$$\frac{\lambda_1^{\frac{n}{2}}}{K} \leqq (\lambda_1 \dots \lambda_n)^{\frac{1}{2}} \leqq K\lambda_n^{\frac{n}{2}}$$

pentru orice $\xi \in D$. Un difeomorfism $X(\xi)$ al unui domeniu D se numește $(\alpha; q, r)$-pseudoconform $(1 \leqq \alpha < \infty)$ dacă are loc inegalitatea

$$\lambda_r^{\frac{q}{2}} \leqq \alpha(\lambda_r \dots \lambda_{r+q-1})^{\frac{1}{2}}$$

pentru orice $\xi \in D$, iar r, q sint intregi verificind condițiile $1 \leqslant r \leqslant n-1$ și $2 \leqslant q \leqslant n-r+1$. Evident, un difeomorfism K-cvasiconform este (α, q, r)-pseudoconform cu $\alpha = K$. Această nouă clasă de difeomorfisme generalizează pe cea a lui G. Cimmino [1] (ce se obține pentru $r=1$) și pe cea definită de autor [4] (și care se obține pentru $r=n-q+1$).

Ca urmare a rezultatelor principale ale lucrării (teoremele 1 și 2), autorul obține criterii de $(\alpha; q, r)$-pseudoconformitate exprimate cu ajutorul deformării modulului $M_q^{V_q}(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{V_q} \rho^q \, dV_q$ a familiei de curbe local rectificabile conținute intr-o varietate q-dimensională V_q din R^n.

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NONLINEAR VARIATION OF CONSTANTS FORMULA AND A FUNCTIONAL INEQUALITY

BY

P. S. KAMALA *

1. Introduction. Bellman-Gronwall-Reid inequalities have been generalized by many authors. In this paper, we wish to give a more general result, which includes many recent works [1, 2, 3] as particular cases. We develop a nonlinear variation of constants formula for a vector differential equation of a specific form. Utilizing this formula, we show that it is possible to obtain various bounds on the solutions under a variety of conditions. In some situations, this variation of constants formula reduces to a general type of Bellman-Gronwall-Reid inequality. With this as motivation we discuss a rather general functional vectorial inequality which includes several known results as special cases [1, 2, 3].

Vectorial inequalities will be used freely with the understanding that the same inequality holds component-wise.

2. Variation of Constants Formula. In this section we obtain a nonlinear variation of constants formula for a vector differential equation

$$u' = A(t)\, g(u) + R(t,u), \qquad u(t_0) = v_0 \geqslant 0 \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix, $A = [a_{ij}]$, $a_{ij} = 0$ for $i \neq j$, $a_{ii} = a_i$, $a_i \in C[I, R]$, $g = \prod\limits_{i=1}^{n} g_i$, $g(u)$ is a column vector $g_i \in C[R_+, R_+]$, $g_i(u_i) > 0$ for $u_i > 0$, $g_i(0) = 0$, $R(t, u)$ is a column vector where $R \in C[I \times R_+^n, R_+^n]$, $I = [t_0, T_0]$, $t_0 \geqslant 0$. The following theorem gives a variation of constants formula for the vector differential equation (2.1) and the bounds on the solution $v(t)$ assuming various conditions on R, a and g.

Theorem 2.1. *Let A and g be as defined earlier. Then a solution $v(t)$ of (2.1) can be exhibited by*

$$v(t) = G^{-1}\left[\int_{t_0}^{t} a(s)\,ds + G\left(v_0 + \int_{t_0}^{t} \Phi^{-1}\left\{G^{-1}(G(v(s)) - \int_{t_0}^{s} a(\xi)\,d\xi\right\} . \Phi(v(s)) . R(s, v(s))\,ds\right)\right] \tag{2.2}$$

Proof. It is known that a solution $v(t)$ of

$$v'(t) = A(t)\, g(v), \quad v(t_0) = v_0 \geqslant 0 \tag{2.3}$$

can be expressed as

$$G(v(t)) = \int_{t_0}^{t} a(s)\,ds + G(v_0), \quad t \in I_0 \tag{2.4}$$

where $G = \prod\limits_{i=1}^{n} G_i$ and

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$$G_i(u_i)=\int_{u_{i0}}^{u_i}\frac{ds}{g_i(s)},\quad u_i,\ u_{i0}>0,$$

$$I_0=\left[t\in I: G(v_0)+\int_{t_0}^{t} a(s)\,ds\in \mathrm{Dom}(G^{-1})\right].$$

