$$+\Gamma_j^r\cdot\left(\frac{\partial F}{\partial x}(E_{r'}+\sum\sum C.)\right).+\frac{D^2F}{DVDW}\left(\frac{\partial F}{\partial x}(E_{v'}+\sum\sum C.)(F(E_{v'}+\sum\sum B.)+\right.$$

$$+F(E_{v'}+\sum\sum B.)\left(\frac{\partial F}{\partial x}\right)(E_{v'}+\sum\sum C.)\Big)\cdot+\frac{DF}{DV}\left\{\frac{1}{2}\left(\frac{DF}{D\tau}E_{v'}\frac{DF}{DV}FE_{v'}+\right.\right.$$

$$+\frac{DF}{DW}\frac{\partial F}{\partial x}E_{v'}\Big)+\sum\sum\frac{D^rK}{DH}\Big|_{H=0}B.\Big\}E_{v'}+\frac{DF}{DW}\left\{\frac{1}{2}\left(\frac{\partial}{\partial x}\frac{DF}{D\tau}E_{v'}+\right.\right.$$

$$+\frac{\partial}{\partial x}\left(\frac{DF}{DV}\right)FE_{v'}+\frac{\partial}{\partial x}\left(\frac{DF}{DW}\right)\frac{\partial F}{\partial x}E_{v'}\Big)+M^r_{j1n}\left(\frac{\partial F}{\partial x}(E_{v'}+\sum\sum C.)\right)\Big\}E_{v'}.$$

For $N=1$ and V_2^{n+m} defined by

$$V_i^{n+m}=U(\tau_n+E_pH,x_i)+\sum F(\tau_n+\Gamma^r.H,x_i.U(\tau_n+E_vH,x_i),R^r_{in}U)A^r_mH \tag{56}$$

identifying the development of (56) and (53) we have:

$$\sum_{j=1}^{2}f_{ij}m_jh_j=\sum_{j=1}^{2}f_{ij}p_jh_j+\sum_{\nu<m}\sum_{j=1}^{2}f_{ij}\sum_{k=1}^{2}a^{\nu}_{jk}\lambda_k\quad(i=1.2)$$

for H and for H^2

$$n_jm_kh_jh_k=p_jp_kh_jh_k+2\sum_{\nu=m}\sum_{l=1}^{2}\Big[\sum_{s=1}^{2}\gamma^{\nu}_{kp}a^{\nu}_{js}h_lh_s \tag{57}$$

From (57) we deduce

$$p_i^2+2\sum_{\nu}a^{\nu}_{ii}\gamma^{\nu}_{ii}=m_i^2,\quad i=1.2$$

$$p_ip_j+2\sum\gamma^{\nu}_{ii}a^{\nu}_{jj}+2\sum\gamma^{\nu}_{ij}a^{\nu}_{ji}=m_im_j,\quad i\neq j$$

$$\sum(\gamma^{\nu}_{ii}a^{\nu}_{ij}+a^{\nu}_{ii}\gamma^{\nu}_{ij})=0,\quad i\neq j$$

$$\sum\gamma^{\nu}_{ii}a^{i}_{jr}=0,\quad i\neq j,$$

$$\sum\gamma^{\nu}_{ij}a^{\nu}_{jj}=0,\quad i\neq j,$$

$$\sum\gamma^{\nu}_{ij}a^{t}_{ij}=0,\quad i\neq j$$

$$m_i^2=p_i^2+2\sum v_ia^{\nu}_{ii},\quad i=1.2$$

$$\sum v_ka^2_{ij}=0,\quad i\neq j$$

$$m_im_i=p_1p_2+2\sum v_ka_{jj},\quad k\neq j.$$

One can obtain two R-K methods of local order two if we consider

$$m=(1,1)\quad p=(1,0),\quad \nu=(1,1),\quad (1,0),$$
$$m=(1,1)\quad p=(0,1)\quad \nu=(1,1),\quad (0,1),$$

