OPERATORI ŞI FUNCŢIONALE ÎN SPAŢII BANACH TOPOLOGICI INVARIANŢI LA ACŢIUNEA UNUI GRUP σ-COMPACT, MEDIABIL, UNIMODULOR

Rezumat

După ce se definesc noţiunile de operator şi de funcţională pe un spaţiu Banach $E(X)$ topologic invariante în raport cu un grup G mediabil şi σ-compact, se indică unele exemple remarcabile de asemenea operatori (a se vedea, spre exemplu, operatorii pe $P^p(X)$ $(1 \leqslant p \leqslant \infty)$, spaţiul funcţiilor aproape periodice în sens Chatard).

Prinicipalul rezultat al lucrării îl constituie teorema 3.1. de tip Converse-Namioka-Phelps, pentru genul de operatori considerat.

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PSEUDOCONFORMAL MAPPINGS IN SPACE (I)

BY

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1. Introduction and definitions. Suppose that $X = X(\xi)$ is a diffeomorphism of a domain $D \subset R^n (n \geqslant 2)$ onto a domain $D' \subset R^n$. For $\xi \in D$ let J be the jacobian matrix of $X(\xi)$, $\left(J_{ih} = \dfrac{\partial x_i}{\partial \xi_h} \right)$, and let $\lambda_1^2 \geqslant \dots \geqslant \lambda_n^2$ be the proper values of J^*J. The diffeomorphism $X(\xi)$ is ([5] pag. 48) α-quasiconformal, (abbreviated α-qc), if the double inequality:

$$\alpha^{-1}\lambda_1^n \leqslant (\lambda_1 \dots \lambda_n) \leqslant \alpha\lambda_n^n,$$

holds for every $\xi \in D$.

Let r and q be integers, with $1 \leqslant r \leqslant (n-1)$, $2 \leqslant q \leqslant (n-r+1)$, and let $\alpha \geqslant 1$ be a constant.

Definition 1. *A diffeomorphism $X(\xi)$ of a domain D is said to be $(\alpha : r, q)$-pseudoconformal, (abbreviated $(\alpha : r, q)$-pc), if the inequality:*

$$\lambda_r^q \leqslant \alpha(\lambda_r \dots \lambda_{r+q-1})$$

holds for every $\xi \in D$. $X(\xi)$ is (r, q)-pc if it is $(\alpha : r, q)$-pc for some α finite. Equivalently, a diffeomorphism $X(\xi)$ is (r, q)-pc if it is

$$\binom{n}{r+q-1}\binom{n}{r-1}^{q-1} S_r^q \leqslant \alpha^q \binom{n}{r}^q S_{r+q-1} S_{r-1}^{q-1}$$

where $S_p = \sum\limits_{1 \leqslant i_1 < \dots < i_p \leqslant n} \lambda_{i_1} \dots \lambda_{i_p}$, $S_0 = 1$.

The proof of this statement is given in Appendix.

We remark that for $r = 1$ we obtain the classes $K_{pq}(\alpha)$ introduced by G. Cimmino in [2]. Obviously, if a diffeomorphism $X(\xi)$ is qc then it is (r, q)-pc for $1 \leqslant r \leqslant (n-1)$, $2 \leqslant q \leqslant (n-r+1)$, and if $X(\xi)$ is $(1, n)$-pc then it is qc.

Let V_q be a C^1 q-dimensional manifold in R^n. If Γ is a curve family in V_q, we denote by $F(\Gamma)$ the family of all non-negative Borel functions $\rho : V_q \to \dot{R}^1$ such that $\int_\gamma \rho \, ds \geqslant 1$ for every locally rectifiable $\gamma \in \Gamma$. The modulus of Γ with respect to V_q is defined by ([5] pag. 28):

$$M_q^{V_q}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{V_q} \rho^q \, dV_q.$$

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If $X(\xi)$ is our diffeomorphism of D onto D', for $\xi \in D$ let $X'(\xi)$ be the derivative of $X(\xi)$ at ξ. The image of the unit ball B^n under the linear mapping $X'(\xi)$ is an ellipsoid $E(\xi)$ with semi-axes $\lambda_1 \geq \ldots \geq \lambda_n$.

