infty} M(\varphi_n * f\varphi_m * g)=M(f)\, M(\varphi_m * g)=M(f)\, M(g)$$

and

$$\lim_{n\to\infty} M(\varphi_n * f\varphi_n * g)=M(f)\, M(g).$$

By a diagonalization process, one may determine a sequence in $L^1(G)$ for which the equality holds for all $g \in E(X)$. It suffices to consider, for every $m \in N^*$, a sequence $\{\varphi_{m,1}, \varphi_{m,2}, \ldots\}$ in $L^1(G)$ such that, for every $g \in E(X)$, $\|g\| \leqslant m$,

$|M(\varphi_{m,n} * f\varphi_{m,n} * g)-M(f)\, M(g)|<\dfrac{1}{n}$; $n \in N^*$. Let $\psi_n=\varphi_{n,n}$ for $n \in N^*$ and consider $\{\psi_n\}$. Then, for any $g \in E(X)$,

$$\lim_{n\to\infty} M(\psi_n * f\psi_n * g)=M(f)\, M(g).$$

(iii)$\Longrightarrow$(i).

If M is not an extreme point in $K_G(E(X))$, there exist extreme points M', $M'' \in K_G(E(X))$ and $\alpha \in]0,1[$ such that $M=\alpha M'+(1-\alpha)M''$.

Consider a real-valued $f_0 \in E(X)$ such that $M(f_0)=0$; by hypothesis, given $\varepsilon>0$, there exists $m \in N^*$ such that $0 \leqslant M(|\psi_m * f_0|^2)<\varepsilon$. By Schwarz's inequality,

$$0 \leqslant |M'(f_0)|^2=|M'(\psi_m * f_0)|^2 \leqslant M'(1_X) M'(|\psi_m * f_0|^2) \leqslant \frac{1}{\alpha} M(|\psi_m * f_0|^2)<\frac{\varepsilon}{\alpha}.$$

Thus $|M'(f_0)|<\sqrt{\varepsilon/\alpha}$. As $\varepsilon>0$ is arbitrary, one may conclude that, if $M(f)=0$ for $f \in E(X)$, then also $M'(f)=0$. There must then exist a constant $k \geqslant 0$ such that $M'=kM$. But $k=kM(1_X)=M'(1_X)=1$. So one has $M=M'$.

Recall that if G is a locally compact group, for $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, $\varphi * f$ belongs to the space $\mathcal{UC}_r(G)$ of bounded right uniformly continuous functions on G. As an amenable group admits left invariant means on $L^\infty(G)$, $\mathcal{C}(G)$, $\mathcal{UC}_r(G)$, one deduces from 3.1. the following corollary.

Corollary 3.2. *Let G be a σ-compact unimodular amenable group. If $M \in K_G(L^\infty(G))$ [resp. $M \in K_G(\mathcal{C}(G))$, $M \in K_G(\mathcal{UC}_r(G))$], the following assertions are equivalent:*

(i) *M is an extreme point in $K_G(L^\infty(G))$ [resp. $K_G(\mathcal{C}(G))$, $K_G(\mathcal{UC}_r(G))$],*

(ii) *for every $f \in L^\infty(G)$ [resp. $f \in \mathcal{C}(G)$, $f \in \mathcal{UC}_r(G)$], and every $g \in L^\infty(G)$ [$g \in \mathcal{C}(G)$, $g \in \mathcal{UC}_r(G)$],*

$$\lim_{n\to\infty} M(\theta_n * fg)=M(f)\, M(g),$$

(iii) *for every $f \in L^\infty(G)$ [resp. $f \in \mathcal{C}(G)$, $f \in \mathcal{UC}_r(G)$], there exists a sequence $\{\psi_n\}$ in $L^1(G)$ such that*

$$\lim_{n\to\infty} M(\psi_n * f\psi_n * g)=M(f)\, M(g)$$

for every $g \in L^\infty(G)$ [resp. $g \in \mathcal{C}(G)$, $g \in \mathcal{UC}_r(G)$].

