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# $\mathcal{J}$-CONGRUENCES ON SEMINEARRINGS

BY

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In this article the notion of „admissible set of subsets" and „normal set of subsets" of a semigroup $S$, introduced by Clifford and Preston in [2], is extended to the theory of seminearrings.

**1. Introduction. Definition** *1.1. A (right) seminearring* $(S, +, .) = S$ *is a non-empty set* $S$ *in which two binary operations* $(+)$ *and* $(\,.\,)$ *are defined such that*

1°. $(S, +)$ *and* $(S, .)$ *are semigroups,*
2°. $\forall (x, y, z) \in \mathrm{cart}^3(S)$ [1] $[(x + y) . z = x . z + y . z]$.

Let $S$ be any seminearring. We consider the following sets of mappings of $S$ into itself

$$\mathcal{L}_0 = \{\lambda_x : S \to S, \quad x \in S \cup \{\square\} \text{ and } \forall s \in S[\lambda_x s = x . s]\}$$
$$\mathcal{L}_1 = \{\lambda'_x : S \to S, \quad x \in S \cup \{\square\} \text{ and } \forall s \in S[\lambda'_x s = x + s]\}$$
$$\mathcal{R}_0 = \{\rho_x : S \to S, \quad x \in S \cup \{\square\} \text{ and } \forall s \in S[\rho_x s = s . x]\}$$
$$\mathcal{R}_1 = \{\rho'_x : S \to S, \quad x \in S \cup \{\square\} \text{ and } \forall s \in S[\rho'_x s = s + x]\}$$
$$\mathcal{Z}_0 = \{\zeta_{xy} : S \to S, \quad (x, y) \in \mathrm{cart}^2(S \cup \{\square\}) \text{ and } \zeta_{xy} = \lambda_x \circ \rho_y\}$$
$$\mathcal{Z}_1 = \{\zeta'_{xy} : S \to S, \quad (x, y) \in \mathrm{cart}^2(S \cup \{\square\}) \text{ and } \zeta'_{xy} = \lambda'_x \circ \rho'_y\}.$$

$\square$ is the notation for the „empty symbol" introduced by Ljapin [3]. Thus $\lambda_\square = \lambda'_\square = \rho_\square = \rho'_\square = \zeta_{\square\square} = \zeta'_{\square\square} = I_S$, the identity operator on $S$.

Furthermore we consider:

$$\mathcal{L}_{01} = \{\lambda'' : S \to S \text{ such that } \lambda'' = \prod_{i=1}^{n}{}' \lambda'_{x'_i} \circ \lambda_{x_i} \text{ with } n \in \mathrm{N}' \,^{2)}$$
$$\text{and } \forall i \in \{1, \ldots, n\}[(x'_i, x_i) \in \mathrm{cart}^2(S \cup \{\square\})]\}$$
$$\mathcal{R}_{01} = \{\rho'' : S \to S \text{ such that } \rho'' = \prod_{i=1}^{n}{}' \rho'_{y'_i} \circ \rho_{y_i} \text{ with } n \in \mathrm{N}'$$
$$\text{and } \forall i \in \{1, \ldots, n\}[(y'_i, y_i) \in \mathrm{cart}^2(S \cup \{\square\})]\}.$$

The product sign $\prod'$ refers to the composition of functions and is considered as „left product", this means:

$$\prod_{i=1}^{n}{}' \mu_i = \mu_n \circ \mu_{n-1} \circ \ldots \circ \mu_2 \circ \mu_1.$$

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1) $\mathrm{cart}^n(S) = \{(s_1, \ldots, s_n) \parallel \forall i \in \{1, \ldots, n\}\ s_i \in S\}$.
2) $\mathrm{N}' = \mathrm{N} \setminus \{0\}$ where $\mathrm{N}$ is the set of the natural numbers.