Definition 2. *The C^1 q-dimensional manifold $V_q \subset D$ is said to belong to the family $N(r, q)$, if $V'_q = X(V_q)$ is, at every point, orthogonal to the semiaxes $\lambda_1, \ldots, \lambda_{r-1}, \lambda_{r+q}, \ldots, \lambda_n$ of $E(\xi)$.*

In this work we show that if $X(\xi)$ is $(\alpha; r, q)-pc$ and if $V_q \in N(r, q)$ we have:

$$\sup_{\Gamma \subset V_q} \left[M_q^{V_q}(\Gamma) / M_q^{V'_q}(\Gamma') \right] \leq \alpha$$

where the supremum is taken over all path families Γ in V_q such that $M_q^{V_q}(\Gamma)$ and $M_q^{V'_q}(\Gamma')$, $(\Gamma' = X(\Gamma))$, are not simultaneously 0 or ∞. Moreover, if every $\xi \in D$ is contained in a C^1 q-dimensional manifold V_q such that $V_q \in N(r, q)$ and if it is

$$\sup_{V_q \in N(r,q)} \left\{ \sup_{\Gamma \subset V_q} [M_q^{V_q}(\Gamma) / M_q^{V'_q}(\Gamma')] \right\} = \alpha,$$

then, $X(\xi)$ is $(\alpha; r, q)-pc$.

Remark. Several of the following results are analogous to those obtained in [5].

2. Preliminary lemmas. Lemma 1. *Let G be a Borel set in a C^1 q-dimensional manifold V_q. Suppose that the curves of a family Γ lie in G and that length of $\gamma \geq K > 0$ for every locally rectifiable $\gamma \in \Gamma$. Then we have:*

$$M_q^{V_q}(\Gamma) \leq m_q(G) K^{-q},$$

where $m_q(G)$ denotes q-dimensional measure of G.

Proof. Define $\rho: V_q \to R^1$ by $\rho(\xi) = K^{-1}$ for $\xi \in G$ and $\rho(\xi) = 0$ otherwise. Then $\rho \in F(\Gamma)$ and the inequality follows.

Lemma 2. *Let V_q be a C^1 q-dimensional manifold passing through the origin O and that in O it is orthogonal to the axes $\xi_1, \ldots, \xi_{r-1}, \xi_{r+q}, \ldots, \xi_n$. Let $I_q = V_q \cap I$ where $I = \{\xi \in R^n, -\delta/2 < \xi_i < \delta/2;\ i = 1, \ldots, n\}$ with δ so small that the border of I_q lies on the hyperplanes $\xi_i = \pm\delta/2$, $i = r, \ldots, (r+q-1)$. Let E and F be the parts of the border of I_q that lie respectively on $\xi_r = -\delta/2$ and $\xi_r = +\delta/2$. Let $\Gamma = \Delta(E, F, I_q)$, i. e. the family of all closed paths which join E and F in I_q. Then*

$$(1 + \varphi(\delta)) M_q^{V_q}(\Gamma) \geq 1, \qquad \lim_{\delta \to 0} \varphi(\delta) = 0.$$

Proof. For each $\zeta = (\zeta_1, \ldots, \zeta_n) \in E$ let γ_ζ be the curve obtained by intersection of I_q with the hyperplanes $\xi_i = \zeta_i$, $i = (r+1), \ldots, (r+q-1)$. Then $\gamma_\zeta \in \Gamma$. If $\rho \in F(\Gamma)$, we obtain by Hölder's inequality:

$$1 \leq \left(\int_{\gamma_\zeta} \rho\, ds \right)^q \leq S(\zeta)^{q-1} \int_{\gamma_\zeta} \rho^q ds \leq S_{\sup}^{q-1} \int_{\gamma_\zeta} \rho^q\, ds,$$

where $S(\zeta)$ denotes the lenght of γ_ζ and $S_{\sup} = \sup_{\zeta \in E} S(\zeta)$. Integration over $\zeta \in E$ yields, by Fubini's theorem

$$\delta^{q-1} \leq S_{\sup}^{q-1} \int_{I_q} \rho^q\, dI_q.$$

Since this holds for every $\rho \in F(\Gamma)$, we obtain

$$M_q^{I_q}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{I_q} \rho^q dI_q \geq \frac{\delta^{q-1}}{S_{\sup}^{q-1}} = \frac{1}{1 + \varphi(\delta)}$$

with $\lim_{\delta \to 0} \varphi(\delta) = 0$.

