$$y_n(\lambda) = \frac{\displaystyle\int_{-\infty}^{-1} \frac{e^{nt}}{V_2(t)} \exp\left\{\int^{t} \frac{V_1 - \lambda e^{r\tau}}{V_2}\, d\tau\right\} dt}{\displaystyle\int_{-\infty}^{-1} \frac{1}{V_2(t)} \exp\left\{\int^{t} \frac{V_1 - \lambda e^{r\tau}}{V_2}\, d\tau\right\} dt}\,.$$

which is analytic in the half-plane bounded by $\lambda = u_1$ where u_1 is the first zero of the integral $Z_0(\lambda)$, in the denominator. However if $Z_0(\lambda)$ has no zero, then $y_n(\lambda)$ is analytic in the whole λ-plane.

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RUNGE-KUTTA METHODS FOR INITIAL VALUE PROBLEMS

BY

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In this paper we deduce some Runge-Kutta type methods for approximation of the solution for initial-value problems.

Let us consider the problem

$$\text{(I)}\quad \begin{cases} (1) & \dfrac{DU}{D\tau} = F\left(\tau, X, U, \dfrac{DU}{DX}\right), \\ (2) & U(0, X) = \Phi(X), \\ (3) & U(\tau, X)|_{X\in\Gamma} = 0, \end{cases}$$

where

$$U : [0, T]^m \times \prod_{i=1}^{q} [a_i, b_i] \to \mathrm{R}^p;$$

$$\tau \in [0, T]^m;\ X \in \prod_{i=1}^{q} [a_i, b_i];\ \Gamma = \partial\left(\prod_{i=1}^{q} [a_i, b_i]\right);$$

$$\Phi : \prod_{i=1}^{q} [a_i, b_i] \to \mathrm{R}^p;$$

$$F : [0, T]^m \times \prod_{i=1}^{q} [a_i, b_i] \times \mathrm{R}^p \times \mathrm{R}^{p\cdot q} \to \mathrm{R}^{p\cdot m}$$

and $DU/D\tau$ (resp. DU/DX) is the Fréchet derivative with respect to τ (resp X) for each X (resp τ).

Suppose that (I) has an unique analytical solution. This results by an argument analogous with Cauchy-Kowalevska theorem [2].

For simplicity we consider only two cases: i) $\tau \in [0, T]$ and $U \in R^2$, $X \in R^2$ and ii) $\tau \in [0, T]^2$.

1. Runge-Kutta methods for Cauchy-Kowalevska problem.

Consider the system of partial differential equations:

$$(4)\qquad \frac{\partial U}{\partial t} = F\left(t, X, U, \frac{DU}{DX}\right);\quad U \in \mathrm{R}^2,\ x \in \mathrm{R}^2$$

with initial condition

$$(5)\qquad U(0, x) = \Phi(x),$$

and boundary condition

$$(6)\qquad U(t, X)|_{X\in\Gamma} = 0,\quad \Gamma = \partial([0, 1]\times[0, 1]).$$

Consider in the set $[0, T] \times [0, 1] \times [0, 1]$ the net $\mathcal{R}_h = \{P_{n\nu}\}$.

$$P_{n,\nu} = (t_n, x_\nu) \text{ with } t_{n+1} = t_n + a_n h \quad n = 0, 1, \dots, t_0 = 0,$$
$$x_\nu = (x_1, \nu_1, x_2, \nu_2), \quad x_j, \nu_j = \sigma_j \nu_j h, \quad \nu_j \text{ integers.}$$

This net has variable step for t.

Let us observe that if we know the values of the solution in the plane $t = t_n$, one can determine the values of the solution for the $(n+1)$-level by

(7) $$U(t_{n+1}, x_\alpha) = U(t_n, x_\alpha) + \frac{a_n h}{1!} \frac{\partial U}{\partial t}(t_n, x_\alpha) + \dots + \varepsilon_s.$$

But to use this method it is necessary to estimate the values of F and its derivatives for $P_{n,\nu} \in \mathcal{R}_h$, t_n fixed.

Using the general idea of Runge-Kutta methods, we try to approximate the values $U(t_n, x_\alpha)$ by V^n_α defined by

(8) $$V^{n+1}_\alpha = V^n_\alpha + \sum_{q=n}^{n+1} \sum_{i=1}^{r} A^q_{i\alpha} K^q_{i\alpha}(h),$$

where

(9) $$K^q_{i\alpha} = h F\Big(t_q + \gamma^q_i h, X_\alpha, V^q_\alpha + \sum_{j=1}^{i-1} B^q_{ij\alpha} K^q_{j\alpha}(h), R^q_{(l)i\alpha}\big(V^q_\alpha + \sum_{j=1}^{i-1} B^q_{ij\alpha} K^q_{j\alpha}(h)\big)\Big),$$

and

(10) $$\gamma^q_1 = 0.$$

$A^q_{i\alpha}$, $B^q_{ij\alpha}$ are constant square matrices, and $R^q_{(l)i\alpha}$ are difference operators with respect to spatial variables which approximate the derivative with respect to $x_l (l = 1, 2)$.

We shall choose γ^q_i, $A^q_{i\alpha}$, $B^q_{ij\alpha}$ and $R^q_{(l)i\alpha}$ such that the coefficients of h^s $0 \leq s \leq l$ in the developments (7) and (8) with $V^n_\alpha = U(t_n, x_\alpha)$ coincide independently with respect to F. Here l will be the largest integer having the above property.

The relations (8) are of implicit type if at least one among the matrices $A^{n+1}_{i\alpha} \neq 0$.

2. Derivation formulas. Let $G(t, X, U, V_1, V_2)$ be a vector function with $U(t, x)$ solution of (4) and $V_j = \partial U / \partial X_j$. The total derivative with respect to t of G is

$$\frac{\partial^* G}{\partial t} = \frac{\partial G}{\partial t} + \sum_{j=1}^{2} \frac{\partial G}{\partial U_j} \frac{\partial U_j}{\partial t} + \sum_{j=1}^{2} \sum_{k=1}^{2} \frac{\partial G}{\partial U_{j(k)}} \frac{\partial U_{j(k)}}{\partial t}.$$

Set $U(i, j) = \partial^2 U / \partial x_i \partial x_j$, and obtain

$$\frac{\partial U_{i(k)}}{\partial t} = \frac{\partial F_i}{\partial x_k} + \sum_{j=1}^{2} \frac{\partial F_i}{\partial U_j} \frac{\partial U_i}{\partial x_k} + \sum_{j=1}^{2} \sum_{l=1}^{2} \frac{\partial F_i}{\partial U_{j(l)}} \frac{\partial U_{i(l)}}{\partial x_k}.$$

