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# A NOTE ON ORTHOGONALITY PRESERVING OPERATORS

BY

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1. **Introduction.** In a recent paper [1], W. A. Al-Salam and A. Verma have considered the set $S$ of all differential operators of the form

$$J(D)=\sum_{k=0}^{\infty} a_k D^k, \quad a_0 \neq 0, \quad D=\frac{d}{dx},$$

where $a_0, a_1, \ldots$ are constants independent of $x$, and investigated those operators $J_1 \in S_1 \subset S$ which preserve orthogonality, i. e., if $\{P_n(x)\}$ is a sequence of simple orthogonal polynomials (s. o. p.), then $\{J_1 P_n(x)\}$ is also s.o. p.

The object of the present note is first to remark that the investigation of Al-Salam and Verma can be stated in the following alternative and more explicit form:

„An operator $J \in S$ is orthogonality preserving for a sequence of orthogonal polynomials $\{P_n(x)\}$ if and only if $J$ is either of the following two forms

(i) $J=\alpha e^{\beta D}$

(ii) $J=\exp(a_1 D+\lambda(D))$,

where $\lambda(t)=u\int B(t)\,dt$ and $B(D)$ is such that

$$B(D)\, P_n(x)=nP_{n-1}(x).$$

Next we wish to show that either of the following two operators

(a) $\alpha e^{\beta D}\, T_\nu$

(b) $\exp(a_1 D+\lambda(D)).T_\nu$

is orthogonality preserving for a sequence of orthogonal polynomials $\{P_n(x)\}$,

(a′) $$\lambda(t)=\frac{\nu}{b}\int B(t)dt,$$

and $B(D)$ is such that

(b′) $$B(D)\, P_n(\nu x)=nP_{n-1}(\nu x),$$

where the operator $T_\nu$ is such that $T_\nu f(x)=f(\nu x)$ and in particular, we note that if

$$T_\nu=\sum_{n=0}^{\infty}\frac{(\nu-1)^n}{n!}x^n D^n=(\nu)^\delta, \quad \text{where } \delta=xD$$

then $T_\nu f(x)=f(\nu x)$.

2. **Theorem.** *If an operator $J.T_\nu$ is orthogonality preserving for a sequence of orthogonal polynomials $\{P_n(x)\}$, then $J.T_\nu$ is either of the following two forms:*

(a) $\alpha e^{\beta D}\, T_\nu$

(b) $\exp(a_1 D + \lambda(D)).\, T_\nu$,

*where* $T_\nu$ *is such that* $T_\nu f(x) = f(\nu x)$, $J = J(D)$, and $\lambda(t) = \frac{\nu}{b}\int B(t)\,dt$ *and* $B(D)$ *is such that*

$$B(D)\, P_n(\nu x) = n P_{n-1}(\nu x).$$

*Proof.* Suppose that $\{P_n(x)\}$ and $\{J\, T_\nu (P_n(x))\}$ are sequences of orthogonal polynomials. Then we can have

$$P_n(x) = (x - \alpha_n) P_{n-1}(x) - \beta_n P_{n-2}(x) \tag{2.1}$$

and

$$J\, T_\nu P_n(x) = (\nu x - \bar{\alpha}_n) J\, T_\nu\, P_{n-1}(x) - \bar{\beta}_n J\, T_\nu P_{n-2}(x). \tag{2.2}$$

Operating on (2.1) by $J T_\nu$ we obtain

$$\nu J' P_{n-1}(\nu x) = (\alpha_n - \bar{\alpha}_n) J P_{n-1}(\nu x) + (\beta_n - \bar{\beta}_n)\, J P_{n-2}(\nu x). \tag{2.3}$$

Now comparing coefficients of $x^{n-1}$ and $x^{n-2}$ in (2.3) we have

$$\alpha_n - \bar{\alpha}_n = \nu a_1 \quad \text{and} \quad \beta_n - \bar{\beta}_n = (n-1)\nu^2 (2a_2 - a_1^2) \tag{2.4}$$

(taking, without any loss of generality, $a_0 = 1$).
It follows therefore from (2.3) and (2.4) that

$$\bar{B}(D)\, P_n(\nu x) = n\nu (2a_2 - a_1^2) P_{n-1}(\nu x), \tag{2.5}$$

$$\text{where } \bar{B}(D) = \frac{1}{J(D)}\, J'(D) - a_1.$$

Now two cases will arise.