Lemma 3. *Suppose that $X = X(\xi)$ is a diffeomorphism of $D \subset R^n$ onto $D' \subset R^n$, and $\lambda_1^2 \geq \ldots \geq \lambda_n^2$ are the numbers defined in the Introduction. If $V_q \subset D$ is a C^1 q-dimensional manifold and $\rho(x)$ is a non-negative Borel function $\rho: V'_q \to R^1$ we have:*

$$\int_{V_q} \rho[x(\xi)]\, (\lambda_{n-q+1} \ldots \lambda_n)\, dV_q \leq \int_{V'_q} \rho(x) dV'_q \leq \int_{V_q} \rho[x(\xi)] (\lambda_1 \ldots \lambda_q)\, dV_q.$$

Proof. Let

$$\overset{i}{\xi} = \overset{i}{\xi}(u_1, \ldots, u_q), \qquad i = 1, \ldots, n, \qquad (u_1, \ldots, u_q) \in O,$$

be the equations of V_q with $\overset{i}{\xi}(u_1, \ldots, u_q) \in C^1$, and let

$$\overset{i}{x} = \overset{i}{x}\{\overset{1}{\xi}(u_1, \ldots, u_q), \ldots, \overset{n}{\xi}(u_1, \ldots, u_q)\}, \qquad i = 1, \ldots, n,$$

be the corresponding equations of V'. Then we have:

$$dV'_q = \left| \begin{pmatrix} \overset{1}{x}_{u_1} & \cdots & \overset{n}{x}_{u_1} \\ \cdots & \cdots & \cdots \\ \overset{1}{x}_{u_q} & \cdots & \overset{n}{x}_{u_q} \end{pmatrix}^2 \right|^{\frac{1}{2}} du_1 \ldots du_q = \left| \begin{matrix} \overset{i}{x}_{u_1}\overset{i}{x}_{u_1} & \cdots & \overset{i}{x}_{u_1}\overset{i}{x}_{u_q} \\ \cdots & \cdots & \cdots \\ \overset{i}{x}_{u_q}\overset{i}{x}_{u_1} & \cdots & \overset{i}{x}_{u_q}\overset{i}{x}_{u_q} \end{matrix} \right|^{\frac{1}{2}} du_1 \ldots du_q =$$

$$= \left| \begin{matrix} \overset{i}{x}_{\overset{h}{\xi}} \overset{h}{\xi}_{u_1} \overset{i}{x}_{\overset{k}{\xi}} \overset{k}{\xi}_{u_1} & \cdots & \overset{i}{x}_{\overset{h}{\xi}} \overset{h}{\xi}_{u_1} \overset{i}{x}_{\overset{k}{\xi}} \overset{k}{\xi}_{u_q} \\ \cdots & \cdots & \cdots \\ \overset{i}{x}_{\overset{h}{\xi}} \overset{h}{\xi}_{u_q} \overset{i}{x}_{\overset{k}{\xi}} \overset{k}{\xi}_{u_1} & \cdots & \overset{i}{x}_{\overset{h}{\xi}} \overset{h}{\xi}_{u_q} \overset{i}{x}_{\overset{k}{\xi}} \overset{k}{\xi}_{u_q} \end{matrix} \right|^{\frac{1}{2}} du_1 \ldots du_q,$$

where square of matrix is made by rows. Let ξ_j be a vector whose components are $\overset{h}{\xi}_j = \frac{\partial \xi^h}{\partial u_j}$ $(h = 1, \ldots, n)$. Then we have:

$$dV'_q = \left| \begin{matrix} \xi_1^* J^* J \xi_1 & \cdots & \xi_1^* J^* J \xi_q \\ \cdots & \cdots & \cdots \\ \xi_q^* J^* J \xi_1 & \cdots & \xi_q^* J^* J \xi_q \end{matrix} \right| du_1 \ldots du_q = \left| \begin{matrix} \bar{\zeta}_1^* \Lambda \bar{\zeta}_1 & \cdots & \bar{\zeta}_1^* \Lambda \bar{\zeta}_q \\ \cdots & \cdots & \cdots \\ \bar{\zeta}_q^* \Lambda \bar{\zeta}_1 & \cdots & \bar{\zeta}_q^* \Lambda \bar{\zeta}_q \end{matrix} \right| du_1 \ldots du_q,$$

where $\Lambda = \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2)$, $\bar{\zeta}_j = C\xi_j$; with $J^*J = C^*\Lambda C$. If ζ_j^h are the components of the vector ζ_j we have:

$$dV'_q = \left| \begin{pmatrix} \overset{1}{\zeta}_1 & \dots & \overset{n}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{1}{\zeta}_q & \dots & \overset{n}{\zeta}_q \end{pmatrix} \begin{pmatrix} \lambda_1^2 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n^2 \end{pmatrix} \begin{pmatrix} \zeta_1^1 & \dots & \zeta_q^1 \\ \dots & \dots & \dots \\ \overset{n}{\zeta}_1 & \dots & \overset{n}{\zeta}_q \end{pmatrix} \right|^{\frac{1}{2}} du_1 \dots du_q =$$

$$= \left(\sum_{1 \leqslant i_1 < \dots < i_q < n} \lambda_{i_1}^2 \dots \lambda_{i_q}^2 \begin{vmatrix} \overset{i_1}{\zeta}_1 & \dots & \overset{i_q}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{i_1}{\zeta}_q & \dots & \overset{i_q}{\zeta}_q \end{vmatrix}^2 \right)^{\frac{1}{2}},$$

that is

$$dV'_q \geqslant (\lambda_{n-q+1} \dots \lambda_n) \left| \begin{pmatrix} \overset{1}{\zeta}_1 & \dots & \overset{n}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{1}{\zeta}_q & \dots & \overset{n}{\zeta}_q \end{pmatrix}^2 \right|^{\frac{1}{2}} du_1 \dots du_q, \quad dV'_q \leqslant (\lambda_1 \dots \lambda_q) \left| \begin{pmatrix} \overset{1}{\zeta}_1 & \dots & \overset{n}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{1}{\zeta}_q & \dots & \overset{n}{\zeta}_q \end{pmatrix}^2 \right|^{\frac{1}{2}} du_1 \dots du_q.$$

On the other hand.

$$dV_q = \left| \begin{pmatrix} \overset{1}{\xi}_1 & \dots & \overset{n}{\xi}_1 \\ \dots & \dots & \dots \\ \overset{1}{\xi}_q & \dots & \overset{n}{\xi}_q \end{pmatrix}^2 \right|^{\frac{1}{2}} du_1 \dots du_q =$$

$$= \left| \left(\begin{pmatrix} \overset{1}{\zeta}_1 & \dots & \overset{n}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{1}{\zeta}_q & \dots & \overset{n}{\zeta}_q \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \dots & \dots & \dots \\ C_{n1} & \dots & C_{nn} \end{pmatrix} \right)^2 \right|^{1/2} du_1 \dots du_q = \left| \begin{pmatrix} \overset{1}{\zeta}_1 & \dots & \overset{n}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{1}{\zeta}_q & \dots & \overset{n}{\zeta}_q \end{pmatrix}^2 \right|^{1/2} du_1 \dots du_q ,$$

whence

$$(\lambda_{n-q+1} \dots \lambda_n) dV_q \leqslant dV'_q \leqslant (\lambda_1 \dots \lambda_q) dV_q ,$$

and the lemma follows.

Lemma 4. *If the hypotheses of lemma 3 are verified and if $V_q \in N(r, q)$, we have*

$$\int_{V'_q} \rho(x)\, dV'_q = \int_{V_q} \rho[X(\xi)]\, (\lambda_r \dots \lambda_{r+q-1}) dV_q .$$

Proof. Under these hypotheses, the terms

$$\begin{vmatrix} \overset{i_1}{\zeta}_1 & \dots & \overset{i_q}{\zeta}_1 \\ \dots & \dots & \dots \\ \overset{i_1}{\zeta}_q & \dots & \overset{i_q}{\zeta}_q \end{vmatrix}$$

that appear in the proof of Lemma 3 vanish except those corresponding to $(i_1, \dots, i_q) = (r, \dots, (r+q-1))$.