We introduce the operator

(11) $$H_k = \frac{\partial}{\partial x_k} + \sum_{i=1}^{2} (U_{(i,k)}, \nabla_{U(i)}) = \frac{\partial}{\partial x_k} + \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 U}{\partial x_i \partial x_k} \frac{\partial}{\partial u_{j(i)}}$$

and obtain

(12) $$\frac{\partial^* G}{\partial t} = \frac{\partial G}{\partial t} + (F, \nabla_U) G + \sum_{k=1}^{2} (H_k F, \nabla_{U(k)}) G = \frac{\partial G}{\partial t} + \sum_{i=1}^{2} \frac{\partial G}{\partial U_i} F_i + \sum_{k=1}^{2} \sum_{j=1}^{2} H_k(F_j) \frac{\partial G}{\partial U_{j(k)}}.$$

We define the operator

(13) $$D_s = \left\{ \frac{\partial}{\partial t} + (F, \nabla_U) + \sum_{k=1}^{2} (H_k F, \nabla_{U(k)}) \right\}^{(s)}$$

in the sense of symbolic power. Obviously, if A_1 and A_2 are constant matrices, we have

(14) $$D_s(A_1 G_1 + A_2 G_2) = A_1 D_s(G_1) + A_2 D_s(G_2).$$

The relation

(15) $$D_1[(G_1, G_2)] = (G_1, D_1(G_2)) + (D_1(G_1), G_2),$$

where $(\cdot, \cdot)$ denotes the inner product, follows immediately.

Proposition 1. *For the operators D_s defined by (13), we have:*

(16) $$D_1[D_s(G)] = D_{s+1}(G) + s \left\{ D_{s-1}\left(\frac{DG}{DU}\right) \cdot D_1(F) + \sum_{k=1}^{2} D_{s-1}\left(\frac{DG}{DU_{(k)}}\right) D_1(H_k(F)) \right\},$$

with $$\frac{DG}{DU_{(k)}} = \left(\frac{\partial G_i}{\partial U_{j(k)}}\right) i, j = 1, 2, \ k = 0, 1, 2, \ U_{(0)} = U.$$

The proof is immediate.

Using (14), (15) and (16) we have the formulas for the derivatives of U:

(17)
i) $$\frac{\partial^2 U}{\partial t^2}(t, x) = \frac{\partial^*}{\partial t} F = D_1(F),$$

ii) $$\frac{\partial^3 U}{\partial t^3}(t, x) = \frac{\partial^*}{\partial t}(D_1 F) = D_1[D_1(F)] = D_2(F) + \frac{DF}{DU} D_1(F) + \sum_{k=1}^{2} \frac{DF}{DU_{(k)}} D_1(H_k F),$$

iii) $$\frac{\partial^4 U(t, x)}{\partial t^4} = D_3(F) + 3 D_1\left(\frac{DF}{DU}\right) D_1(F) + 3 \sum_{k=1}^{2} D_1\left[\frac{DF}{DU_{(k)}}\right] D_1[H_k(F)] + \frac{DF}{DU}\left\{D_2(F) + \frac{DF}{DU} D_1(F) + \sum_{k=1}^{2} \frac{DF}{DU_{(k)}} D_1(H_k(F))\right\} + \sum_{k=1}^{2} \frac{DF}{DU_{(k)}}\left\{D_2[H_k(F)] + \frac{D(H_k(F))}{DU} D_1(F) + \sum_{l=1}^{2} \frac{DH_k(F)}{DU_{(l)}} D_1(H_l F)\right\}.$$

Let $\psi(h)$ be any function of the form $\psi(h) = \psi(t + \gamma h, x(h), u_{(k)}(h))$ $k = 0, 1, 2$, x linear function with respect to h. For the total derivative with respect to h we have

(18) $$\frac{d}{dh} \psi = \left\{ \gamma \frac{\partial}{\partial t} + \sum_{k=1}^{2} \omega_k \frac{\partial}{\partial x_k} + \sum_{j=0}^{2} \left(\frac{d}{dh} U_{(j)}(h), \nabla_{U(j)} \right) \right\} \psi(h).$$

If we set

(19) $$\mathcal{L}_q = \left\{ \gamma \frac{\partial}{\partial t} + \sum_{k=1}^{2} \omega_k \frac{\partial}{\partial x_k} + \sum_{j=0}^{2} \left(\frac{d}{dh} U_{(j)}(h), \nabla_{U(j)} \right) \right\}^{(q)},$$

we can derive a relation analogous to (16)

$$\frac{d}{dh}(\mathcal{L}_q \psi) = \mathcal{L}_{q+1}\psi + q\sum_{j=0}^{2} \mathcal{L}_{q-1}\left(\frac{D\psi}{DU_{(j)}}\right)\frac{d^2U_{(j)}}{dh^2}, \tag{20}$$

and then we can easily obtain the derivatives of ψ.

Finite difference operators $R^q_{(l)l,\alpha}$ are 2×2 matrices whose elements are

$$r^q_{(l)i,\alpha;j,k} = \frac{1}{h}\sum_{\beta} r^{q,\beta}_{(l)i,\alpha;j,k}\, e(\beta), \tag{21}$$

where $e(\beta)$ is the translation operator:

$$e(\beta)\, V(x_\alpha) = V(x_{1\alpha} + \beta_1\sigma_1 h,\ x_{2\alpha} + \beta_2\sigma_2 h) = V(x_{\alpha+\beta}). \tag{22}$$

It is necessary to impose the condition

$$\begin{aligned} R^q_{(l)l,\alpha} U(t_q, x_\alpha) &= \frac{\partial}{\partial x_p} U(t_q, x_\alpha) + \\ &+ \sum_{|\nu|\geq l,\ \nu\geq 0,\ \nu_l\geq 1} h^{|\nu|} <\sigma>^{\nu} M^q_{(l)i,\alpha;\nu} \frac{\partial^{|\nu|+1}}{\partial x^{\nu}\partial x_p} U(t_p, x_\alpha) \end{aligned} \tag{23}$$

where $<\sigma>^{\nu} = \Pi\, \sigma_i^{\nu_i}$ and $M^q_{(l)l,\alpha;\nu}$ are square matrices. It follows from (23)

$$\begin{aligned} &\text{i)}\ \sum_{\beta} r^{q,\beta}_{(l)l,\alpha;j,k} = 0 \quad l = 1.2,\ i = 1.2,\ j,k = 1.2,\ q = n,\ n+1; \\ &\text{ii)}\ \sum_{\beta} r^{q,\beta}_{(l)i,\alpha;j,k}\cdot \sigma_s\,\beta_s = \delta_{sl}\,\delta_{jk} \\ &\text{iii)}\ \sum_{\beta} r_{(l)j,k;\beta_{q_1}\beta_{q_2}}\cdots\beta_{q_n} = \frac{1}{\sigma_{q_N}} m^q_{(l)i,\alpha,\nu_{q_1q_2}\cdots q_n,j,k}\,\delta_{l q_N} \end{aligned} \tag{24}$$

$N \geqslant 2$, $l, q_1, \ldots, q_N = 1, 2$; $j, k = 1, 2$, $q = n$, $n+1$ and $\nu_{q_1q_2\cdots q_N}$ is a 2-index whose components ν_s are equal to number of indices $q_j = s$.