Finally we consider:

$$\mathcal{L} = \{\lambda : S \to S \text{ such that } \lambda = \prod_{i=1}^{n}{}' \lambda_{x_i'} \circ \rho_{y_i'} \circ \lambda_{x_i} \text{ with } n \in \mathbf{N}'$$

$$\text{and } \forall i \in \{1, \ldots, n\}[(x_i', y_i', x_i) \in \operatorname{cart}^3(S \cup \{\square\})]\},$$

$$\mathcal{L}' = \{\lambda' : S \to S \text{ such that } \lambda' = \prod_{i=1}^{n}{}' \lambda_{x_i'} \circ \lambda_{x_i} \circ \rho_{y_i} \text{ with } n \in \mathbf{N}'$$

$$\text{and } \forall i \in \{1, \ldots, n\}[(x_i', x_i, y_i) \in \operatorname{cart}^3(S \cup \{\square\})]\},$$

$$\mathcal{R} = \{\rho : S \to S \text{ such that } \rho = \prod_{i=1}^{n}{}' \lambda_{x_i'} \circ \rho_{y_i'} \circ \rho_{y_i} \text{ with } n \in \mathbf{N}'$$

$$\text{and } \forall i \in \{1, \ldots, n\}[(x_i', y_i', y_i) \in \operatorname{cart}^3(S \cup \{\square\})]\},$$

$$\mathcal{R}' = \{\rho' : S \to S \text{ such that } \rho' = \prod_{i=1}^{n}{}' \rho_{y_i'} \circ \lambda_{x_i} \circ \rho_{y_i} \text{ with } n \in \mathbf{N}'$$

$$\text{and } \forall i \in \{1, \ldots, n\}[(y_i', x_i, y_i) \in \operatorname{cart}^3(S \cup \{\square\})]\},$$

$$\mathcal{Z} = \{\zeta : S \to S \text{ such that } \zeta = \prod_{i=1}^{n}{}' \lambda_{x_i'} \circ \rho_{y_i'} \circ \lambda_{x_i} \circ \rho_{y_i} \text{ with } n \in \mathbf{N}'$$

$$\text{and } \forall i \in \{1, \ldots, n\}[(x_i', y_i', x_i, y_i) \in \operatorname{cart}^4(S \cup \{\square\})]\}.$$

The considered sets of mappings are semigroups with respect to the composition of functions.

$\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_0, \mathcal{R}_1, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{L}_{01}, \mathcal{R}_{01}, \mathcal{L}, \mathcal{L}', \mathcal{R}, \mathcal{R}'$ are subsemigroups of $\mathcal{Z}$, for the identity operator is an element of each of those sets. Similarly $\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_1, \mathcal{Z}_1, \mathcal{L}_{01}$ are subsemigroups of $\mathcal{L}$

$\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_0, \mathcal{Z}_0, \mathcal{L}_{01}$ are subsemigroups of $\mathcal{L}'$.

$\mathcal{L}_1, \mathcal{R}_0, \mathcal{R}_1, \mathcal{Z}_1, \mathcal{R}_{01}$ are subsemigroups of $\mathcal{R}$.

$\mathcal{L}_0, \mathcal{R}_0, \mathcal{R}_1, \mathcal{Z}_0, \mathcal{R}_{01}$ are subsemigroups of $\mathcal{R}'$.

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$\mathcal{L}_0$ and $\mathcal{R}_0$ are subsemigroups of $\mathcal{Z}_0$.

$\mathcal{L}_1$ and $\mathcal{R}_1$ are subsemigroups of $\mathcal{Z}_1$.

Using the right distributive law in $S$ we have:

$$\forall \zeta \in \mathcal{Z}\,[\zeta = (\prod_{i=1}^{n}{}' \lambda_{x_i''} \circ \rho_{y_i''} \circ \lambda_{x_i}) \circ \rho_y \text{ with } n \in \mathbf{N}' \text{ and}$$

$$\forall i \in \{1, \ldots, n\}\,(x_i'', y_i'', x_i, y) \in \operatorname{cart}^4(S \cup \{\square\})]$$

$$\forall \rho' \in \mathcal{R}'\,[\rho' = (\prod_{i=1}^{n}{}' \rho_{y_i''} \circ \lambda_{x_i}) \circ \rho_y \text{ with } n \in \mathbf{N}' \text{ and}$$



$$\forall i \in \{1, \ldots, n\}\,(y_i'', x_i, y) \in \operatorname{cart}^3(S \cup \{\square\})]$$

$$\forall \rho \in \mathcal{R}\,[\rho = \lambda_{x'} \circ \rho_{y'} \circ \rho_y \text{ with } (x', y', y) \in \operatorname{cart}^3(S \cup \{\square\})]$$

$$\forall \rho'' \in \mathcal{R}_{01}\,[\rho'' = \rho_{y'} \circ \rho_y \text{ with } (y', y) \in \operatorname{cart}^2(S \cup \{\square\})].$$