that is

$$V_i^{(n_1+1,\,n_2+1)}=V_i^{(n_1+1,\,n_2)}+F\left(\tau_n+\begin{pmatrix}h_1\\0\end{pmatrix},x_i,V_2^{(n_1+1,\,n_2)},R_x(V_2^{(n_1+1,\,n_2)})\right)\begin{pmatrix}0\\h_2\end{pmatrix},$$

$$V_i^{(n_1+1,\,n_2+1)}=V_i^{(n_1,\,n_2+1)}+F\left(\tau_n+\begin{pmatrix}0\\h_2\end{pmatrix},x_i,V_i^{(n_1,\,n_2+1)},R_x(V_i^{(n_1,\,n_2+1)})\right)\begin{pmatrix}h_1\\0\end{pmatrix}.$$

This method permits us a paralel processing in the sens of I. M i k l o s k o [1].

We can prove that this method has the global order of approximation h_1+h_2 by the same argument utilized in the one-dimensional case.

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THE GENERAL PROBLEM OF THE ORTHOTROPIC ALMOST CYLINDRICAL BEAMS

BY

C. I. BORŞ and I. CRĂCIUN

1. General equations and statement of the problem. We shall consider a beam limited by two planes $x_3=0$, $x_3=h$ $(h>0)$ and by a surface $\mathcal{F}$ given by the equation

$$f[x_1(1-\varepsilon x_3),\ x_2(1-\varepsilon x_3)]=0, \tag{1.1}$$

where ε is a small parameter, the square and higher powers of which can be neglected. Such beams are called „almost cylindrical beams", and many results are known in connection with them, specially for the problems of Saint-Venant [1], [2], [5], [7]—[9], etc.*).

In this paper we shall study the almost cylindrical beams loaded on the lateral surface and on the ends.

Let $\mathcal{V}$ be the domain occupied by the beam. We suppose that there are not body forces so that the equations of equilibrium are given by

$$\frac{\partial\sigma_{ij}}{\partial x_j}=0\qquad\text{in } \mathcal{V}^{**}). \tag{1.2}$$

The tractions applied on the lateral surface are

$$\sigma_{ij}n_j=\sum_{k=0}^{n}\tau_i^{(k)}(x_1,x_2)\,x_3^k\quad\text{on } \mathcal{F}, \tag{1.3}$$

where the functions $\tau_i^{(k)}(x_1,x_2)$ are given and n_1, n_2, n_3 are the components of the unit vector of the outward normal to $\mathcal{F}$.

On the end $x_3=0$, there are given the resultant and the resultant moment

$$\mathbf{R}=(R_1,R_2,R_3),\quad \mathbf{M}=(M_1,M_2,M_3), \tag{1.4}$$

where R_1, R_2, R_3 and M_1, M_2, M_3 are the given components of the resultant and of the resultant moment.

On the end $x_3=h$, there are applied tractions so that the beam is in equilibrium. We shall call the problem defined by (1.2), (1.3), (1.4) as the *general problem of the almost cylindrical beams.*

The material of the beam is orthotropic so that the relations between the stress components σ_{ij} and the strain components γ_{ij} are given by

*) A more complete bibliography may be found in [2].
**) We use the summation convention over repeated indices.

$$
(1.5)\qquad \begin{cases} \sigma_{11}=C_{11}\gamma_{11}+C_{12}\gamma_{22}+C_{13}\gamma_{33}, & \sigma_{22}=C_{12}\gamma_{11}+C_{22}\gamma_{22}+C_{23}\gamma_{33}, \\ \sigma_{33}=C_{13}\gamma_{11}+C_{23}\gamma_{22}+C_{33}\gamma_{33}, & \sigma_{23}=C_{44}\gamma_{23}, \quad \sigma_{31}=C_{55}\gamma_{31}, \quad \sigma_{12}=C_{66}\gamma_{12}, \end{cases}
$$

where C_{ij} are constants which characterize the elastic properties of the material of the beam and the strain components γ_{ij} are connected with the displacements components u_i by

$$
\gamma_{ii}=\frac{\partial u_i}{\partial x_i}\ \text{(not summed)}, \quad \gamma_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\ (i\neq j).
$$