3. R. K. - methods for $r = 1$. From (8) we have

$$V^{n+1}_\alpha = V^n_\alpha + A^{n+1}_{1\alpha} K^{n+1}_{1\alpha}(h) + A^n_{1\alpha} K^n_{1\alpha}(h). \tag{25}$$

From

$$K^{n'}_{1\alpha}(0) = F\left(t_n, x_\alpha, U(t_n, x_\alpha), \frac{\partial U}{\partial x_{(p)}}(t_n, x_\alpha)\right), \tag{26}$$

$$K^{n''}_{1\alpha}(0) = 2\sum_{j=1}^{2}\sigma_j \frac{DF}{DU_{(j)}} M^n_{(j)1\alpha(\delta_{lj})}\frac{\partial^2 U^n_\alpha}{\partial x_j^2}, \tag{27}$$

$$k^{n+1}_{1\alpha}(0) = F\left(t_n, x_\alpha, U^n_\alpha, \frac{\partial U^n_\alpha}{\partial x_j}\right); \tag{28}$$

$$\begin{aligned} k^{n+1}_{1\alpha}(0) = 2\Bigg\{ &a_n\frac{\partial F}{\partial t} + \frac{DF}{DU}(A^n_{1\alpha} + A^{n+1}_{1\alpha})F + \\ &+ \sum_{j=1}^{2}\frac{DF}{DU_{(j)}}\left[\sigma_j M^{n+1}_{j1\alpha(\delta lj)}\frac{\partial^2 U^n_\alpha}{\partial x_j^2} + (A^n_{1\alpha} + A^{n+1}_{1\alpha}) H_j(F)\right]\Bigg\} \end{aligned} \tag{29}$$

we derive

$$\begin{aligned} \widetilde{U}(t_{n+1}, x_\alpha) = &\ U(t_n, x_\alpha) + h(A^n_{1\alpha} + A^{n+1}_{1\alpha})F + \\ &+ h^2\Bigg\{A^n_{1\alpha}\sum_{j=1}^{2}\frac{DF}{DU_j}\sigma_j M^n_{(j)1\alpha(\delta lj)}\frac{\partial^2 U^n_\alpha}{\partial x_j^2} + A^{n+1}_{1\alpha}\Bigg[a_n\frac{\partial F}{\partial t} + \frac{DF}{DU}(A^n_{1\alpha} + A^{n+1}_{1\alpha})F + \\ &+ \sum_{j=1}^{2}\frac{DF}{DU_{(j)}}\sigma_j M^{n+1}_{(j)1\alpha(\delta lj)}\frac{\partial^2 U^n_\alpha}{\partial x_j^2} + (A^n_{1\alpha} + A^{n+1}_{1\alpha}) H_j(F)\Bigg]\Bigg\} + \sigma(h^2). \end{aligned} \tag{30}$$

From (30), (7) and (17) we obtain for the coefficient of h

$$A^n_{1\alpha} + A^{n+1}_{1\alpha} = a_n E \tag{31}$$

and for that of h^2

$$A^{n+1}_{1\alpha} = \frac{a_n}{2}E, \tag{32}$$

$$M^n_{(j)1,\alpha(\delta lj)} + M^{n+1}_{(j)1,\alpha(\delta lj)} = 0. \tag{33}$$

The coefficients of h^3 in (30) and (7) in general are not equal.

We deduce that the local discretization error is of order h^3. Observe that this order is obtained for $M^{(n)}_{(j),1\alpha} \neq 0$.

Therefore the $R-K$ formula for $r = 1$ is, in the implicit case:

$$V^{n+1}_\alpha = V^n_\alpha + \frac{a_n h}{2}[F(t_n, x_\alpha, V^n_\alpha, R^n_{(j)1\alpha}V^n_\alpha) + F(t_{n+1}, x_\alpha, V^{n+1}_\alpha, R^{n+1}_{(j)1,\alpha}V^{n+1}_\alpha)] \tag{34}$$

and for the explicit case, $A^{n+1}_1 = 0$, is

$$V^{n+1}_\alpha = V^n_\alpha + a_n h F(t_n, x_\alpha, V^n_\alpha, R^n_{(j)1\alpha}V^n_\alpha). \tag{35}$$

Note that the formula (35) has the local discretization error of order h^2.

The system (34) with respect to V^{n+1}_α is solved by iteration.

This processes is convergent for small h, if we take for the first approximation the explicit solution of (35). It is convenient to use the methods (34) and (35) in the form predictor-corrector:

$$\begin{aligned} Z^{n+1}_\alpha &= V^n_\alpha + a_n h F(t_n, x_\alpha, V^n_\alpha, R^n_{U,1\alpha}V^n_\alpha), \\ V^{n+1}_\alpha &= V^n_\alpha + \frac{a_n h}{2}[F(t_n, x_\alpha, V^n_\alpha, R^n_{(j)1\alpha}V^n_\alpha) + F(t_{n+1}, x_\alpha, Z^{n+1}_\alpha, R^{n+1}_{(j)1,\alpha}Z^{n+1}_\alpha)], \end{aligned} \tag{36}$$

Proposition 2. *The local order of approximation for R-K - method (34) is 3 and for (35) is 2.*

Proof. Let $U(t_n, x_\alpha)$ be the true values of the solution for (4) and $\widetilde{V}^{n+1}_\alpha$ defined by:

$$\begin{aligned} \widetilde{V}^{n+1}_\alpha = U(t_n, x_\alpha) + \frac{a_n h}{2}\{&F(t_n, x_\alpha, U(t_n, x_\alpha), R^n_{(j)1\alpha}U(t_n, x_\alpha)) + \\ &+ F(t_{n+1}, x_\alpha, \widetilde{V}^{n+1}_\alpha, R^{n+1}_{(j)1,\alpha}\widetilde{V}^{n+1}_\alpha)\}. \end{aligned}$$

Consider the development of $\widetilde{V}^{n+1}_\alpha$

$$\tilde{V}_{\alpha}^{n+1} = U(t_n, x_\alpha) + a_n h F(t_n, x_\alpha, U(t_n, x_\alpha), U_{(j)}(t_n, x_\alpha)) +$$

$$+ a_n h^2 \Big\{ \sum_{j=1}^{2} \sigma_j \frac{DF}{DU_{(j)}} M^n_{(j)1,\alpha,(\delta j)} \frac{\partial^2 U(t_n, x_\alpha)}{\partial x_j^2} + a_n \frac{\partial F}{\partial t} + a_n \frac{DF}{DU} F +$$

$$+ \sum_{j=1}^{2} \sigma_j \frac{DF}{DU_{(j)}} M^{n+1}_{(j)1\alpha(\delta j)} \frac{\partial^2 U(t_n, x_\alpha)}{\partial x_j^2} + a_n \sum_{j=1}^{2} \frac{DF}{DU_{(j)}} H_j(F) \Big\} +$$

$$+ \frac{a_n}{2} \frac{h^3}{2!} \Big\{ \sum_{j=1}^{2} \Big\{ \Big(\sigma_j M^n_{(j)1\alpha(\delta, j)} \frac{\partial^2 U(t_n, x_\alpha)}{\partial x_j^2}, \nabla_{U(j)} \Big)^{(2)} F +$$