**Definition 1.2.** *A $\mathcal{T}$-congruence $q$ on $S$, where $\mathcal{T}$ is one of the transformation semigroups above-mentioned, is an equivalence relation $q$ on $S$ such that*

$$\forall \varphi \in \mathcal{T}\,[(\varphi, \varphi)_* q \subseteq q]^{1)}.$$

Let $F$ be the set of all $\mathcal{T}$-congruences on $S$. Then $F \in C(S) = \{L_0, L_1, R_0, R_1, Z_0, Z_1, L_{01}, R_{01}, L, L', R, R', Z\}$ where $\mathcal{T}$ is the „corresponding“ element of the set

$$\mathcal{T}(S) = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{R}_0, \mathcal{R}_1, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{L}_{01}, \mathcal{R}_{01}, \mathcal{L}, \mathcal{L}', \mathcal{R}, \mathcal{R}', \mathcal{Z}\}.$$

It is clear that for all $F$ of $C(S)$, $F$ is a complete lattice with respect to the inclusion and the usual definition of inf and sup. Let $(q_i : q_i \in F, i \in I)$ be a family of $\mathcal{T}$-congruences and consider

$$\tau_I = \{(a, b) \in \operatorname{cart}^2(S) \,\|\, \exists m < \omega^{2)}, \exists f : m \to I[(a, b) \in \prod_{k=0}^{m} q_{f(k)}]\},$$

where $f : m \to I$ is a function from $m$ into $I$ and

$$\prod_{k=0}^{m} q_{f(k)} = q_{f(0)} \circ q_{f(1)} \circ q_{f(m)}.$$

Then it is a straightforward exercise to show that

$$\tau_I = \sup(q_i : q_i \in F, \; i \in I).$$

This shows that

$$\forall (\mathcal{T}, \mathcal{T}') \in \operatorname{cart}^2(\mathcal{T}(S))\,[\mathcal{T} \subseteq \mathcal{T}' \Rightarrow F' \text{ complete sublatice of } F].$$

**Definition 1.3.** *Let $q$ be any relation on $S$. The maximal $\mathcal{T}$-congruence contained in $q$ is a $\mathcal{T}$-congruence $\mu$ such that $\mu \subseteq q$ and $\forall r \in F[r \subseteq q \Rightarrow r \subseteq \mu]$. The minimal $\mathcal{T}$-congruence containing $q$ is a $\mathcal{T}$-congruence $\nu$ such that $q \subseteq \nu$ and $\forall r \in F[q \subseteq r \Rightarrow \nu \subseteq r]$.*

Let $E$ be the set of all equivalences on $S$. We define an operator $\mathcal{T}^* : E \to F$ such that for all $q$ of $E$ $\mathcal{T}^* q = \bigcap_{\varphi \in \mathcal{T}} (\varphi, \varphi)^* q$.

Then

$$\forall q \in E[\mathcal{T}^* q \subseteq q \text{ and } [q \in F \text{ iff } \mathcal{T}^* q = q]]$$

$$\text{and if } q = \bigcap_{i \in I} q_i \text{ with } q_i \in E \text{ for all } i \in I \;\; \mathcal{T}^* q = \bigcap_{i \in I} \mathcal{T}^* q_i.$$

**Theorem 1.4.** *$\mathcal{T}^*$ is an idempotent, intersection preserving mapping of $E$ onto $F$. If $q \in E$ then $\mathcal{T}^* q$ is the unique maximal $\mathcal{T}$-congruence contained in $q$.*

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1) $(\varphi, \varphi)_* q = \{(\varphi a, \varphi b)$ *such that* $(a, b) \in q\}$.
$(\varphi, \varphi)^* q = \{(a', b') \in \operatorname{cart}^2 S$ such that $(\varphi a', \varphi b') \in q\}$.

2) $\omega$: first limit ordinal.

*Proof.* The theorem stated here is the analogue of the proposition stated by Clifford and Preston [2] in Lemma 10.3 about the mappings $L$, $R$ and $C$. Its proof proceeds on similar lines. □

Let $\Sigma$ be the set of all reflexive-symmetric relations on $S$. We define an operator $\mathcal{F}_*: \Sigma \to F$ such that for all $q$ of $\Sigma$ $\mathcal{F}_* q = T(\bigcap_{\varphi \in \mathcal{F}} (\varphi, \varphi)_* q)$[1]. Then

$$\forall q \in \Sigma [q \subseteq \mathcal{F}_* q \text{ and } [q \in F \text{ iff } q = \mathcal{F}_* q]].$$