The inverse relations of (1.5) are taken in the form

$$
\gamma_{11}=\frac{1}{E}(\nu_{11}\sigma_{11}+\nu_{12}\sigma_{22}-\nu_1\sigma_{33}), \quad \gamma_{23}=\frac{1}{C_{44}}\sigma_{23},
$$

$$
(1.5')\qquad \gamma_{22}=\frac{1}{E}(\nu_{12}\sigma_{11}+\nu_{22}\sigma_{22}-\nu_2\sigma_{33}), \quad \gamma_{31}=\frac{1}{C_{55}}\sigma_{31},
$$

$$
\gamma_{33}=\frac{1}{E}(-\nu_1\sigma_{11}-\nu_2\sigma_{22}+\sigma_{33}), \quad \gamma_{12}=\frac{1}{C_{66}}\sigma_{12}.
$$

The components γ_{ij} must satisfy the conditions of compatibility of Saint-Venant

$$
(1.6)\qquad \frac{\partial^2\gamma_{11}}{\partial x_2^2}+\frac{\partial^2\gamma_{22}}{\partial x_1^2}=\frac{\partial^2\gamma_{12}}{\partial x_1\,\partial x_2},\ \ldots\ \frac{\partial}{\partial x_1}\left(-\frac{\partial\gamma_{23}}{\partial x_1}+\frac{\partial\gamma_{31}}{\partial x_2}+\frac{\partial\gamma_{12}}{\partial x_3}\right)=2\frac{\partial^2\gamma_{11}}{\partial x_2\,\partial x_3},\ \ldots
$$

If we make up the transformation (see also [2], ch. X)

$$
(1.8)\qquad \begin{cases} \xi_1=x_1(1-\varepsilon x_3), & \xi_2=x_2(1-\varepsilon x_3), & \xi_3=x_3, \\ x_1=\xi_1(1+\varepsilon\xi_3), & x_2=\xi_2(1+\varepsilon\xi_3), & x_3=\xi_3, \end{cases}
$$

then the surface (1.1) becomes

$$
(1.8)\qquad f(\xi_1,\ \xi_2)=0,
$$

which is a cylindrical surface $\mathcal{F}^*$ in the space ξ_1, ξ_2, ξ_3. We shall denote by $\mathcal{V}^*$ the domain of the transformed beam, by S the domain of a cross-section of the cylinder (1.8) and by Γ the boundary of S.

We shall take the axes of coordinates so that

$$
(1.9)\qquad \int_S \xi_1\,d\sigma=0, \quad \iint_S \xi_2\,d\sigma=0, \quad \iint_S \xi_1\xi_2\,d\sigma=0.
$$

It is easy to prove the following formulae

$$
(1.10)\qquad \frac{\partial}{\partial x_1}=(1-\varepsilon\xi_3)\frac{\partial}{\partial\xi_1}, \quad \frac{\partial}{\partial x_2}=(1-\varepsilon\xi_3)\frac{\partial}{\partial\xi_2}, \quad \frac{\partial}{\partial x_3}=(1-\varepsilon\xi_3)\frac{\partial}{\partial\xi_3}-\varepsilon\nabla,
$$

where by ∇ we denote the differential operator

$$
\nabla=\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}-\xi_3\frac{\partial}{\partial\xi_3}.
$$

If we denote the unit vector of the outward normal to the cylindrical surface $\mathcal{F}^*$ by $n^*=(n_1^*,\ n_2^*,\ 0)$, then between n_i and n_α^* we have the realtions [2]

$$
(1.11)\qquad n_1=n_1^*, \quad n_2=n_2^*, \quad n_3=-\varepsilon(\xi_1 n_1^*+\xi_2 n_2^*).
$$