(37)

$$+ 2 \frac{DF}{DU_{(j)}} \sum_{\substack{|\nu|=2 \\ \nu_j \geq 1}} <\sigma>^{\nu} M^n_{(j)1\alpha\nu} \frac{\partial^3 U}{\partial x^{\nu} \partial x_{(j)}} \Big\} + \Big\{ a_n \frac{\partial}{\partial t} + a_n (F, \nabla_n) +$$

$$\sum_{j=1}^{2} \Big(\frac{d}{dh} \Big(R^n_{(j)1\alpha} \Big(U^n_\alpha + \frac{a_n}{2} k^n_{1\alpha} + \frac{a_n}{2} k^{n+1}_{1\alpha} \Big) \Big|_{h=0}, \nabla u_{(j)} \Big) \Big\}^{(2)} F +$$

$$+ \frac{DF}{DU} \frac{d^2}{dh^2} \Big(\frac{a_n}{2} (k^n_{1\alpha} + k^{n+1}_{1\alpha}) \Big) \Big|_{h=0} + \sum_{j=1}^{2} \frac{DF}{DU_{(j)}} \cdot \frac{d^2}{dh^2} \Big(R^{n+1}_{1\alpha} \Big(U^n_\alpha + \frac{a_n}{2} k^n_{1\alpha} +$$

$$+ \frac{a_n}{2} k^{n+1}_{1\alpha} \Big) \Big) \Big|_{h=0} + Sh^4 .$$

By (33) and (7) we obtain :

$$U(t_{n+1}, x_\alpha) - \tilde{V}_{\alpha}^{n+1} = \frac{(a_n h)^3}{3!} \Big[D_2(F) + \frac{DF}{DU} D_1(F) + \sum_{j=1}^{2} \frac{DF}{DU_{(j)}} D_1(H_j F) -$$

$$- \frac{a_n h^3}{4} \Big\{ a_n^2 \Big(D_1 + \frac{1}{a_n} \sum_{j=1}^{2} \Big(\sigma_j M^{n+1}_{(j)1\alpha(\delta, ij)} \frac{\partial^2 U^n_\alpha}{\partial x_j^2}, \nabla_{U(j)} \Big) \Big)^{(2)} F +$$

$$+ \sum_{j=1}^{2} \Big(\sigma_i M^n_{(j)1\alpha\delta ij} \frac{\partial^2 U^n_\alpha}{\partial x_j^2}, \nabla_{U(j)} \Big)^{(2)} F + 2 \sum_{j=1}^{2} \frac{DF}{DU_{(j)}} \Big(\sum_{i=1}^{2} \sigma_i \sigma_j (M^n_{(j)1,\alpha(i,j)} +$$

$$+ M^{n+1}_{(j),1\alpha(i,j)}) \frac{\partial^3 U}{\partial x_i \partial x_j^2} \Big) + 2 a_n \sum_{j=1}^{2} \frac{DF}{DU_j} M^{n+1}_{(j)1\alpha(\delta ij)} \frac{\partial^{*2}}{\partial x_j^2} F +$$

$$+ a_n \frac{DF}{DU} \Big(\sum_{j=1}^{2} \frac{DF}{DU_{(j)}} \sigma_j M^n_{(j)1\alpha(\delta ij)} \frac{\partial^2 U^n_\alpha}{\partial x_j^2} \Big) + a_n^2 \frac{DF}{DU} \Big[\Big(D_1 + \frac{1}{a_n} \sum_{j=1}^{2} (\sigma_j M^{n+1}_{(j)1\alpha(\delta ij)}) \frac{\partial^2 U}{\partial x_j^2},$$

$$\nabla_{U(j)} \Big) \Big) F \Big] \Big\} + Qh^4$$

that is the order of (34) is 3.

Denote by

(38)
$$\varepsilon^k_\alpha = U(t_k, x_\alpha) - Z^k_\alpha,$$
$$\eta^k_\alpha = U(t_k, x_\alpha) - V^k_\alpha.$$

From proposition 2 we have

(39) $$U(t_{k+1}, x_\alpha) = U(t_k, x_\alpha) + a_k h F(t_k, x_\alpha, U(t_k, x_\alpha), R^k_{j1\alpha} U(t_k, x_\alpha)) + h^2 \mu_{\alpha k}$$

and

$$U(t_{k+1}, x_\alpha) = U(t_k, x_\alpha) + \frac{a_k h}{2} [F(t_k, x_\alpha, U(t_k, x_\alpha), R^k_{(j)1\alpha} U(t_k, x_\alpha)) +$$

(40)

$$+ F(t_{k+1}, x_\alpha, U(t_{k+1}, x_\alpha), R^{k+1}_{(j),1\alpha} U(t_{k+1}, x_\alpha))] + \nu_{\alpha k} h^3.$$

From (36), (39) and (40) we deduce

$$\varepsilon^{k+1}_\alpha = U(t_{k+1}, x_\alpha) - Z^{k+1}_\alpha = U(t_k, x_\alpha) - V^k_\alpha +$$

$$+ a_k h [F(t_k, x_\alpha, V^k_\alpha + \eta^k_\alpha, R^k_{(j)1\alpha}(V^k_\alpha + \eta^k_\alpha)) - F(t_k, x_\alpha, V^k_\alpha, R^k_{(j)1\alpha} V^k_\alpha)] + \mu_{\alpha k} h^2$$

and

$$\eta^{k+1}_\alpha = U(t_{k+1}, x_\alpha) - V^{k+1}_\alpha = U(t_k, x_\alpha) - V^k_\alpha +$$

$$+ \frac{a_k h}{2} [F(t_k, x_\alpha, V^k_\alpha + \eta^k_\alpha, R^k_{(j)1\alpha}(V^k_\alpha + \eta^k_\alpha)) - F(t_k, x_\alpha, V^k_\alpha, R^k_{(j)1\alpha} V^k_\alpha) +$$

$$+ F(t_{k+1}, x_\alpha, Z^{k+1}_\alpha + \varepsilon^{k+1}_\alpha, R^{k+1}_{(j)1\alpha}(Z^{k+1}_\alpha + \varepsilon^{k+1}_\alpha)) - F(t_{k+1}, x_\alpha Z^{k+1}_\alpha R^{k+1}_{(j)1\alpha} Z^{k+1}_\alpha)] +$$

$$+ V_{\alpha k} h^3,$$

or

$$\varepsilon^{k+1}_\alpha = \eta^k_\alpha + a_k h \Big(\frac{D^* F^k_\alpha}{DU} \eta^k_\alpha + \sum_{j=1}^{2} \frac{D^* F^k_\alpha}{DU_{(j)}} R^k_{(j)1\alpha} \eta^k_\alpha \Big) + h^2 \mu_{\alpha k},$$