**Theorem 1.5.** *$\mathcal{F}_*$ is an idempotent mapping of $\Sigma$ onto $F$. If $q \in \Sigma$ then $\mathcal{F}_* q$ is the unique minimal $\mathcal{F}$-congruence containing $q$.*

*Proof.* The statement of the theorem is the analogue of the assertions in Lemma 10.3 [2] about the mappings $L^*T$, $R^*T$, $C^*T$. The proof follows in the same way. □

**Corollary 1.6.** *Let $q_0$ be any relation on $S$. $\mathcal{F}_*(q_0 \cup q_0^{-1} \cup \Delta_S)$ is the minimal $\mathcal{F}$-congruence containing $q_0$.* □

**2. Admissible and normal sets. Definition 2.1.** *Let $\mathcal{A}$ be a disjoint set of subsets of a seminearring $S$. In accordance with* Clifford *and* Preston [2] *$\mathcal{A}$ is said to be $F$ admissible if $\mathcal{A}$ is a subset of the set of congruence classes of some $\mathcal{F}$-congruence on $S$; and if $q$ be an equivalence such that the elements of $\mathcal{A}$ are $q$-classes, then $q$ is said to admit $\mathcal{A}$.*

**Theorem 2.2.** *Let $\mathcal{A} = \{A_i : i \in I,\ A_i \subseteq S,\ A_i \cap A_j = \emptyset$ if $i \neq j\}$ be a $F$-admissible set. Then the set $F(\mathcal{A}) = \{q \in F \,\|\, \mathcal{A} \subseteq \mathrm{Clq}(S)$[2]$\}$ of $\mathcal{F}$-congruences admitting $\mathcal{A}$ forms a closed sublattice[3] of $F$.*

*Proof.* This follows immediately from the definitions. □ Suppose that $\mathcal{A} = \{A_i : i \in I\}$. is a disjoint set of subsets of $S$. Then $\alpha(\mathcal{A}) = \bigcup_{i \in I} \mathrm{cart}^2(A_i) \cup \Delta_{S \setminus A}$ and $\beta(\mathcal{A}) = \bigcup_{i \in I} \mathrm{cart}^2(A_i) \cup \mathrm{cart}^2(S \setminus A)$, where $A = \bigcup_{i \in I} A_i$, are respectively the minimal and maximal equivalence on $S$ admitting $\mathcal{A}$. An equivalence $q$ has the property that each $A_i$ is a union of $q$-classes if and only if $q \subseteq \beta(\mathcal{A})$ and each $A_i$ is contained in some $q$-class if and only if $\alpha(\mathcal{A}) \subseteq q$. Hence, from Theorem 1.4 and Theorem 1. 5 we have

**Corollary 2.3.** *Each $A_i$ is the union of $\mathcal{F}^* \beta(\mathcal{A})$-classes and each $A_i$ is contained in some $\mathcal{F}_* \alpha(\mathcal{A})$-class and they are respectively the maximal and the minimal $\mathcal{F}$-congruence with the property.* □ Now we can state the analogue of Theorem 10.4 of Clifford and Preston [2]. Its proof proceeds on similar lines.

**Theorem 2.4.** *Let $\mathcal{A} = \{A_i : i \in I,\ A_i \subseteq S,\ A_i \cap A_j = \emptyset$ if $i \neq j\}$ be a disjoint set of subsets of $S$.*

(i) *Then $\mathcal{A}$ is $F$-admissible if and only if*

$$\mathcal{F}_* \alpha(\mathcal{A}) \subseteq \mathcal{F}^* \beta(\mathcal{A});$$

*and then $q$ is a $\mathcal{F}$-congruence admitting $\mathcal{A}$ if and only if*

$$\mathcal{F}_* \alpha(\mathcal{A}) \subseteq q \subseteq \mathcal{F}^* \beta(\mathcal{A}),$$

*and so $F(\mathcal{A})$ is a complete lattice.*

(ii) *$\mathcal{A}$ is $F$-admissible if and only if*

($K$) $[\varphi \in \mathcal{F}$ and $(i, j) \in \mathrm{cart}^2(I)$ such that $\varphi_* A_i \cap A_j \neq \emptyset] \Rightarrow \varphi_* A_i \subseteq A_j$. □

**Definition 2.5.** *A $F$-admissible set of subsets $\mathcal{A}$ is said to be $F$-normal in $S$, if there is only one $\mathcal{F}$-congruence on $S$ which admits $\mathcal{A}$.*

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[1] $T(r)$ is the transitive closure of the relation $r$.