*Case I.* If $2a_2 = a_1^2$, then it follows from (2.5) that

$$\left[\frac{1}{J(D)}\, J'(D) - a_1\right] P_n(\nu(x)) = 0,$$

whence we have

$$J(D) = \alpha \exp(a_1 D). \tag{2.6}$$

*Case II.* If $2a_2 - a_1^2 = b_0 \neq 0$, then it follows from (2.5) that

$$\frac{b}{\nu}\, B(D)\, P_n(\nu x) = n P_{n-1}(\nu x), \qquad b = \frac{1}{b_0}.$$

Now we note that I. M. Sheffer [2] proved that there exists an operator $B(D)$ of the form

$$B(D) = \sum_{k=0}^{\infty} C_k \left(\frac{1}{\nu}\right)^{k+1} D^{k+1}, \quad C_0 \neq 0, \tag{2.7}$$

such that $B(D) P_n(\nu x) = n P_{n-1}(\nu x)$, if and only if $\{P_n(x)\}$ is of Meixner type i. e. generating relation of the form

$$\sum_{n=0}^{\infty} P_n(x) t^n / n! = A(t) \exp(x H(t)),$$

where



$$A(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \dots,$$

$$H(t) = t + h_1 t^2 + h_2 t^3 + \dots,$$

Thus in this case, when the polynomials $\{P_n(x)\}$ are of Meixner type, we then have

$$\frac{b}{\nu}\, \bar{B}(D) = B(D), \tag{2.8}$$

whence we have

$$J(t) = \exp\left(a_1 t + \frac{\nu}{b}\int B(t)\,dt\right), \quad J(0) = 1, \tag{2.9}$$

where $B(D)$ is such that $B(D) P_n(\nu x) = n P_{n-1}(\nu x)$

**Corollary** (i). If $\nu = 1$, then $T_1 f(x) = f(x)$. Thus when $\nu = 1$, our theorem reduces to the theorems of Al-Salam and Verma.

(ii): If $T_\nu = (\nu)^\delta$ where $\delta \equiv xD$, then the following two operators

(a) $\alpha e^{\beta D}.(\nu)^\delta$

(b) $\exp(a_1 D + \lambda(D)).(\nu)^\delta$

are orthogonality preserving for a sequence of orthogonal polynomials $\{P_n(x)\}$ where $\lambda(t) = \frac{\nu}{b}\int B(t)\,dt$ and $B(D)$ is such that $B(D)\, P_n(\nu(x)) = n P_{n-1}(\nu x)$.

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## REFERENCES


1. Al-Salam W. A. and A. Verma — *Some orthogonality preserving operators.* Proc Amer. Math. Soc., 23 (1969), pp. 136—139.
2. Sheffer I. M. — *Some properties of polynomials of type zero.* Duke Math. J., 5 (1939), pp. 590—622.


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## NOTĂ ASUPRA OPERATORILOR CARE CONSERVĂ ORTOGONALITATEA

### Rezumat

Obiectul Notei prezente este 1) a demonstra că rezultatele lui Al-Salam și Verma se pot formula astfel: un operator $J \in S$ conservă ortogonalitatea unui șir de polinoame $\{P_n(x)\}$ dacă și numai dacă are una din formele (i), (ii) și 2) operatorii (a) și (b) conservă ortogonalitatea unui șir de polinoame $\{P_n(x)\}$ dat prin relațiile (a'), (b').