As in [6] we have

$$
(1.12)\qquad \tau_i^{(k)}(x_1,\ x_2)=\tau_i^{*(k)}(\xi_1,\ \xi_2)+\varepsilon\left(\xi_1\frac{\partial\tau_i^{*(k)}}{\partial\xi_1}+\xi_2\frac{\partial\tau_i^{*(k)}}{\partial\xi_2}\right)\xi_3,
$$

where $\tau_i^{*(k)}$ have an obvious meaning. It results

$$
(1.13)\qquad \sum_{k=0}^{n}\tau_i^{(k)}(x_1,\ x_2)x_3^k=\sum_{k=0}^{n}\tau_i^{(k)}(\xi_1,\ \xi_2)\xi_3^k+\varepsilon\sum_{k=0}^{n}\left(\xi_1\frac{\partial\tau_i^{*(k)}}{\partial\xi_1}+\xi_2\frac{\partial\tau_i^{*(k)}}{\partial\xi_2}\right)\xi_3^{k+1}.
$$

2. Solution of the stated problem. In an analogous way as in [6] we shall try to find the solution of our problem in the form

$$
(2.1)\qquad u_i(x_1,\ x_2,\ x_3)=u_i^*(\xi_1,\ \xi_2,\ \xi_3)+\varepsilon u_i^0(\xi_1,\ \xi_2,\ \xi_3),
$$

where u_i^0 are additional unknown displacements and u_i^* are the displacements of the following problem

$$
(2.2)\qquad \begin{cases} \sigma_{ij,j}^*=0 \text{ in } \mathcal{V}^*, \quad \sigma_{i\beta}^* n_\beta^*=\sum_{k=0}^{n}\tau_i^{*(k)}(\xi_1,\ \xi_2)\xi_3^k \text{ on } \mathcal{F}^*, \\ \mathbf{R}=(R_1,\ R_2,\ R_3),\ \mathbf{M}=(M_1,\ M_2,\ M_3) \text{ on } \xi_3=0\ ^{*)}. \end{cases}
$$

The problem defined by (2.2) is the general problem for the orthotropic cylindrical beam (1.8) and we know to solve it [3]. Therefore we suppose that the solution of this problem is known.

The stresses σ_{ij} corresponding to the displacements (2.1) are given by

$$
(2.3)\qquad \begin{cases} \sigma_{ii}=(1-\varepsilon\xi_3)\,\sigma_{ii}^*-\varepsilon C_{i3}\nabla u_3^*+\varepsilon\sigma_i^0\ \text{(not summed)}, \\ \sigma_{\alpha3}=(1-\varepsilon\xi_3)\,\sigma_{\alpha3}^*-\varepsilon C_{\mu\mu}\nabla u_\alpha^*+\varepsilon\sigma_{\alpha3}^0\ (\mu=6-\alpha), \\ \sigma_{12}=(1-\varepsilon\xi_3)\sigma_{12}^*+\varepsilon\sigma_{12}^0, \end{cases}
$$

where σ_{ij}^* are the known stresses of the problem (2.2), σ_{ij}^0 are the stresses of the unknown displacements u_i^0.

Taking into account (2.3), (2.2) and (1.10), form the equilibrium equations (1.2) we get

$$
(2.5)\qquad \sigma_{ij,j}^0+X_i=0 \text{ in } \mathcal{V}^*,
$$

where

$$
\begin{cases} X_\alpha=-[\sigma_{\alpha3}^*+\nabla\sigma_{\alpha3}^*+C_{\alpha3}(\nabla u_3^*)_{,\alpha}+C_{\mu\mu}(\nabla u_\alpha^*)_{,3}]=\sum_{k=0}^{n+3}g_\alpha^{(k)}(\xi_1,\ \xi_2)\xi_3^k, \quad (\mu=6-\alpha), \\ X_3=-[\sigma_{33}^*+\nabla\sigma_{33}^*+C_{44}(\nabla u_1^*)_{,1}+C_{55}(\nabla u_2^*)_{,2}+C_{33}(\nabla u_3^*)_{,3}]=\sum_{k=0}^{n+3}g_3^{(k)}(\xi_1,\ \xi_2)\xi_3^k. \end{cases}
$$

*) Everywhere in this paper the index j after comma indicates partial differentiation with respect to ξ_j ; $i, j=1, 2, 3$; $\alpha, \beta=1, 2$.