[2] $\mathrm{Clq}: S \to Q(S) \,\|\, \forall s \in S\ \mathrm{Clq}(s) = \{x \in S \,\|\, (x, s) \in q\}$.

[3] See Birkhoff [1] p. 50.



**Corollary 2.6.** *$\mathcal{A}$ is a $F$-normal set of disjoint subsets of $S$, if and only if*

$$\mathcal{F}_* \alpha(\mathcal{A}) = \mathcal{F}^* \beta(\mathcal{A}). \ \square$$

**3. Congruences in regular and inverse seminearrings. Definition 3.1.** *A seminearring $S$ is called additive regular or $(+)$-regular [multiplicative regular or $(.)$-regular] when the additive semigroup $(S, +)$ [the multiplicative semigroup $(S, .)$] is regular. We call a seminearring $S$ an additive inverse seminearring or $(+)$-inverse seminearring [a multiplicative inverse seminearring or $(.)$-inverse seminearring] when the additive semigroup $(S, +)$ [the multiplicative semigroup $(S, .)$] is an inverse semigroup.*

Let $\bar{x}$ be the unique $(+)$-inverse of any element $x$ of an $(+)$-inverse seminearring, then $\forall (x, s) \in \mathrm{cart}^2(S)\ \bar{x}.s = \overline{x.s}$ and $s.\bar{x} = \overline{s.x}$.

**Theorem 3.2.** *Let $S$ be a $(+)$-regular [$(.)$-regular] seminearring and $q$ a $\mathcal{F}$-congruence on $S$ with $\mathcal{F} \in \{\mathcal{Z}_1, \mathcal{L}, \mathcal{R}, \mathcal{Z}\}$, [$\mathcal{F} \in \{\mathcal{Z}_0, \mathcal{L}', \mathcal{R}', \mathcal{Z}\}$]. Let $\mathcal{A}$ be the set of $q$-classes which contain $(+)$-indempotents [$(.)$-idempotents]. Then each $\mathcal{F}'$-congruence $r$ on $S$ with $\mathcal{F}' \in \{\mathcal{Z}_1, \mathcal{L}, \mathcal{R}_0, \mathcal{Z}\}$ [$\mathcal{F}' \in \{\mathcal{Z}_0, \mathcal{L}', \mathcal{R}', \mathcal{Z}\}$] admitting $\mathcal{A}$ coincides with $q$.*

*Proof.* The considered $\mathcal{F}$-congruence $q$ and the $\mathcal{F}'$-congruence $r$ are at least compatible with the $(+)$-structure [$(.)$-structure] of the seminearring. Hence the proof of the theorem follows from the proof of Theorem 7. 33 of Clifford and Preston [2]. □ From the proof of Theorem 7.39 in [2] we get a stronger result.

**Theorem 3.3.** *Let $S$ be an $(+)$-inverse [$(.)$-inverse] seminearring and $q$ a $\mathcal{F}$-congruence on $S$ with $\mathcal{F} \in \{\mathcal{L}_1, \mathcal{L}_{01}, \mathcal{Z}_1, \mathcal{L}, \mathcal{R}, \mathcal{Z}\}$ [$\mathcal{F} \in \{\mathcal{L}_0, \mathcal{L}_{01}, \mathcal{Z}_0, \mathcal{L}', \mathcal{R}', \mathcal{Z}\}$]. Let $\mathcal{A}$ be the set of $q$-classes which contain $(+)$-idempotents [$(.)$-idempotents]. Then each $\mathcal{F}'$-congruence $r$ on $S$ with $\mathcal{F}' \in \{\mathcal{L}_1, \mathcal{L}_{01}, \mathcal{Z}_1, \mathcal{L}, \mathcal{R}, \mathcal{Z}\}$ [$\mathcal{F}' \in \{\mathcal{L}_0, \mathcal{L}_{01}, \mathcal{Z}_0, \mathcal{L}', \mathcal{R}', \mathcal{Z}\}$] admitting $\mathcal{A}$ coincides with $q$.*

*Proof.* Since $q$ and $r$ are at least left compatible with the $(+)$-structure [$(.)$-structure] of the seminearring, the proof of the theorem follows from the proof of Theorem 7.39 in [2]. □