Remark 3.3. It is known that, for a non-compact σ-compact amenable group G, card $K_G(L^\infty(G)) \geqslant 2^c$ ([2]); hence the sets of topologically invariant means considered in 3.2. are non trivial. In order to show the interest of 3.2. let us mention that a nontrivial group has no multiplicative left invariant mean on the space $\mathcal{UC}(G)$ of bounded uniformly continuous functions on G ([5]) and thus no topologically invariant means neither on $\mathcal{C}(G)$ nor on $L^\infty(G)$. Therefore, the corollary characterizes the topologically invariant means on $\mathcal{C}(G)$ or $L^\infty(G)$ that are „approximately multiplicative"; this characterization is given in a manner that generalizes the fact that extreme points of the convex set of positive normalized operators on an algebra of functions are just the multiplicative ones ([7]).

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OPERATORI ȘI FUNCȚIONALE ÎN SPAȚII BANACH TOPOLOGICI INVARIANȚI LA ACȚIUNEA UNUI GRUP σ-COMPACT, MEDIABIL, UNIMODULOR

Rezumat

După ce se definesc noțiunile de operator și de funcțională pe un spațiu Banach $E(X)$ topologic invariante în raport cu un grup G mediabil și σ-compact, se indică unele exemple remarcabile de asemenea operatori (a se vedea, spre exemplu, operatorii pe $P^p(X)$ $(1 \leqslant p \leqslant \infty)$, spațiul funcțiilor aproape periodice în sens Chatard).

Prinicipalul rezultat al lucrării îl constituie teorema 3.1. de tip Converse-Namioka-Phelps, pentru genul de operatori considerat.

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PSEUDOCONFORMAL MAPPINGS IN SPACE (I)

BY

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1. Introduction and definitions. Suppose that $X = X(\xi)$ is a diffeomorphism of a domain $D \subset R^n (n \geqslant 2)$ onto a domain $D' \subset R^n$. For $\xi \in D$ let J be the jacobian matrix of $X(\xi)$, $\left(J_{ih} = \dfrac{\partial x_i}{\partial \xi_h} \right)$, and let $\lambda_1^2 \geqslant \dots \geqslant \lambda_n^2$ be the proper values of J^*J. The diffeomorphism $X(\xi)$ is ([5] pag. 48) α-quasiconformal, (abbreviated α-qc), if the double inequality:

$$\alpha^{-1}\lambda_1^n \leqslant (\lambda_1 \dots \lambda_n) \leqslant \alpha\lambda_n^n,$$

holds for every $\xi \in D$.

Let r and q be integers, with $1 \leqslant r \leqslant (n-1)$, $2 \leqslant q \leqslant (n-r+1)$, and let $\alpha \geqslant 1$ be a constant.

Definition 1. *A diffeomorphism $X(\xi)$ of a domain D is said to be $(\alpha : r, q)$-pseudoconformal, (abbreviated $(\alpha : r, q)$-pc), if the inequality:*

$$\lambda_r^q \leqslant \alpha(\lambda_r \dots \lambda_{r+q-1})$$

holds for every $\xi \in D$. $X(\xi)$ is (r, q)-pc if it is $(\alpha : r, q)$-pc for some α finite. Equivalently, a diffeomorphism $X(\xi)$ is (r, q)-pc if it is

$$\binom{n}{r+q-1}\binom{n}{r-1}^{q-1} S_r^q \leqslant \alpha^q \binom{n}{r}^q S_{r+q-1} S_{r-1}^{q-1}$$

where $S_p = \sum\limits_{1 \leqslant i_1 < \dots < i_p \leqslant n} \lambda_{i_1} \dots \lambda_{i_p}$, $S_0 = 1$.

The proof of this statement is given in Appendix.

We remark that for $r = 1$ we obtain the classes $K_{pq}(\alpha)$ introduced by G. Cimmino in [2]. Obviously, if a diffeomorphism $X(\xi)$ is qc then it is (r, q)-pc for $1 \leqslant r \leqslant (n-1)$, $2 \leqslant q \leqslant (n-r+1)$, and if $X(\xi)$ is $(1, n)-pc$ then it is qc.

Let V_q be a C^1 q-dimensional manifold in R^n. If Γ is a curve family in V_q, we denote by $F(\Gamma)$ the family of all non-negative Borel functions $\rho : V_q \to \dot{R}^1$ such that $\int_\gamma \rho ds \geqslant 1$ for every locally rectifiable $\gamma \in \Gamma$. The modulus of Γ with respect to V_q is defined by ([5] pag. 28):

$$M_q^{V_q}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{V_q} \rho^q dV_q.$$

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