We now apply variation of constants method and determine v_0 as a function of t such that (vt) will be a solution of (2.1). There results the equation

$$\frac{\partial G}{\partial u}(v(t)).v'(t)=a(t)+\frac{\partial G}{\partial u}(v_0).v_0'. \tag{2.5}$$

on view of the definition of G and (2.1), (2.5) can be reduced to

$$\frac{\partial G}{\partial u}(v(t)).R(t,\ v)=\frac{\partial G}{\partial u}(v_0).v_0'. \tag{2.6}$$

Consequently

$$v_0'=\Phi^{-1}(V_0).\Phi(v(t)).R(t,\ v(t)), \tag{2.7}$$

where $\Phi(v)$ is an $n\times n$ matrix, $\Phi(v)=[\Phi_{ij}(v)]$, $\Phi_{ij}(v)=0$ for $i\neq j$, $\Phi_{ij}(v)=1/g_i(v_i)$ for $i=j$. Using the equation (2.4) to eliminate $v(t)$, there results the differential equation

$$v_0'=\Phi^{-1}(v_0).\Phi\left(G^{-1}\left\{\int_{t_0}^{t} a(s)\,ds+G(v_0)\right\}\right).R\left(t,\ G^{-1}\left(G(v_0)+\int_{t_0}^{t} a(s)\,ds\right)\right). \tag{2.8}$$

Let $v_0(t)$ be the solution of (2.8) existing on I_0. Then (2.4) gives the integral equation satisfied by a solution $v(t)$ of (2.1) in the form, for $t\in I_0$,

$$v(t)=G^{-1}\left[\int_{t_0}^{t} a(s)\,ds+G\left(v_0+\int_{t_0}^{t}\Phi^{-1}\left\{G^{-1}\left(G(v(s))-\int_{t_0}^{s} a(\xi)\,d\xi\right)\right\}.\Phi(v(s)).R(s,\ v(s))\,ds\right)\right].$$

which proves the theorem.

Corollary 2.1. *Let the hypothesis of theorem 2.1 hold. Further, let $R(t,v)\geq 0$, $a(t)\geq 0$ and g be nondecreasing in v. Then the following estimate on $v(t)$ is valid:*

$$v(t)\leq G^{-1}\left[\int_{t_0}^{t} a(s)\,ds+G\left(v_0+\int_{t_0}^{t} R(s,\ v(s))\,ds\right)\right]. \tag{2.9}$$

Proof. Since $a\geq 0$, we have $v_0(t)\leq v(t)$. Using the hypothesis on g and the equality (2.2) we arrive at the estimate (2.9).

Corollary 2.2. *Assume that all the hypotheses of corollary 2.1 are satisfied except that the conditions on R and a are replaced by*

$$R(t,\ v)\leq 0,\qquad a\leq 0. \tag{2.10}$$

Then the conclusion of corollary 2.1 is true.

Proof: In view of the hypothesis on g, (2.2) and the fact that $R(t,\ v)\leq 0$, there results the inequality (2.9).

Corollary 2.3. *Let the assumptions of corollary 2.2 be satisfied except that (2.10) is replaced by*

(2.11 a) $\qquad (a)\,R(t,\ v)\geq 0$ and $a(t)\leq 0$,

or

(2.11 b) $\qquad (b)\,R(t,\ v)\leq 0$ and $a(t)\geq 0$.

Then the following bound for $v(t)$ is true:

$$v(t)\geq G^{-1}\left[\int_{t_0}^{t} a(s)\,ds+G\left(v_0+\int_{t_0}^{t} R(s,\ v(s))\,ds\right)\right]. \tag{2.12}$$

Proof. In the case when (2.11 a) is satisfied, using the hypotheses on g, R, a and the estimate (2.2) we get the bound (2.12). Similarly we arrive at the inequality (2.12) in the other case when (2.11 b) is satisfied.