Lemma 5. *Suppose that $X = X(\xi)$ is a diffeomorphism of $D \subset R^n$ onto $D' \subset R^n$, and that $\lambda_1 \geqslant \dots \geqslant \lambda_n$, are the semiaxes of the ellipsoid $E(\xi)$ defined in the Introduction. Let $V_q \subset D$ be a C^1 q-dimensional manifold such that $V'_q = X(V_q)$ is at every point, orthogonal to the semiaxes $\lambda_1, \dots, \lambda_{r-1}$, where $r \geqslant 2$. Then, if γ is a rectifiable curve in V_q, and if $\rho(x)$ is a non-negative Borel function $\rho : V'_q \to R^1$ we have :*

$$\int_{X(\gamma)} \rho(x) ds' \leqslant \int_{\gamma} \rho[X(\xi)]\, \lambda_r\, ds .$$

Proof. This lemma follows from theorem 5.3 of [5] pag. 12 if we observe that, when $\xi \in V_q$ and $(\xi + h) \in V_q$

$$L(\xi, X) = \limsup_{h \to 0} \frac{|X(\xi + h) - X(\xi)|}{|h|} = \lambda_r .$$

3. Main results Theorem 1. *Let $X = X(\xi)$ be a $(\alpha : r, q)$ — pc diffeomorphism of $D \subset R^n$, onto $D' \subset R^n$, and let $N(r, q)$ be the family introduced by Definition 2. Then we have :*

$$\sup_{V_q \in N(r,q)} \left\{ \sup_{\Gamma \subset V_q} \left[M_q^{V_q}(\Gamma) / M_q^{V'_q}(\Gamma') \right] \right\} \leqslant \alpha$$

where the curve families Γ are such that $M_q^{V_q}(\Gamma)$ and $M_q^{V'_q}(\Gamma')$, are not simultaneously 0 or ∞.

Proof. We have to prove that $M_q^{V_q}(\Gamma) \leqslant \alpha M_q^{V'_q}(\Gamma')$ for an arbitrary curve family Γ in an arbitrary C^1 q-dimensional manifold V_q (in D) belonging to $N(r, q)$. Let $\rho \in F(\Gamma')$. Define $\tilde{\rho} : V_q \to R^1$ by $\tilde{\rho}(\xi) = \rho(X(\xi))\lambda_r$, where $\lambda_1^2 \geqslant \dots \geqslant \lambda_n^2$ are the proper values of $J^* J$. If γ is a locally rectifiable curve in Γ, for Lemma 5 we obtain :

$$\int_{\gamma} \rho[X(\xi)] \lambda_r\, ds \geqslant \int_{X(\gamma)} \rho(x) \cdot ds' \geqslant 1 .$$

Hence $\tilde{\rho} \in F(\Gamma)$, which implies, with Lemma 4 :

$$M_q^{V_q}(\Gamma) \leqslant \int_{V_q} \tilde{\rho}^q dV_q = \int_{V_q} \rho[X(\xi)]^q \lambda_r^q dV_q \leqslant \alpha \int_{V_q} \rho[X(\xi)]^q (\lambda_r \dots \lambda_{r+q-1}) dV_q = \alpha \int_{V'_q} \rho(x)^q dV'_q .$$

Since this holds for every $\rho \in F(\Gamma')$, we obtain $M_q^{V_q}(\Gamma) \leqslant \alpha M_q^{V'_q}(\Gamma')$.

Theorem 2. *Let $X = X(\xi)$ be a diffeomorphism of $D \subset R^n$ onto $D' \subset R^n$, and let $\lambda_1^2 \geqslant \dots \geqslant \lambda_n^2$ be the proper values of $J^* J$ for $\xi \in D$. If V_q is a C^1 q-dimensional manifold belonging to $N(r, q)$ we have*

$$\sup_{\xi \in V_q} [\lambda_r^q / (\lambda_r \dots \lambda_{r+q-1})] \leqslant \sup_{\Gamma \subset V_q} \left[M_q^{V_q}(\Gamma) / M_q^{V'_q}(\Gamma') \right] .$$

Proof. Let $\sup_{\Gamma \subset V_q} [M_q^{V_q}(\Gamma) / M_q^{V'_q}(\Gamma')] = \beta$. *We may assume $\beta < +\infty$. We have to show that for $\bar{\xi} \in V_q$:*

$$\lambda_r^q \leqslant \beta (\lambda_r \dots \lambda_{r+q-1}) .$$

By performing a preliminary similarity transformation, we may assume that $\bar{\xi} = 0 = X(\bar{\xi})$, and that $X'(0) = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$. Let $0 < \varepsilon < \lambda_r$ and let I_q be defined as in Lemma 2 with δ so small that $\| X(\xi) - X(0) \| \leqslant \varepsilon\delta$ for $\xi \in I_q$, where $\| . \|$ denotes euclidean norm. Let E and F be the faces $\xi_r = -\delta/2$ and $\xi_r = +\delta/2$ of I_q, respectively. Set $\Gamma = \Delta(E, F, I_q)$. By Lemma 2)