(41)

$$\eta^{k+1}_\alpha = \eta^k_\alpha + \frac{a_k h}{2} \Big(\frac{D^* F^k_\alpha}{DU} \eta^k_\alpha + \sum_{j=1}^{2} \frac{D^* F^k_\alpha}{DU_{(j)}} R^k_{(j)1\alpha} \eta^k_\alpha +$$

$$+ \frac{D^{**} F^k_\alpha}{DU} \varepsilon^{k+1}_\alpha + \sum_{j=1}^{2} \frac{D^{**} F^k_\alpha}{DU_{(j)}} R^{k+1}_{(j)1\alpha} \varepsilon^{k+1}_\alpha \Big) + \nu_{\alpha k} h^3.$$

Suppose that the finite difference operator R is

$$R^k_{(j)1\alpha} = \frac{1}{h} \sum_{\beta \in B^k_{j,\alpha}} R^k_{(j)1\alpha\beta} e(\beta)$$

with $\sigma_i = 0(h)$ and $R^k_{(j)1\alpha\beta}$ square matrices. Now, we have

(42) $$\varepsilon^{k+1}_\alpha = \eta^k_\alpha + a_k h \frac{D^* F^k_\alpha}{DU} \eta^k_\alpha + \sum_{j=1}^{2} \sum_{\beta \in B^k_\alpha} \frac{D^* F^k_\alpha}{DU_{(j)}} R^k_{(j)1\alpha\beta} \eta^k_{\alpha+\beta} + \mu_{\alpha k} h^2,$$

$$\eta^{k+1}_\alpha = \eta^k_\alpha + \frac{a_k h}{2} \Big(\frac{D^* F^k_\alpha}{DU} \eta^k_\alpha + \frac{D^{**} F^k_\alpha}{DU} \varepsilon^{k+1}_\alpha \Big) + \frac{a_k}{2} \sum_{j=1}^{2} \sum_{\beta \in B^k_{j\alpha}} \frac{D^* F^k_\alpha}{DU_{(j)}} R^k_{J1\alpha\beta} \eta^k_{\alpha+\beta} +$$

(43)

$$+ \frac{a_k}{2} \sum_{J=1}^{2} \sum_{\beta \in B^k_{1\alpha}} \frac{D^{**} F^k_\alpha}{DU_{(j)}} R^{k+1}_{j1\alpha\beta} \varepsilon^{k+1}_{\alpha+\beta} + V_{\alpha k} h^3.$$

Let $\varepsilon^k = \max_\alpha |\varepsilon^k_\alpha|$ and $\eta^k = \max_\alpha |\eta^k_\alpha|$.

We derive

$$\varepsilon^{k+1} \leqslant \eta^k (1 + Lh) + Mh^2,$$

(44) $$\eta^{k+1} \leqslant \eta^k (1 + L'h) + L''h \varepsilon^{k+1} + Nh^3,$$

with L, L', L'', M, N obtained considering the norms of the operators DF/DU, $DF/DU_{(j)}$, $R^k_{(j)1\alpha\beta}$ and setting $M = \sup_{k,\alpha} |\mu_{k\alpha}|$, $N = \sup_{k,\alpha} |\nu_{k\alpha}|$, $t_k \in [0, T]$.

Finally we obtain

(45)
$$\tau_i^{k+1} \leq e^{QT}\tau_i^0 + e^{QT}(N + L''M)h^2,$$

and thus we have proved

Proposition 3. *The predictor-corrector R-K-method (36) has the global order of approximation h^2 if the function F has bounded derivatives until the 4^{th} order on $[0, T]\times[0, 1]^2\times \mathbf{R}^n$ and σ_i are of order h.*

4. The R. K-method for $r = 2$.

In this case for (8) we have the corrections

(46)
$$\begin{aligned}
&\text{i) } K^n_{1\alpha} = hF(t_n, x_\alpha, V^n_\alpha, R^n_{1\alpha}V^n_\alpha),\\
&\text{ii) } K^{n+1}_{1\alpha} = hF(t_{n+1}, x_\alpha, V^{n+1}_\alpha, R^{n+1}_{1\alpha}V^{n+1}_\alpha)\\
&\text{iii) } K^n_{2\alpha} = hF(t_n + \gamma^n_{2\alpha}h, x_\alpha, V^n_\alpha + B^n_{21\alpha}K^n_{1\alpha}, R^n_{2\alpha}V^n_\alpha + C^n_{21\alpha}Q^n_{1\alpha}K^n_{1\alpha}),\\
&\text{iv) } K^{n+1}_{2\alpha} = hF(t_{n+1} + \gamma^{n+1}_{2\alpha}h, x_\alpha, V^{n+1}_\alpha + B^{n+1}_{21\alpha}K^{n+1}_{1\alpha}, R^{n+1}_{2\alpha}V^{n+1}_\alpha + C^{n+1}_{21\alpha}Q^{n+1}_{1\alpha}K^{n+1}_{1\alpha})
\end{aligned}$$

$x_\alpha \in [0, 1]$, $R^q_{i\alpha}$, $Q^q_{i\alpha}$ finite difference operators such that

(47)
$$R^q_{p\alpha}g(x_\alpha, h) = \sum_{S=0}^{\infty}\sum_{j=0}^{\infty} \frac{h^{s+j}}{j:s} M^q_{p\alpha j} \frac{\partial^{s+j+1} g(x_\alpha, h)}{\partial x^{j+1}\,\partial h^s}, \quad M^q_{p\alpha 0} = E,$$
$$(p = 1,\ 2;\ q = n,\ n+1)$$
$$Q^q_{1\alpha}g(x_\alpha, h) = \sum_{s=0}^{\infty}\sum_{j=0}^{\infty} \frac{h^{s+j}}{j:s} N^q_{1\alpha j} \frac{\partial^{s+j+1} g(x_\alpha, h)}{\partial x^{j+1}\,\partial h^s}, \quad N^q_{1\alpha 0} = E.$$
$$(q = n, n+1).$$

Identifying the development of (7) and (8) we derive:

(48)
$$\begin{aligned}
a_n F\left(t_n, x_\alpha, U^n_\alpha, \frac{\partial U^n_\alpha}{\partial x}\right) &= A^n_{1\alpha}F(t_n, x_\alpha, U^n_\alpha, (R^n_{1\alpha}U^n_\alpha)_0) +\\
&+ A^n_{2\alpha}F(t_n, x_\alpha, U^n_\alpha, (R^n_{2\alpha}U^n_\alpha + C^n_{21\alpha}Q^n_{1\alpha}K^n_{1\alpha})_0) +\\
&+ A^{n+1}_{1\alpha}F(t_n, x_\alpha, U^n_\alpha, (R^{n+1}_{1\alpha}(U^n_\alpha + \sum_p\sum_q A^q_{p\alpha}K^q_{p\alpha}))_0) +\\
&+ A^{n+1}_{2\alpha}F(t_n, x_\alpha, U^n_\alpha, (R^{n+1}_{2\alpha}(U^n_\alpha + \sum_p\sum_q A^q_{p\alpha}K^q_{p\alpha}) + C^{n+1}_{21\alpha}Q^{n+1}_{1\alpha}K^{n+1}_{1\alpha})_0),
\end{aligned}$$

and

$$\begin{aligned}
&B^n_{21\alpha} = C^n_{21\alpha} = \gamma^n_{2\alpha}E, \quad B^{n+1}_{21\alpha} = C^{n+1}_{21\alpha} = \gamma^{n+1}_{2\alpha}E,\\
&\gamma^n_{2\alpha}\gamma^{n+1}_{2\alpha}(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha}) \neq 0,\\
&A^n_{1\alpha} = \{a_n(2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 - \gamma^{n+1}_{2\alpha}(a_n + 2\gamma^{n+1}_{2\alpha}))]/6\gamma^n_{2\alpha}(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2\}E,\\
&A^n_{2\alpha} = \{a^2_n\gamma^{n+1}_{2\alpha}/6\gamma^n_{2\alpha}(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})\}E,\\
&A^{n+1}_{1\alpha} = \{a_n[2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 + \gamma^n_{2\alpha}(a_n - 2\gamma^n_{2\alpha})]/6\gamma^{n+1}_{2\alpha}(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})\}E,\\
&A^{n+1}_{2\alpha} = -\{a^2_n\gamma^n_{2\alpha}/6\gamma^{n+1}_{2\alpha}(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})\}E,
\end{aligned}$$

(49)
$$\begin{aligned}
&M^n_{1\alpha 1} = a_n\gamma^n_{2\alpha}(a_n + \gamma^{n+1}_{2\alpha} - \gamma^n_{2\alpha})N_\alpha,\\
&M^n_{2\alpha 1} = -\frac{\gamma^n_{2\alpha}}{\gamma^{n+1}_{2\alpha}}\{(a_n + \gamma^{n+1}_{2\alpha})[2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 -\\
&- \gamma^{n+1}_{2\alpha}(a_n + 2\gamma^{n+1}_{2\alpha})] + \gamma^{n+1}_{2\alpha}[2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 -\\
&- \gamma^n_{2\alpha}(a_n - 2\gamma^n_{2\alpha})]\}N_\alpha,\\
&M^{n+1}_{1\alpha 1} = a_n\gamma^n_{2\alpha}(a_n + \gamma^{n+1}_{2\alpha} - \gamma^n_{2\alpha})N_\alpha,\\
&M^{n+1}_{2\alpha 1} = -\frac{\gamma^{n+1}_{2\alpha}}{\gamma^n_{2\alpha}}\{\gamma^n_{2\alpha}[2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 - \gamma^{n+1}_{2\alpha}(a_n + 2\gamma^{n+1}_{2\alpha})] +\\
&+ (\gamma^n_{2\alpha} - a_n)[2(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})^2 + \gamma^n_{2\alpha}(a_n - 2\gamma^n_{2\alpha})]\}N_\alpha,\\
&\gamma^{n+1}_{2\alpha}N^{n+1}_{1\alpha 1} - \gamma^n_{2\alpha}N^n_{1\alpha 1} = -2a_n(\gamma^n_{2\alpha} + \gamma^{n+1}_{2\alpha})[(\gamma^n_{2\alpha})^2 + (\gamma^{n+1}_{2\alpha})^2 +\\
&+ \gamma^n_{2\alpha}\gamma^{n+1}_{2\alpha})]N_\alpha,\\
&N_\alpha\{(\gamma^n_{2\alpha})^3(5\gamma^{n+1}_{2\alpha} + 2a_n) + (\gamma^n_{2\alpha})^2[14(\gamma^{n+1}_{2\alpha})^2 + 4a_n\gamma^{n+1}_{2\alpha}] +\\
&+ \gamma^n_{2\alpha}[5(\gamma^{n+1}_{2\alpha})^3 - 4a_n(\gamma^{n+1}_{2\alpha})^2 - a^2_n\gamma^{n+1}_{2\alpha}] - 2a_n(\gamma^{n+1}_{2\alpha})^3\} = 0.
\end{aligned}$$

We can choose, for example

$$\gamma^n_{2\alpha} = 2a_n, \quad \gamma^{n+1}_{2\alpha} = -\frac{a_n}{2}$$

Then,

(50)
$$\begin{aligned}
&B^n_{21\alpha} = C^n_{21\alpha} = 2a_nE\\
&B^{n+1}_{21\alpha} = C^{n+1}_{21\alpha} = -\frac{a_n}{2}E\\
&A^n_{1\alpha} = \frac{a_n}{4}E\\
&A^n_{2\alpha} = -\frac{a_n}{36}E,\\
&A^{n+1}_{1\alpha} = \frac{a_n}{3}E,\\
&A^{n+1}_{2\alpha} = \frac{4a_n}{9}E,\\
&M^n_{1\alpha 1} = -3a^3_nN_\alpha,\\
&M^n_{2\alpha 1} = 12a^3_nN_\alpha,\\
&M^{n+1}_{1\alpha 1} = \frac{3a^3_n}{4}N_\alpha,\\
&M^{n+1}_{2\alpha 1} = \frac{15a^3_n}{8}N_\alpha,\\
&N^{n+1}_{1\alpha 1} + 4N^n_{1\alpha 1} = -\frac{39a^3_n}{2}N_\alpha
\end{aligned}$$

and we obtain the 4^{th} order R - K - method

$$V_\alpha^{n+1} = V_\alpha^n + \frac{a_nh}{36}\{9F(t_n, x_\alpha V_\alpha^n, R_{1\alpha}^n V_\alpha^n) -$$

$$- F(t_n+2a_nh, x_\alpha, V_\alpha^n+2a_nhF(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n), R_{2\alpha}^n(V_\alpha^n+2a_nh F(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n)))$$

$$+ 12F(t_{n+1}, x_\alpha, V_\alpha^{n+1}, R_{1\alpha}^{n+1} V_\alpha^{n+1}) +$$

(51)
$$+ 16F(t_{n+1} - \frac{a_nh}{2}, x_\alpha, V_\alpha^{n+1} - \frac{a_nh}{2} F(t_{n+1}, x_\alpha, V_\alpha^{n+1}, R_{1\alpha}^{n+1} V_\alpha^{n+1}),$$

$$R_{2\alpha}^{n+1}(V_\alpha^{n+1} - \frac{a_nh}{2} F(t_{n+1}, x_\alpha, V_\alpha^{n+1}, R_{1\alpha}^{n+1} V_\alpha^{n+1})))\}.$$

The corresponding predictor-corrector R − K - method is

$$Z_\alpha^{n+1} = V_\alpha^n + \frac{a_nh}{2}[F(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n(V_\alpha^n)) +$$

$$+ F(t_n + a_nh, X_\alpha, V_\alpha^n + a_nhF(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n), R_{2\alpha}^n[V_\alpha^n + a_nhF(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n)])],$$

$$V_\alpha^{n+1} = V_\alpha^n + \frac{a_nh}{36}\{9F(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n) -$$