Under our supposition we can consider the functions $g_i^{(k)}(\xi_1, \xi_2)$ as known. Making use of (2.3), (2.2) and (1.11), from the boundary conditions (1.3) we have

$$
\left\{
\begin{aligned}
&\sigma^0_{\alpha\beta}\, n^*_\beta = \sum_{k=0}^{n+3} f_\alpha^{(k)}(\xi_1, \xi_2)\,\xi_3^k, \\
&\sigma^0_{3\beta}\, n^*_\beta = \sum_{k=0}^{n+4} f_3^{(k)}(\xi_1, \xi_2)\,\xi_3^k \quad \text{on } \mathcal{F}^*,
\end{aligned}
\right. \tag{2.6}
$$

where the functions $f_i^{(k)}(\xi_1, \xi_2)$ depend on the solution of the problem (2.2), so that they are known.

The end conditions (1.4) will be satisfied if the resultant components and the resultant moment components of the stresses on the end $\xi_3=0$ are

$$
\begin{aligned}
&R^0_i = -C_{\mu\mu}\iint_S \nabla u^*_i\, d\sigma \qquad (\mu = 6 - i), \\
&M^0_\alpha = (-1)^\alpha\, C_{33}\iint_S \xi_\beta \nabla u^*_3\, d\sigma \qquad (\alpha \neq \beta), \\
&M^0_3 = \iint_S (C_{55}\,\xi_2 \nabla u^*_i - C_{44}\,\xi_1 \nabla u^*_1)\,d\sigma.
\end{aligned} \tag{2.7}
$$

In this way for the determination of the stresses σ^0_{ij} we obtained a general problem for the cylindrical beam (1.8) defined by the equations (2.5), (2.6), (2.7). The equations (2.5) are with body forces.

It is easy to see that the stresses

$$
\left\{
\begin{aligned}
&\tilde\sigma_{11} = 0, \quad \tilde\sigma_{22} = 0, \quad \tilde\sigma_{12} = 0, \\
&\tilde\sigma_{33} = \sum_{k=0}^{n+3} \frac{1}{(k+1)(k+2)}\,[g_{1,1}^{(k)} + g_{2,2}^{(k)}]\,\xi_3^{k+2} - \sum_{k=0}^{n+2} \frac{1}{k+1}\, g_3^{(k)}\,\xi_3^{k+1}, \\
&\tilde\sigma_{23} = -\sum_{k=0}^{n+3} \frac{1}{k+1}\, g_2^{(k)}\,\xi_3^{k+3}, \qquad \tilde\sigma_{31} = -\sum_{k=0}^{n+3} \frac{1}{k+1}\, g_1^{(k)}\,\xi_3^{k+1},
\end{aligned}
\right. \tag{2.8}
$$

satisfy the equations (2.5), but they do not verify the compatibility conditions.

In order to solve the problem (2.5), (2.6), (2.7), we shall make use by a method developed in [4] for the same problem but without body forces. In this purpose let us put down

$$
\left\{
\begin{aligned}
&\sigma^0_{11} = \sum_{k=0}^{n+3} \Phi_{k,22}\,\xi_3^k - \sum_{k=0}^{n+4} k\psi^1_{k,2}\,\xi_3^{k-1} - \sum_{k=0}^{n+5} k(k-1)\chi^1_{k,2}\,\xi_3^{k-2}, \\
&\sigma^0_{22} = \sum_{k=0}^{n+3} \Phi_{k,11}\,\xi_3^k + \sum_{k=0}^{n+4} k\psi^2_{k,1}\,\xi_3^{k-1} + \sum_{k=0}^{n+5} k(k-1)\chi^2_{k,1}\,\xi_3^{k-2}, \\
&\sigma^0_{33} = 2\sum_{k=0}^{n+5} (\Omega_k - \theta_k)_{,12}\,\xi_3^k + \sum_{k=0}^{n+3} \frac{1}{(k+1)(k+2)}\,[g_{1,1}^{(k)} + g_{2,2}^{(k)}]\,\xi_3^{k+2} - \sum_{k=0}^{n+2} \frac{1}{k+1}\, g_3^{(k)}\,\xi_3^{k+1} \\
&\sigma^0_{23} = -\sum_{k=0}^{n+4} \psi_{k,1}\,\xi_3^k - \sum_{k=0}^{n+5} k(\chi_k - \theta_k + \Omega_k)_{,1}\,\xi_3^{k-1} - \sum_{k=0}^{n+3} \frac{1}{k+1}\, g_2^{(k)}\,\xi_3^{k+1},
\end{aligned}
\right. \tag{2.9}
$$