$M_q^{I_q}(\Gamma)(1+\varphi(\delta))\geqslant 1$, with $\lim\limits_{\delta\to 0}\varphi(\delta)=0$. We next estimate $M(\Gamma')$. Since $X(E)$ lies between the hyperplanes $\xi_r=-(\lambda_r\pm\varepsilon)\delta/2$ and since $X(F)$ lies between the hyperplanes $\xi_r=+(\lambda_r\pm\varepsilon)\delta/2$ then $l(\gamma)\geqslant(\lambda_r-\varepsilon)\delta/2$ for every $\gamma\in\Gamma'$. On the other hand, $I'_q=X(I_q)$ is contained in the strip $\{X\in R^n \mid -(\lambda_r+\varepsilon)\,\delta/2\leqslant x_i\leqslant(\lambda_r+\varepsilon)\,\delta/2,\ i=r,\dots,(r+q-1)\}$, and

$$m_q(I'_q)\leqslant(\lambda_r+\varepsilon)\delta\dots(\lambda_{r+q-1}+\varepsilon)\delta(1+\psi(\delta)),$$

where m_q denotes q-dimensional measure and $\lim\limits_{\delta\to 0}\psi(\delta)=0$. Hence Lemma 1 implies:

$$M_q^{I'_q}(\Gamma')\leqslant(\lambda_r+\varepsilon)\dots(\lambda_{r+q-1}+\varepsilon)\,(1+\psi(\delta))\,(\lambda_r-\varepsilon)^{-q},$$

hence,

$$(\lambda_r-\varepsilon)^q\leqslant(1+\varphi(\delta))\,(\lambda_r+\varepsilon)\dots(\lambda_{r+q-1}+\varepsilon)\,(1+\psi(\delta))M_q^{I_q}(\Gamma)/M_q^{I'_q}(\Gamma')\leqslant$$
$$\leqslant(1+\varphi(\delta))\,(\lambda_r+\varepsilon)\dots(\lambda_{r+q-1}+\varepsilon)\,(1+\psi(\delta))\,\beta.$$

Since ε and δ are arbitrary, the theorem follows.

Corollary 1. *Let $X=X(\xi)$ be a diffeomorphism of $D\subset R^n$ onto $D'\subset R^n$. If every $\xi\in D$ is contained in a C^1 q-dimensional manifold V_q such that $V_q\in N(r,q)$ and if it is*

$$\sup_{V_q\in N(r,q)}\left\{\sup_{\Gamma\subset V_q}\left[M_q^{V_q}(\Gamma)/M_q^{V'_q}(\Gamma')\right]\right\}=\alpha,$$

then $X(\xi)$ is $(\alpha;\ r,\ q)-pc$.

Proof. This result follows immediately from Theorem 2.

Corollary 2. *Let $X=X(\xi)$ be a conformal diffeomorphism of $D\subset R^n$ onto $D'\subset R^n$, and let V_q be a C^1 q-dimensional manifold in D. Then*

$$M_q^{V_q}(\Gamma)=M_q^{V'_q}(\Gamma').$$

Proof. As $\lambda_1=\dots=\lambda_n$, $X(\xi)$ is $(1,\ 1,\ q)-pc$ and $V_q\in N(1,\ q)$. By Theorem 1, $M_q^{V_q}(\Gamma)\leqslant\leqslant M_q^{V'_q}(\Gamma')$. Since the inverse diffeomorphism is conformal too, the reverse inequality is also true.

Corollary 3. *Suppose $X=X(\xi)$ be a diffeomorphism of $D\subset R^n$ onto $D'\subset R^n$. If for every $\xi\in D$ there exist a C^1 q-dimensional manifold V_q and a C^1 q'-dimensional manifold $V_{q'}$ such that $\xi\in V_q\cap V_{q'}$, $V_q\in N(1,\ q)$, $V_{q'}\in N(q,\ q')$, $q+q'=(n+1)$, and if it is*

$$\sup_{V_q\in N(1,q)}\left\{\sup_{\Gamma\subset V_q}\left[M_q^{V_q}(\Gamma)/M_q^{V'_q}(\Gamma')\right]\right\}\leqslant\alpha,\quad \sup_{V_{q'}\in N(q,\,q')}\left\{\sup_{\Gamma\subset V_{q'}}\left[M_{q'}^{V_{q'}}(\Gamma)/M_{q'}^{V'_{q'}}(\Gamma')\right]\right\}\leqslant\alpha,$$ then

we have

$$M(\Gamma)\leqslant\alpha^{2n}M(\Gamma')$$

for every curve family $\Gamma\subset D$.