**Corollary 3.4.** *Let $S$ be a $(+)$-regular [$(.)$-regular] seminearring and $q$ a $\mathcal{F}$-congruence on $S$, with $\mathcal{F} \in \{\mathcal{Z}_1, \mathcal{L}, \mathcal{R}, \mathcal{Z}\}$ [$\mathcal{F} \in \{\mathcal{Z}_0, \mathcal{L}', \mathcal{R}', \mathcal{Z}\}$]. Then the set $\mathcal{A}$ of $q$-classes which contain $(+)$-idempotents [$(.)$-idempotents] is $F$-normal. If $S$ be an $(+)$-inverse [$(.)$-inverse] seminearring the statement also holds with $\mathcal{F} \in \{\mathcal{L}_1, \mathcal{L}_{01}\}$ [$\mathcal{F} \in \{\mathcal{L}_0, \mathcal{L}_{01}\}$].* □

**Theorem 3.5.** *Let $S$ be a $(+)$-regular seminearring, and consider*

*$\mathcal{A} = \{A_i \subseteq S,\ i \in I\}$ satisfying*

($R_1$) *$A_i \cap A_j = \emptyset$ if $i \neq j$,*

($R_2$) $[\varphi \in \mathcal{F}$, $(i, j) \in \mathrm{cart}^2(I)$ such that $\varphi_* A_i \cap A_j \neq \emptyset] \Rightarrow \varphi_* A_i \subseteq A_j$,

($R_3$) *Each $(+)$ idempotent is contained in some element of $\mathcal{A}$ and each element of $\mathcal{A}$ contains at least one $(+)$-idempotent. Then $\mathcal{A}$ is $F$-normal if $\mathcal{F} \in \{\mathcal{Z}_1, \mathcal{L}, \mathcal{R}, \mathcal{Z}\}$ and $q_{\mathcal{A}} = \mathcal{F}_* \alpha(\mathcal{A}) = \mathcal{F}^* \beta(\mathcal{A})$ is the unique $\mathcal{F}$-congruence defined by $\mathcal{A}$.*

*Proof.* That $\mathcal{A}$ is $F$-admissible follows from Theorem 2.4 (ii). Let $q$ be a $\mathcal{F}$ - congruence admitting $\mathcal{A}$, then $\mathcal{A}$ is the set of $q$-classes which contain $(+)$-idempotents. Hence from the preceding corollary, $\mathcal{A}$ is $F$-normal. From Corollary 2.6, $\mathcal{F}_* \alpha(\mathcal{A}) = \mathcal{F}^* \beta(\mathcal{A})$ is the unique $\mathcal{F}$-congruence admitting $\mathcal{A}$. □ A similar theorem holds for $(.)$-regular seminearrings.

**Theorem 3.6.** *Let $S$ be an $(+)$-inverse seminearring. Then $\mathcal{A} = \{A_i \subseteq S,\ i \in I\}$ is a $Z$-normal set of subsets of $S$ if $\mathcal{A}$ satisfies the following conditions*

($I_1$) *each $A_i$ is an $(+)$-inverse subsemigroup of $S$,*

($I_2$) $A_i \cap A_j = \emptyset$ if $i \neq j$,

($I_3$) *each $(+)$-idempotent in $S$ is contained in some element of $\mathcal{A}$,*

($I_4$) *for each* $a \in S$ *and* $i \in I$, $\overline{a} + A_i + a \subseteq A_j$ *for some* $j$; *we shall write* $j = i + a$, *so that* $\overline{a} + A_i + a \subseteq A_{i+a}$,

($I_5$) *if* $(a,\ a+\overline{b},\ b+\overline{b}) \in \mathrm{cart}^3(A_i)$ *then* $b \in A_i$,

($I_6$) *for each* $s \in S$ *and* $i \in I$, $A_i . s \subseteq A_j$ *for some* $j$; *denote* $j =$ *is then* $A_i . s \subseteq A_{is}$

($I_7$) *if* $(a+\overline{a},\ b+\overline{b},\ a+\overline{b}) \in \mathrm{cart}^3(A_i)$ *then for each* $j \in I_j$

$$ja = jb \text{ and } [s + \overline{s} \in A_j \Rightarrow a.s + \overline{s.b} \in A_{ja} = A_{jb}].$$

*Then* $q_{\mathcal{A}} = \{(a,\ b) \in \mathrm{cart}^2(S):\ (a+\overline{a}, b+\overline{b}, a+\overline{b},\ b+\overline{a}) \in \mathrm{cart}^4 A_i$ *for some* $i \in I\}$, *is the unique* $\mathscr{Z}$*-congruence on* $S$ *admitting* $\mathcal{A}$. *Conversely, let* $q$ *be a* $\mathscr{Z}$*-congruence and* $\mathcal{A}$ *be the set of* $q$*-classes which contain* $(+)$*-idempotents. Then* $\mathcal{A}$ *satisfies* $I_1 - I_7$ *and* $q = q_{\mathcal{A}}$.