$$
\left|
\begin{aligned}
&\sigma^0_{31} = \sum_{k=0}^{n+4} \psi_{k,2}\,\xi_3^k + \sum_{k=0}^{n+5} k(\chi_k + \theta_k - \Omega_k)_{,2}\,\xi_3^{k-1} - \sum_{k=0}^{n+3} \frac{1}{k+1}\, g_1^{(k)}\,\xi_3^{k+1}, \\
&\sigma^0_{12} = -\sum_{k=0}^{n+3} \Phi_{h,12}\,\xi_3^k + \sum_{k=0}^{n+5} k(k-1)\,(\Omega_k - \theta_k)\,\xi_3^{k-2},
\end{aligned}
\right.
$$

where

$$
\begin{aligned}
2(\Omega_k - \theta_k)_{,12} = {}& \nu_1 \Phi_{k,22} + \nu_2 \Phi_{k,11} + (k+1)\Big\{\Big(\nu_2 - \frac{E}{2C_{44}}\Big)(\psi_{k+1} + (k+2)\chi_{k+2})^{,2}_{,1} - \\
& - \Big(\nu_1 - \frac{E}{2C_{55}}\Big)(\psi_{k+1} + (k+2)\chi_{k+2})^{,1}_{,2} + \frac{E(k+2)}{2}\Big[\frac{1}{C_{66}}\Phi_{k+2} - \frac{1}{C_{44}}(\Omega_{k+2} - \theta_{k+2})^{,2}_{,1} - \\
& - \frac{1}{C_{55}}(\Omega_{k+2} - \theta_{k+2})^{,1}_{,2} - \frac{(k+3)(k+4)}{C_{66}}(\Omega_{k+4} - \theta_{k+4})^{,12}\Big]\Big\} - \frac{E}{2C_{44}}\, g_2^{(k),2} - \\
& - \frac{E}{2C_{55}}\, g_1^{(k),1} + \frac{1}{k}\, g_3^{(k-1)} - \frac{1}{k(k-1)}\,[g_{1,1}^{(k-2)} + g_{2,2}^{(k-2)}],
\end{aligned}
$$

$$
(k = 0,\ 1,\ 2, \ldots, n+5;\ g_i^{(-p)} = 0, \text{ for } p = 1,\ 2, \ldots),
$$

$$
\theta_k = \frac{1}{4}\,(a_k^{(1)}\,\xi_1^2\,\xi_2 + a_k^{(2)}\,\xi_1\,\xi_2^2) - \frac{1}{2}\,A_k\,\xi_1\,\xi_2 = \theta^*_k - \frac{1}{2}\,A_k\,\xi_1\,\xi_2,
$$

$$
\chi_k = a_k^{(1)}\Pi_1 + a_k^{(2)}\Pi_2 + c_k\Pi, \qquad \psi^{,i} = \int \psi\, d\xi_i,
$$

$\Phi_h(\xi_1, \xi_2)$, $\psi_k(\xi_1, \xi_2)$ being unknown functions and $a_k^{(\alpha)}$, C_h, A_k constants to be determined.