(52)
$$- F(t_n+2a_nh, x_\alpha, V_\alpha^n+2a_nhF(t_n, x_\alpha, V_\alpha^n, R_{1\alpha}^n V_\alpha^n + R_{2\alpha}^n(V_\alpha^n+2a_nhF(t_n, x_\alpha, V_\alpha^n, R_\alpha^n V_\alpha^n)$$

$$+ 12F(t_{n+1}, x_\alpha, Z_\alpha^{n+1}, R_{1\alpha}^{n+1} Z_\alpha^{n+1}) +$$

$$+ 16F(t_{n+1} - \frac{a_nh}{2}, x_\alpha Z_\alpha^{n+1} - \frac{a_nh}{2} F(t_{n+1}, x_\alpha, Z_\alpha^{n+1}, R_{1\alpha}^{n+1} Z_\alpha^{n+1}),$$

$$R_{2\alpha}^{n+1}(Z_\alpha^{n+1} - \frac{a_nh}{2} F(t_{n+1}, x_\alpha, Z_\alpha^{n+1}, R_{1\alpha}^{n+1} Z_\alpha^{n+1})))\}$$

It is easy to prove by the same argument as for proposition 3 that the method (52) has the global order of approximation h^4 with the same assumption for F and σ_i.

5. The R-K-method for generalized Cauchy-Kovalevska problems.

Let us consider

$$\tau = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in [0, T]^2, \ x \in [0, 1], \ U \subset \mathbf{R}^2,$$

We introduce the net $\{P_i^n\}$ with $n = (n_1, n_2)$ $P_j^n = (n_1h_1, n_2h_2, j\Delta x)$. Let

(53)
$$U(\tau + H, x_i) = U(\tau, x_i) + \frac{DU(\tau, x_i)}{D\tau} \cdot H + \frac{1}{2!}\frac{D^2U}{D\tau^2} H \cdot H + \ldots$$

By the same argument atilized in the one dimensional case we introduce the approximation

(54)
$$V_i^{n+m} = V_i^{n+r} + \sum_{j=1}^{N} \sum_{0<p<m} K_{jin}^p A_{jm}^p H, \quad r < m,$$

where r, n, m, p are 2-indices and $n + m = (n_1 + m_1, n_2 + m_2)$. A_{jm}^p are square matrices and the corrections K are defined by

(55)
$$K_{jin}^\nu = F(\tau_n + \Gamma_j^\nu H, x_i, V_i^{n+\nu} + \sum_{l<j}\sum_{\mu<\nu} K_{lin}^\mu B_{jlin}^{\mu\nu} H,$$

$$R_{jin}^\nu(V_i^{n+\nu} + \sum_{l<j}\sum_{\mu<\nu} K_{lin}^\mu(C_{jlin}^{\mu\nu} H)).$$

In (55) Γ_j^ν, $B_{jln}^{\mu\nu}$, $C_{jpin}^{\mu\nu}$ are (2×2) − matrices and R is an finite difference matrix operator

Denote $E_\nu H = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ and consider

$$U(\tau_n + E_\nu H, x_i) = U(\tau_n, x_i) + F\left(\tau_n, x_i, U(\tau_n, x_i), \frac{\partial U}{\partial x}\right) E_\nu H +$$

$$+ \frac{1}{2!}\left(\frac{DF}{D\tau}(E_\nu H) + \frac{DF}{DV}(FE_\nu H) + \frac{DF}{DW}\left(\frac{\partial}{\partial x} FE_\nu H\right)\right) E_\nu H +$$

$$+ \frac{1}{3!}\left\{\frac{D^2F}{D\tau^2}(E_\nu H)(E_\nu H) + \frac{D^2F}{DV^2}(FE_\nu H)(FE_\nu H) + 2\frac{D^2F}{D\tau DV}(E_\nu H)(FE_\nu H) +\right.$$

$$+ 2\frac{D^2F}{D\tau DW}(E_\nu H)\left(\frac{\partial F}{\partial x} E_\nu H\right) + 2\frac{D^2F}{DVDW}(FE_\nu H)\left(\frac{\partial}{\partial x} FE_\nu H\right) +$$

$$+ \frac{D^2F}{DW^2}\left(\frac{\partial}{\partial x} FE_\nu H\right)\left(\frac{\partial}{\partial x} FE_\nu H\right) + \frac{DF}{DV}\left\{\frac{DF}{D\tau}(E_\nu H)(E_\nu H) + \frac{DF}{DV}(FE_\nu H)(E_\nu H) +\right.$$

$$+ \frac{DF}{DW}\left(\frac{\partial F}{\partial x} E_\nu H\right)(E_\nu H)\}(E_\nu H) +$$

$$+ \frac{DF}{DW}\left[\frac{\partial}{\partial x}\left(\frac{DF}{D\tau}\right)(E_\nu H)(E_\nu H) + \frac{\partial}{\partial x}\left(\frac{DF}{DV}\right)(FE_\nu H)(E_\nu H) +\right.$$

$$\left.+ \frac{\partial}{\partial x}\left(\frac{DF}{DW}\right)\left(\frac{\partial F}{\partial x} E_\nu H)(E_\nu H\right)\right] E_\nu H + \ldots$$

Denoting by D^* the total derivative with respect to H we have

$$\frac{D^*}{DH} K_{jl}^r \Big|_{H=0} = \frac{DF}{D\tau}\Gamma_j^\nu \cdot + \frac{DF}{DV}(FE_\nu \cdot + \sum_l\sum_\mu F B_{jpln}^{\mu\nu} \cdot) \cdot +$$

$$+ \frac{DF}{DW}\left(\frac{\partial F}{\partial x} E_r \cdot + \sum_p\sum_\mu \frac{\partial F}{\partial x} C_{jlin}^{\mu r} \cdot\right) \cdot$$

$$\frac{D^{*2}}{DH^2} K_{jin}^r \Big|_{H=0} = \frac{D^2F}{D\tau^2}\Gamma_j^r \cdot \Gamma_j^r \cdot + \frac{D^2F}{DV^2}(F(E_r \cdot + \sum\sum B \cdot)^{(2)}) +$$

$$+ \frac{D^2F}{D\tau DV}(\Gamma_j^r \cdot (F(E_r \cdot + \sum\sum B \cdot)) + F(E_r \cdot + \sum\sum B \cdot)\Gamma_j^r \cdot) +$$

$$+ \frac{D^2F}{DW^2}\left(\frac{\partial F}{\partial x}(E_r \cdot + \sum\sum C \cdot)\right)^{(2)} + \frac{D^2F}{D\tau DW}\left(\frac{\partial F}{\partial x}(E_r \cdot + \sum\sum C \cdot)\Gamma_j^r \cdot +\right.$$