We now consider once again the differential equation (2.8). Assume that $a\geq 0$, $g(u)$ is non decreasing in u. In view of these assumptions, (2.8) may be written as

$$v_0'\leq R\left(t,\ G^{-1}\left(G(v_0)+\int_{t_0}^{t} a(s)\,ds\right)\right)=W(t,\ v_0). \tag{2.13}$$

Setting $\tilde{w}=v_0$, (2.13) can be transformed into

$$\tilde{w}'\leq W(t,\ \tilde{w}) \tag{2.14}$$

with

$$\tilde{w}(t_0)=v_0(t_0). \tag{2.15}$$

We now work with the inequality (2.14) together with (2.15) and obtain an estimate for $v(t)$.

Corollary 2.4. *Let $W(t,\ \tilde{w})$ be quasi monotone nondecreasing in $\tilde{w}$ and assume that (2.14) and (2.15) hold.*

Then,

$$v(t)\leq G^{-1}\left[\int_{t_0}^{t} a(s)\,ds+G(r(t))\right], \tag{2.16}$$

where $r(t)$ is the maximal solution of

$$r'=W(t,\ r),\qquad r(t_0)=\tilde{w}(t_0). \tag{2.17}$$

Proof. From (2.14), (2.15) and by the application of theorem 4.1.1 in [3], there results

$$\tilde{w}(t)\leq r(t,\ t_0,\ \tilde{w}(t_0)) \tag{2.18}$$

where $r(t,\ t_0,\ \tilde{w}(t_0))$ is the maximal solution of (2.17). In view of (2.18) and the definition of $\tilde{w}$, (2.4) gives the integral inequality satisfied by a solution $v(t)$ of (2.1) in the form, for $t\in I_0$,

$$v(t) \leqslant G^{-1}\left[\int_{t_0}^{t} a(s)\,ds + G(r(t))\right],$$

proving the stated result.

Remark 2.1. Our results include the results in [2] as special cases.

Corollary 2.1 is a very useful result from an application point of view. In some mathematical models, though it may not be possible to find the actual solution, some kind of an estimate on the solution may be worth knowing.

We exhibit the following example as an application of corollary 2.1. Of course, the equation that we consider here is a scalar equation.

Example. Consider the equation

$$\frac{du}{dt} = u^2 + t, \quad u(0) = 1.$$

In spite of its simplicity, it does not have a solution that can be expressed in an explicit form. However, by using corollary 2.1 we can get an explicit bound for the solution as $u \leqslant (2+t^2)/(2+2t^2-2t-t^3)$.

3. In this section we consider the functional inequality

(3.1) $$f(x(t)) \leqslant a(t) + B(t)\,h\left(c(t) + \int_{t_0}^{t} K(t, s)\,W(s, x(s))\,ds\right), \quad t \in I$$

where $f = \coprod_{i=1}^{n} f_i$, $f \in C[R_+^n, R_+^n]$, f is a column vector and each f_i is strictly increasing, $f^{-1} = \coprod_{i=1}^{n} f_i^{-1}$, $K(t, s)$, $B(t)$ are $n \times n$ matrices where $K(t, s) = [k_{ij}(t, s)]$, $k_{ij} \in C[I \times I, R_+]$, $B(t) = [b_{ij}(t)]$, $h = \coprod_{i=1}^{n} h_i$, h is a column vector $W \in C \times [I \times R_+^n, R_+^n]$ and $x, a, c \in C[I, R_+^n]$.

In the following theorem we obtain estimates under various hypotheses on h.

Theorem 3.1. *Let $f = \coprod_{i=1}^{n} f_i$, each f_i be strictly increasing, $h = \coprod_{i=1}^{n} h_i$, h be monotone nondecreasing in u, $W_i(t, u_1, u_2, \ldots, u_n)$ be nondecreasing in u_j for each t, $i = 1, 2, \ldots, n$.*

Define $\bar{B}_{ij}(t) = \max_{t_0 \leqslant s \leqslant t} b_{ij}(s)$, $\bar{B}_{ij}(t)$ be nonsingular,

$$A_i(t) = \max_{t_0 \leqslant s \leqslant t} a_i(s), \quad C_i(t) = \max_{t_0 \leqslant s \leqslant t} c_i(s), \quad K_{ij}(t, s) = \max_{s \leqslant \sigma \leqslant t} k_{ij}(\sigma, s).$$

Then

(3.2) $$(1) \quad x(t) \leqslant f^{-1}[a(t) + B(t)\,h(r_1(t, t_0, C(t)))] \quad t \in I_{10},$$

where $r_1(T, t_0, r_{10})$ is the maximal solution of

(3.3) $$r_1' = \bar{K}(T, t)\,W(t, f^{-1}(a(t) + B(t)h(r_1))), \quad r_1(t_0) = r_{10}$$

existing on $I_1 \subset I$.