Proof. By Corollary 1, $X(\xi)$ is $(\alpha;\ 1,\ q)-pc$ and $(\alpha;\ q,\ q')-pc$, hence:

$$\begin{cases}\lambda_1^q\leqslant\alpha(\lambda_1\dots\lambda_q)\\ \lambda_q^{q'}\leqslant\alpha(\lambda_q\dots\lambda_{q+q'-1})\end{cases}\Rightarrow\begin{cases}\lambda_1\leqslant\alpha\lambda_q\\ \lambda_q\leqslant\alpha\lambda_n\end{cases}\Rightarrow\ \lambda_1\leqslant\alpha^2\lambda_n,$$

whence

$$\alpha^{-2n}\lambda_1^n\leqslant(\lambda_1\dots\lambda_n)\leqslant\alpha^{2n}\lambda_n^n,$$

therefore $X(\xi)$ is $\alpha^{2n}-qc$ and the corollary follows.

4. Appendix. With the notations introduced in the Introduction, we show that if it is

$$\lambda_r^q\leqslant\alpha(\lambda_r\dots\lambda_{r+q-1}),$$

then, it is also

$$\binom{n}{r+q-1}\binom{n}{r-1}^{q-1}S_r^q\leqslant\bar{\alpha}^q\binom{n}{r}^q S_{r+q-1}S_{r-1}^{q-1},$$

and if it is

$$\binom{n}{r+q-1}\binom{n}{r-1}^{q-1}S_r^q\leqslant\bar{\alpha}^q\binom{n}{r}^q S_{r+q-1}S_{r-1}^{q-1},$$

then it is also

$$\lambda_r^q\leqslant\bar{\bar{\alpha}}(\lambda_r\dots\lambda_{r+q-1}).$$

For $r=1$, this proof is given in [4]. For $r\geqslant 2$ we have:

$$\lambda_r^q\leqslant\alpha(\lambda_r\dots\lambda_{r+q-1})\Rightarrow(\lambda_1\dots\lambda_r)^q\leqslant\alpha(\lambda_1\dots\lambda_{r-1})^q(\lambda_r\dots\lambda_{r+q-1})\Rightarrow$$

$$\Rightarrow S_r^q\binom{n}{r}^{-q}\leqslant\alpha S_{r-1}^{q-1}(\lambda_1\dots\lambda_{r+q-1})\Rightarrow\binom{n}{r+q-1}\binom{n}{r-1}^{q-1}S_r^q\leqslant$$

$$\leqslant\alpha\binom{n}{r+q-1}\binom{n}{r-1}^{q-1}\binom{n}{r}S_{r-1}^{q-1}S_{r+q-1}.$$

Moreover we have

$$\binom{n}{r+q-1}\binom{n}{r-1}^{q-1}S_r^q\leqslant\bar{\alpha}^q\binom{n}{r}^q S_{r+q-1}S_{r-1}^{q-1}\leqslant(\lambda_1^q\dots\lambda_r^q)\leqslant$$

$$\leqslant\bar{\alpha}^q\binom{n}{r}^q(\lambda_1\dots\lambda_{r+q-1}\lambda_1^{q-1}\dots\lambda_{r-1}^{q-1})\Rightarrow(\lambda_1\dots\lambda_{r-1}\cdot\lambda_r^q)\leqslant\bar{\alpha}^q\binom{n}{r}^q(\lambda_1\dots\lambda_{r-1}\lambda_r\dots$$

$$\dots\lambda_{r+q-1})\Rightarrow\lambda_r^q\leqslant\bar{\alpha}^q\binom{n}{r}^q(\lambda_r\dots\lambda_{r+q-1}).$$

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REPREZENTĂRI PSEUDOCONFORME ÎN SPAȚIU