$$+\Gamma_j^r\cdot\left(\frac{\partial F}{\partial x}(E_{r'}+\sum\sum C.)\right).+\frac{D^2F}{DVDW}\left(\frac{\partial F}{\partial x}(E_{v'}+\sum\sum C.)(F(E_{r'}+\sum\sum B.)+\right.$$

$$+F(E_{r'}+\sum\sum B.)\left(\frac{\partial F}{\partial x}\right)(E_{r'}+\sum\sum C.)\Big).+\frac{DF}{DV}\left\{\frac{1}{2}\left(\frac{DF}{D\tau}E_{r'}\frac{DF}{DV}FE_{r'}+\right.\right.$$

$$+\frac{DF}{DW}\frac{\partial F}{\partial x}E_{v'}\Big)+\sum\sum\frac{D^rK}{DH}|_{H=0}B.\Big\}E_{r'}+\frac{DF}{DW}\left\{\frac{1}{2}\left(\frac{\partial}{\partial x}\frac{DF}{D\tau}E_{v'}+\right.\right.$$

$$+\frac{\partial}{\partial x}\left(\frac{DF}{DV}\right)FE_{i'}+\frac{\partial}{\partial x}\left(\frac{DF}{DW}\right)\frac{\partial F}{\partial x}E_{r'}\Big)+M_{j\nu n}^r\left(\frac{\partial F}{\partial x}(E_{r'}+\sum\sum C.)\right)\Big\}E_{r'}$$

For $N=1$ and V_2^{n+m} defined by

$$V_i^{n+m}=U(\tau_n+E_pH,x_t)+\sum F(\tau_n+\Gamma^r.H,x_i,U(\tau_n+E_rH,x_i),R_{in}^rU)A_m^rH \tag{56}$$

identifying the development of (56) and (53) we have:

$$\sum_{j=1}^{2}f_{ij}m_jh_j=\sum_{j=1}^{2}f_{ij}p_jh_j+\sum_{\nu<m}\sum_{j=1}^{2}f_{ij}\sum_{k=1}^{2}a_{jk}^{\nu}\lambda_k\quad(i=1.2)$$

for H and for H^2

$$n_jm_kh_jh_k=p_jp_kh_jh_k+2\sum_{\nu=m}\sum_{l=1}^{2}\sum_{s=1}^{2}\gamma_{kp}^{\nu}a_{js}^{\nu}h_lh_s \tag{57}$$

From (57) we deduce

$$p_i^2+2\sum_{\nu}a_{ii}^{\nu}\gamma_{ii}^{\nu}=m_i^2,\quad i=1.2$$

$$p_ip_j+2\sum\gamma_{ii}^{\nu}a_{jj}^{\nu}+2\sum\gamma_{ij}^{\nu}a_{ji}^{\nu}=m_im_j,\quad i\neq j$$

$$\sum(\gamma_{ii}^{\nu}a_{ij}^{\nu}+a_{ii}^{\nu}\gamma_{ij}^{\nu})=0,\quad i\neq j$$

$$\sum\gamma_{ii}^{\nu}a_{jt}^{i}=0,\quad i\neq j,$$

$$\sum\gamma_{ij}^{\nu}a_{jj}^{\nu}=0,\quad i\neq j,$$

$$\sum\gamma_{ij}^{\nu}a_{ij}^{t}=0,\quad i\neq j$$

$$m_i^2=p_i^2+2\sum v_ia_{ii}^{\nu},\quad i=1.2$$

$$\sum v_ka_{ij}^2=0,\ i\neq j$$

$$m_im_i=p_1p_2+2\sum v_ka_{jj},\quad k\neq j.$$

One can obtain two R - K methods of local order two if we consider

$$m=(1,1)\quad p=(1,0),\quad \nu=(1,1),\quad (1,0),$$
$$m=(1,1)\quad p=(0,1)\quad \nu=(1,1),\quad (0,1),$$

that is

$$V_i^{(n_1+1,\,n_2+1)}=V_i^{(n_1+1,\,n_2)}+F\left(\tau_n+\begin{pmatrix}h_1\\0\end{pmatrix},x_i,V_2^{(n_1+1,\,n_2)},R_x(V_2^{(n_1+1,\,n_2)})\right)\begin{pmatrix}0\\h_2\end{pmatrix},$$

$$V_i^{(n_1+1,\,n_2+1)}=V_i^{(n_1,\,n_2+1)}+F\left(\tau_n+\begin{pmatrix}0\\h_2\end{pmatrix},x_i,V_i^{(n_1,\,n_2+1)},R_x(V_i^{(n_1,\,n_2+1)})\right)\begin{pmatrix}h_1\\0\end{pmatrix}.$$

This method permits us a paralel processing in the sens of I. M i k l o s k o [1].

We can prove that this method has the global order of approximation h_1+h_2 by the same argument utilized in the one - dimensional case.

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THE GENERAL PROBLEM OF THE ORTHOTROPIC ALMOST CYLINDRICAL BEAMS

BY

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1. General equations and statement of the problem. We shall consider a beam limited by two planes $x_3=0$, $x_3=h$ $(h>0)$ and by a surface $\mathcal{F}$ given by the equation

$$f[x_1(1-\varepsilon x_3),\ x_2(1-\varepsilon x_3)]=0, \tag{1.1}$$

where ε is a small parameter, the square and higher powers of which can be neglected. Such beams are called „almost cylindrical beams", and many results are known in connection with them, specially for the problems of Saint-Venant [1], [2], [5], [7]—[9], etc. *).

In this paper we shall study the almost cylindrical beams loaded on the lateral surface and on the ends.

Let $\mathcal{V}$ be the domain occupied by the beam. We suppose that there are not body forces so that the equations of equilibrium are given by

$$\frac{\partial\sigma_{ij}}{\partial x_j}=0\qquad\text{in }\mathcal{V}\,**). \tag{1.2}$$

The tractions applied on the lateral surface are

$$\sigma_{ij}n_j=\sum_{k=0}^{n}\tau_i^{(k)}(x_1,x_2)\,x_3^k\quad\text{on }\mathcal{F}, \tag{1.3}$$

where the functions $\tau_i^{(k)}(x_1,x_2)$ are given and n_1, n_2, n_3 are the components of the unit vector of the outward normal to $\mathcal{F}$.

On the end $x_3=0$, there are given the resultant and the resultant moment

$$\mathrm{R}=(R_1,R_2,R_3),\quad \mathrm{M}=(M_1,M_2,M_3), \tag{1.4}$$

where R_1, R_2, R_3 and M_1, M_2, M_3 are the given components of the resultant and of the resultant moment.

On the end $x_3=h$, there are applied tractions so that the beam is in equilibrium. We shall call the problem defined by (1.2), (1.3), (1.4) as the *general problem of the almost cylindrical beams.*

The material of the beam is orthotropic so that the relations between the stress components σ_{ij} and the strain components γ_{ij} are given by

*) A more complete bibliography may be found in [2].
**) We use the summation convention over repeated indices.