*Proof.* $I_1 - I_5$ are the conditions $K_1 - K_5$ for a kernel normal system of an $(+)$-inverse semigroup $(S,\ +)$. From Theorem 7.48 in [2], $\mathcal{A}$ is a $Z_1$-normal set of subsets of $S$ and $q_{\mathcal{A}}$ is the unique $\mathscr{Z}_1$-congruence on $S$ admitting $\mathcal{A}$. There remains to show that $q_{\mathcal{A}}$ is compatible with the $(.)$-structure of $S$.

Let $(a+\overline{a},\ b+\overline{b},\ a+\overline{b},\ b+\overline{a}) \in \mathrm{cart}^4(A_i)$ for some $i \in I$. Then by $I_6$, $((a+\overline{a}).s,\ (b+\overline{b}).s,\ (a+\overline{b}).s,\ (b+\overline{a}).s) \in \mathrm{cart}^4(A_{is})$. Whence it follows from the right distributive law that $(a.s+\overline{a.s},\ b.s+\overline{b.s},\ a.s+\overline{b.s},\ b.s+\overline{a.s}) \in \mathrm{cart}^4(A_{is})$. So we have that $(a,\ b) \in q_{\mathcal{A}} \Rightarrow$ $\Rightarrow \forall s \in S[(a.s,\ b.s) \in q_{\mathcal{A}}]$. For each $s \in S$ $\exists j \in I$ such that $s+\overline{s} \in A_j$. Then by $I_7$

$$((s+\overline{s}).a,\ (s+\overline{s}).b,\ s.a+\overline{s.b}) \in \mathrm{cart}^3(A_{ja}) = \mathrm{cart}^3(A_{jb}), \text{ or}$$
$$(s.a+\overline{s.a},\ s.b+\overline{s.b},\ s.a+\overline{s.b}) \in \mathrm{cart}^3(A_{ja}) = \mathrm{cart}^3(A_{jb}).$$

Whence it follows that $(a,\ b) \in q_{\mathcal{A}} \Rightarrow \forall s \in S[(s.a,\ s.b) \in q_{\mathcal{A}}]$. Conversely, if $q$ is a $\mathscr{Z}$-congruence on an $(+)$-inverse seminearring, then it is clear from Lemma 7.40 in [2] that the set $\mathcal{A}$ of $q$-classes which contain $(+)$-idempotents satisfies the conditions $I_1 - I_5$. Moreover $q = q_{\mathcal{A}}$. It remains to show that $\mathcal{A}$ satisfies $I_6$ and $I_7$. For each $i \in I$ there exists an $(+)$-idempotent $e$ such that $A_1 = \mathrm{Clq}(e)$ or $[a \in A_i \Leftrightarrow (e,\ a)_i \in q]$. Then for all $s \in S$ we have $\forall a \in A_i[(e.s,\ a.s) \in q]$, where $e.s = (e+e).s = e.s + e.s$. Whence there exists $j \in I$ such that $A_i . s \subseteq A_j$. This shows that condition $I_6$ holds. Let $(a+\overline{a},\ b+\overline{b},\ a+\overline{b}) \in \mathrm{cart}^3(A_i)$, then $(a,\ b) \in q_{\mathcal{A}} = q$. For all $s \in S$ we have $(s.a,\ s.b) \in q_{\mathcal{A}}$ or $(s.a+\overline{s.a},\ s.b+\overline{s.b},\ s.a+\overline{s.b}) \in \mathrm{cart}^3(A_j)$ for some $j \in I$. Whence it follows that $I_7$ holds. □

From the proof of Theorem 3.6 we have

**Corollaries 3.7.**

(i) $\mathcal{A}$ *is* $Z_1$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_5$,

(ii) $\mathcal{A}$ *is* $R$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_6$,