(2) *If the Jacobian $\partial h/\partial u$ of h exists and is continuous in u, then,*

(3.4) $$x(t) \leqslant f^{-1}[r_2(t, t_0, A(t) + \bar{B}(t)\,h(C(t)))], \quad t \in I_{20},$$

where $r_2(T, t_0, r_{20})$ is the maximal solution of

(3.5) $$r_2' = \bar{B}(T)\frac{\partial h}{\partial u}(h^{-1}[\bar{B}(T)^{-1}(r_2 - A(T))])\bar{K}(T, t)W(t, f^{-1}(r_2)), \quad r_2(t_0) = r_{20}$$

existing on $I_1 \subset I$.

In each of the above cases, $I_{i0}(i = 1, 2)$ is the appropriate interval contained in I subject to the domains of the inverse function involved.

Proof. We observe that for $t_0 \leqslant t \leqslant T \leqslant T_0$, we have, from (3.1),

(3.6) $$f(x)(t)) \leqslant a(t) + B(t)\,h(v(t, T)),$$

where $$v(t, T) = C(T) + \int_{t_0}^{t} \bar{K}(T, s)\,W(s, x(s))\,ds.$$

Consequently it follows that

$$v' \leqslant \bar{K}(T, t)\,W(t, f^{-1}\{a(t) + B(t)\,h(v)\}).$$

By theorem 4.1.1 [3] we get

(3.7) $$v(T, T) \leqslant r_1(T, t_0, v(t_0, T)), \quad T \geqslant t_0,$$

where $r_1(T, t_0, r_{10})$ is the maximal solution of (3.3) with $r_{10} = v(t_0, T)$. The estimates (3.6) and (3.7) yield the inequality

(3.8) $$f(x(T)) \leqslant a(T) + B(T)\,h(r_1(T, t_0, C(T))), \quad T \geqslant t_0$$

implying the stated result (3.2).

The functional inequality (3.1), for $t_0 \leqslant t \leqslant T \leqslant T_0$, can now be written as

(3.9) $$f(x(t)) \leqslant A(T) + \bar{B}(T)\,h(v),$$

where $v(t, T)$ is as defined above in (1). Setting $Z = Z(t, T) = A(T) + \bar{B}(T)\,h(v)$, it can be easily seen that

(3.10) $$Z' \leqslant \bar{B}(T)\frac{\partial h}{\partial v}(h^{-1}(\bar{B}(T)^{-1}(Z - A(T))))\,\bar{K}(T, t)\,W(t, f^{-1}(Z))$$

which yields the estimate, from theorem 4.1.1 [3],

(3.11) $$Z(T, T) \leqslant r_2(T, t_0, Z(t_0, T)), \quad T \geqslant t_0$$

where $r_2(T, t_0, r_{20})$ is the maximal solution of (3.5) with

$$r_{20} = A(T) + \bar{B}(T)\,h(C(T)).$$

In view of the definition of Z, (3.9) and (3.11) there results the inequality (3.4).

Remark 3.1. The results in [2, 3] are particular cases of our results, as the functions that we considered are vector functions. Suppose that $n = 1$.

[1] Let $f(u) = u$, $B(t) = 1$, $h(u) = u$, $c(t) = 0$ and $K(t, s) = 1$, then we deduce corollary 1.9.4 in [3] from (3.2) which is a generalization of Bellman-Gronwall-Reid inequality in a rather general form.

[2] Suppose we let $B(t) = b(t)$, $K(t, s) = k(t, s)$. Then theorem 3.1 reduces to a theorem in [2] which itself is a generalization of many earlier results.