The function χ_k is associated with the function $\omega_k = a_k^{(1)}\varphi_1 + a_k^{(2)}\varphi_2 + C_k\varphi$ by [4]

$$
\left\{
\begin{aligned}
&-(\chi_k - \theta^*_k)_{,1} = C_{44}(\omega_k + C_k\xi_1\,\xi_2)_{,2}, \\
&(\chi_k + \theta^*_k)_{,2} = C_{55}(\omega_k - C_k\xi_1\,\xi_2)_{,1}.
\end{aligned}
\right.
$$

From the above equations we get for the functions χ_k and ω_k the following equation

$$
\nabla\chi_k = C_{55}\theta^*_{k,11} - C_{44}\theta^*_{k,22} - 2C_{44}C_{55}C_k \quad \text{in } S, \tag{2.10}
$$

$$
\nabla\omega_k = 2\theta^*_{k,12}, \quad \text{in } S, \tag{2.11}
$$

where

$$
\Delta = C_{55}\frac{\partial^2}{\partial\xi_1^2} + C_{44}\frac{\partial^2}{\partial\xi_2^2}.
$$

The boundary conditions for the functions χ_k and ω_h are taken in the form

$$
\chi_{k,2}n^*_1 - \chi_{k,1}n^*_2 = -(\theta^*_{k,2}\,n^*_1 + \theta^*_{k,1}\,n^*_2) \text{ on } \Gamma, \tag{2.12}
$$

$$
\mathcal{D}\omega_k = C_k(C_{55}\,\xi_2\,n^*_1 - C_{44}\,\xi_1 n^*_2) \quad \text{on } \Gamma, \tag{2.13}
$$

where

$$
\mathcal{D} = C_{55}\,n^*_1\frac{\partial}{\partial\xi_1} + C_{44}\,n^*_2\frac{\partial}{\partial\xi_2}.
$$

From the equations (2.10), (2.12), and (2.11), (2.13) it results immediately the equations defining the functions H_1, H_2, H and φ_1, φ_2, φ. It is easy to prove that all these functions exist. We shall suppose that we know these functions.

The stresses σ^0_{ti} given by (2.9) satisfy identically the equilibrium equations (2.5). The compatibility conditions of Saint-Venant will be satisfied if the functions Φ_k and ψ_k verify the equations

(2.14) $$\Theta^{(2)}_{k,11}+\Theta^{(1)}_{k,22}=0 \quad \text{in } S, \ (k=0,\ 1,\ 2,\dots,\ n+3),$$

(2.15) $$\nabla\psi_k=(k+1)[-C_{55}\Omega_{k+1,11}+C_{44}\Omega_{k+1,22}+C_{44}C_{55}\Theta^{(1),1}_{k+1,2}]-\frac{1}{k}\,[C_{55}\,g^{(k-1)}_{2,1}-C_{44}\,g^{(k-1)}_{1,2}] \ \text{in } S,$$

$$(k=0,\ 1,\ 2,\dots,\ n+4),$$

where

$$\Theta^{(\alpha)}_k=\frac{1}{C_{66}}\,[\Phi_{k,\alpha\alpha}-(k+1)\,(k+2)\,(\Omega_{k+2}-\theta_{k+2})^{;\beta}_{,\alpha}]+\frac{2}{E}\Big\{\nu_{\alpha\alpha}\Phi_{k,\beta\beta}+\nu_{12}\Phi_{k,\alpha\alpha}-$$

$$-\nu_\alpha\left[2(\Omega_k-\theta_k)_{,12}-\frac{1}{k}\,g^{(k-1)}_3+\frac{1}{k(k-1)}\,(g^{(k-2)}_{1,1}+g^{(k-2)}_{2,2})\right]+$$

$$+(k+1)\,[(-1)^\beta\nu_{12}(\psi_{k+1}+(k+2)\chi_{k+2})^{;\beta}_{,\alpha}+(-1)^\alpha\,\nu_{\alpha\alpha}(\psi_{k+1}+(k+2)\,\chi_{k+2})^{;\alpha}_{,\beta}]\Big\},\quad (\alpha\neq\beta).$$

From (2.6) we obtain the following boundary conditions for the functions Φ_k and ψ_k,