(iii) *Let* $I_7'$ *be the condition:*

*if* $(a+\overline{a},\ b+\overline{b},\ a+\overline{b}) \in \mathrm{cart}^3(A_i)$, *then for each* $s \in S$, *there exists a* $j \in I$ *such that* $(s.a+\overline{s.a},\ s.b+\overline{s.b},\ s.a+\overline{s.b}) \in \mathrm{cart}^3(A_j)$. *Then* $\mathcal{A}$ *is* $L$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_5$ *and* $I_7'$. □

**Theorem 3.8.** *Let* $S$ *be an* $(.)$*-inverse seminearring. Then* $\mathcal{A} = \{A_i \subseteq S,\ i \in I\}$ *is a* $Z$*-normal set of subsets of* $S$ *if* $\mathcal{A}$ *satisfies the following conditions*

($I_1$) *each* $A_i$ *is an* $(.)$*-inverse subsemigroup of* $S$,

($I_2$) $A_i \cap A_j = \emptyset$ *if* $i \neq j$,

($I_3$) *each* $(.)$*-idempotent in* $S$ *is contained in some element of* $\mathcal{A}$,



($I_4$) *for each* $a \in S$ *and* $i \in I$ $\underline{a}.A.a \subseteq A_j$[1], *for some* $j \in I$; *we denote* $j = i.a$, *so that* $\underline{a}.A_i.a \subseteq A_{i.a}$,

($I_5$) *if* $(a,\ a.\underline{b},\ b.\underline{b}) \in \mathrm{cart}^3(A_i)$ *then* $b \in A_i$,

($I_6$) *if* $(a.\underline{a},\ b.\underline{b},\ a.\underline{b}) \in \mathrm{cart}^3(A_i)$ *then for each* $s \in S$ *there exists* $\underline{a}$ $j \in I$ *such that* $((a+s),\ (b+s).\overline{(b+s)},\ (a+s).(b+s)) \in \mathrm{cart}^3(A_i)$,

($I_7$) *if* $(a.\underline{a},\ b.\underline{b},\ a.\underline{b}) \in \mathrm{cart}^3(A_i)$ *then for each* $s \in S$ *there exists* $\underline{a}$ $k \in I$ *such that*

$$((s+a).\underline{(s+a)},\ (s+b).\underline{(s+b)},\ (s+a).\underline{(s+b)}) \in \mathrm{cart}^3(A_k)$$

*then* $q_{\mathcal{A}} = \{(a,\ b) \in \mathrm{cart}^2(S) \,\|\, (a.\underline{a},\ b.\underline{b},\ a.\underline{b},\ b.\underline{a}) \in \mathrm{cart}^4(A_i)$ *for some* $i \in I\}$ *is the unique* $\mathscr{Z}$*-congruence on* $S$ *admitting* $\mathcal{A}$.
*Conversely, let* $q$ *be a* $\mathscr{Z}$*-congruence and* $\mathcal{A}$ *be the set of* $q$*-classes which contain* $(.)$*-idempotents. Then* $\mathcal{A}$ *satisfies* $I_1 - I_7$ *and* $q = q_{\mathcal{A}}$.

*Proof.* From $I_1 - I_5$ we get that $\mathcal{A}$ is a kernel normal system of the $(.)$-inverse semigroup $(S,\ .)$. This shows that $\mathcal{A}$ is a $Z_0$-normal set of subsets of $S$ and $q_{\mathcal{A}}$ is the unique $\mathscr{Z}_0$-congruence on $S$ admitting $\mathcal{A}$. That $q_{\mathcal{A}}$ is compatible with the $(+)$-structure of $S$, immediately follows from its definition using $I_6$ and $I_7$. Conversely, let $q$ be a $\mathscr{Z}$-congruence on $S$. Then the set $\mathcal{A}$ of $q$-classes which contain $(.)$-idempotents satisfies $I_1 - I_5$ by Lemma 7.40 in [2] and $q = q_{\mathcal{A}}$. $I_6$ and $I_7$ follow from the fact that $q$ is compatible with the $(+)$-structure of $S$. □

From the proof of Theorem 3.8 we get

**Corollaries 3.9.**

(i) $\mathcal{A}$ *is* $Z_0$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_5$,

(ii) $\mathcal{A}$ *is* $R'$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_6$,

(iii) $\mathcal{A}$ *is* $L'$*-normal if* $\mathcal{A}$ *satisfies* $I_1 - I_5$ *and* $I_7$. □

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[1] $\underline{a}$ is the $(.)$-inverse of $a$ in the inverse semigroup $(S,\ .)$