Example 3.1. Consider the equations

$$x_1' = \frac{2}{3}x_1^{-1/2}+x_2, \quad x_1\neq 0, \quad x_2\neq 0$$

$$x_2' = \frac{2}{3}x_2^{-1/2}+x_1, \; x_1(0)=0, \; x_2(0)=.$$

Applying theorem 2.1, corollary 2.1 and the theory of differential inequalities, the estimate on the solution of above equations is given by

$$\begin{array}{c} x_1(t) \quad [(h(t))^{3/2}+t]^{2/3} \\ \leqslant \\ x_2(t) \quad [(b(t))^{3/2}+t]^{2/3}. \end{array}$$

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FORMULA VARIAȚIEI CONSTANTELOR (ÎN CAZUL NELINIAR) ȘI O INEGALITATE FUNCȚIONALĂ

Rezumat

Se consideră sistemul neliniar (2.1) unde matricea $A(t)$ de tipul $n\times n$ este așa fel încît $a_{ij}=0$ pentru $i\neq j$, $a_{ii}\in C(I, R)$, $I=[t_0, T_0]$, $t_0\geqslant 0$ iar $g(u)$ și $R(t, u)$ sînt vectori n-dimensionali, componentele lui $g(u)$ satisfăcînd condițiile $g_i(u_i)>0$ pentru $u_i>0$, $g_i(0)=0$. În prima parte a lucrării se demonstrează că o soluție $v(t)$ a sistemului (2.1) se poate exprima sub forma (2.2). În partea a doua a lucrării, plecînd de la inegalitatea funcțională (3.1), se stabilește evaluarea (3.2), care în particular conține cunoscuta inegalitate a lui Bellman-Gronwall.

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ÜBER UNGLEICHUNGEN VON PÓLYA IN DER THEORIE DER GANZEN FUNKTIONEN

VON

L. VOLKMANN

Einleitung. Sei $f(z)$ eine ganze Funktion. Es bezeichne $n(r, 0)$ die Anzahl der Nullstellen von $f(z)$ im Kreis $|z|\leqslant r$ und $M(r)=M(r, f)$ das Maximum des absoluten Betrages dieser Funktion auf dem Kreis $|z|=r$. Mit

$$\lambda=\lim_{r\to\infty}\frac{\log\log M(r)}{\log r} \quad \text{bzw.} \quad \mu=\lim_{r\to\infty}\frac{\log\log M(r)}{\log r}$$

werden die Ordnung bzw. die untere Ordnung von $f(z)$ bezeichnet.

Für ganze Funktionen der Ordnung λ, sind seit 1923 folgende Ungleichungen von G. P ó l y a [7] bekannt:

$$\lim_{r\to\infty}\frac{n(r, 0)}{\log M(r)}\geqslant\frac{\sin\pi\lambda}{\pi}, \quad \text{für } \lambda<1, \tag{1}$$

$$\lim_{r\to\infty}\frac{n(r, 0)}{\log M(r)}\leqslant\lambda. \tag{2}$$

Die Ungleichung (2) wurde 1944 von S. M. S h a h [8] durch

$$\lim_{r\to\infty}\frac{n(r, 0)}{\log M(r)}\leqslant\mu_1 \tag{2'}$$

verschärft, wobei

$$\mu_1=\lim_{r\to\infty}\frac{\log n(r, 0)}{\log r}$$

bedeutet und $\mu_1\leqslant\mu\leqslant\lambda$ gilt.

Die Ungleichung (1) soll in dieser Arbeit durch den folgenden Satz erweitert werden.

Satz 1. *Sei $f(z)$ eine ganze Funktion der unteren Ordnung $\mu<1$, dann gilt für jedes ρ, mit $\mu\leqslant\rho\leqslant\min(\lambda, 1)$,*

$$\varlimsup_{r\to\infty}\frac{n(r, 0)}{\log M(r)}\geqslant\frac{\sin\pi\rho}{\pi}. \tag{3}$$

Mit der gleichen Beweismethode kann auch der nächste Satz bewiesen werden.

Satz 2. *Sei $f(z)$ eine ganze Funktion der unteren Ordnung $\mu<1$, mit reellen negativen Nullstellen, dann gilt für jedes ρ, mit $\mu\leqslant\rho\leqslant\min(\lambda, 1)$,*

$$\varliminf_{r\to\infty}\frac{n(r, 0)}{\log M(r)}\leqslant\frac{\sin\pi\rho}{\pi}. \tag{4}$$