Rezumat

Una din definițiile difeomorfismelor evasiconforme $X(\xi)$ ale unui domeniu $D\subset R^n$ ($n\geqq 2$) pe un domeniu $D'\subset R^n$ este următoarea: Dacă $\lambda_1\geqq\dots\geqq\lambda_n$ sînt autovalorile lui J^2 (unde J e matricea jacobiană a lui X), atunci X se numește K-evasiconform in D dacă are loc dubla inegalitate

$$\frac{\lambda_1^{\frac{n}{2}}}{K} \leqq (\lambda_1 \ldots \lambda_n)^{\frac{1}{2}} \leqq K\lambda_n^{\frac{n}{2}}$$

pentru orice $\xi \in D$. Un difeomorfism $X(\xi)$ al unui domeniu D se numește $(\alpha : q, r)$-pseudoconform $(1 \leqq \alpha < \infty)$ dacă are loc inegalitatea

$$\lambda_r^{\frac{q}{2}} \leqq \alpha(\lambda_r \ldots \lambda_{r+q-1})^{\frac{1}{2}}$$

pentru orice $\xi \in D$, iar r, q sînt întregi verificînd condițiile $1 \leqslant r \leqslant n-1$ și $2 \leqslant q \leqslant n-r+1$. Evident, un difeomorfism K-cvasiconform este (α, q, r)-pseudoconform cu $\alpha = K$. Această nouă clasă de difeomorfisme generalizează pe cea a lui G. Cimmino [1] (ce se obține pentru $r=1$) și pe cea definită de autor [4] (și care se obține pentru $r=n-q+1$).

Ca urmare a rezultatelor principale ale lucrării (teoremele 1 și 2), autorul obține criterii de $(\alpha ; q, r)$-pseudoconformitate exprimate cu ajutorul deformării modulului $M_q^{V_q}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{V_q} \rho^q \, dV_q$ a familiei de curbe local rectificabile conținute într-o varietate q-dimensională V_q din R^n.

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NONLINEAR VARIATION OF CONSTANTS FORMULA AND A FUNCTIONAL INEQUALITY

BY

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1. Introduction. Bellman-Gronwall-Reid inequalities have been generalized by many authors. In this paper, we wish to give a more general result, which includes many recent works [1, 2, 3] as particular cases. We develop a nonlinear variation of constants formula for a vector differential equation of a specific form. Utilizing this formula, we show that it is possible to obtain various bounds on the solutions under a variety of conditions. In some situations, this variation of constants formula reduces to a general type of Bellman-Gronwall-Reid inequality. With this as motivation we discuss a rather general functional vectorial inequality which includes several known results as special cases [1, 2, 3].

Vectorial inequalities will be used freely with the understanding that the same inequality holds component-wise.

2. Variation of Constants Formula. In this section we obtain a nonlinear variation of constants formula for a vector differential equation

$$u' = A(t)\, g(u) + R(t,u), \qquad u(t_0) = v_0 \geqslant 0 \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix, $A = [a_{ij}]$, $a_{ij} = 0$ for $i \neq j$, $a_{ii} = a_i$, $a_i \in C[I, R]$, $g = \prod_{i=1}^{n} g_i$, $g(u)$ is a column vector $g_i \in C[R_+, R_+]$, $g_i(u_i) > 0$ for $u_i > 0$, $g_i(0) = 0$, $R(t, u)$ is a column vector where $R \in C[I \times R_+^n, R_+^n]$, $I = [t_0, T_0]$, $t_0 \geqslant 0$. The following theorem gives a variation of constants formula for the vector differential equation (2.1) and the bounds on the solution $v(t)$ assuming various conditions on R, a and g.

Theorem 2.1. *Let A and g be as defined earlier. Then a solution $v(t)$ of (2.1) can be exhibited by*

$$v(t) = G^{-1}\left[\int_{t_0}^{t} a(s)\, ds + G\left(v_0 + \int_{t_0}^{t} \Phi^{-1}\left\{G^{-1}(G(v(s)) - \int_{t_0}^{s} a(\xi)\, d\xi)\right\} . \Phi(v(s)) . R(s, v(s))\, ds\right)\right] \tag{2.2}$$

Proof. It is known that a solution $v(t)$ of

$$v'(t) = A(t)\, g(v), \quad v(t_0) = v_0 \geqslant 0 \tag{2.3}$$

can be expressed as

$$G(v(t)) = \int_{t_0}^{t} a(s)\, ds + G(v_0), \quad t \in I_0 \tag{2.4}$$

where $G = \prod_{i=1}^{n} G_i$ and

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