(2.16) $$\Phi_{k,\beta\beta}\,n^*_\alpha-\Phi_{k,12}\,n^*_\beta=f^{(k)}_\alpha+(k+1)\,\{(-1)^\beta[\psi_{k+1}+(k+2)\,\chi_{k+2}]^{;\alpha}_{,\beta}\,n^*_\alpha-$$

$$-(k+2)\,(\Omega_{k+2}-\theta_{k+2})\,n^*_\beta,\ \text{on } \Gamma,\ (\alpha\neq\beta),$$

(2.17) $$\psi_{k,2}\,n^*_1-\psi_{k,1}\,n^*_2=f^{(k)}_3-\frac{1}{k}\,(g^{(k-1)}_1\,n^*_1+g^{(k-1)}_2\,n^*_2)+$$

$$+(k+1)\,[\Omega_{k+1,1}\,n^*_2+\Omega_{k+1,2}\,n^*_1]+\frac{k+1}{2}\,A_{k+1}\,(\xi_1\,n^*_1+\xi_2\,n^*_2),\quad \text{on } \Gamma.$$

If we denote with $\mathcal{F}_i(\xi_1,\ \xi_2)$ the expressions from the right hand of (2.16), (2.17), then for the existence of the solution of the problems (2.14), (2.16) and (2.15), (2.17), we must have

(2.18) $$\begin{cases}\displaystyle\int_\Gamma \mathcal{F}_i\,(\xi_1,\ \xi_2)\,ds=0,\\[2ex] \displaystyle\int_\Gamma [\xi_2\,\mathcal{F}_1(\xi_1,\ \xi_2)-\xi_1\,\mathcal{F}_2(\xi_1,\ \xi_2)]ds=0.\end{cases}$$

Taking the obvious identity

(2.19) $$\int_\Gamma \omega_k\,\mathcal{D}\,f ds=\int_\Gamma f\,\mathcal{D}\,\omega_k-\iint_S f\nabla\,\omega_k\,d\sigma,$$

and making here f equal with ξ_1, ξ_2, $\varphi(\xi_1,\ \xi_2)$, then from (2.18) we can derive for the constants $a^{(\alpha)}_{k+2}$, C_{k+2}, A_{k+1} the following formulae

(2.20) $$\begin{cases} I_{\beta\beta}\,a^{(\alpha)}_{k+2}=\dfrac{1}{(k+1)(k+2)}\displaystyle\int_\Gamma [f^{(k)}_\alpha+(-1)^\beta\,(k+1)\,\psi_{k+1}\,n^*_\beta]ds-\iint_S \Omega_{k+2,\beta}\,d\sigma,\quad(\alpha\neq\beta),\\[2ex] D_tC_{k+2}=\dfrac{1}{(k+1)\,(k+2)}\displaystyle\int_\Gamma \{[f^{(k)}_1+(k+1)\,\psi_{k+1,2}^{;1}\,n^*_1]\,\xi_2-[f^{(k)}_2-(k+1)\,\psi_{k+1,1}^{;2}\,n^*_1]\,\xi_1\}\,ds+\\[2ex] \quad+\displaystyle\iint_S\{[(\tilde\chi_{k+2}+\theta_{k+2}-\Omega_{k+2})_{,2}\,\xi_2+(\tilde\chi_{k+2}-\theta_{k+2}+\Omega_{k+2})_{,1}\,\xi_1\}\,d\sigma\\[1ex] \qquad\qquad (k=0,\ 1,\ 2,\dots,\ n+3),\\[2ex] A_{k+1}=\dfrac{-1}{k+1}\displaystyle\int_\Gamma f^{(k)}_3\,ds-\dfrac{1}{k(k+1)}\iint_S [g^{(k-1)}_{1,1}+g^{(k-1)}_{2,2}]\,d\sigma-2\iint_S \Omega_{k+1,12}\,d\sigma\\[1ex] \qquad\qquad (k=0,\ 1,\ 2,\dots,n+4).\end{cases}$$

In the above formulare we made the notations

$$I_{\alpha\alpha}=\iint_S \xi^2_\beta d\sigma\quad(\alpha\neq\beta),$$

(2.21) $$\tilde\chi_k=a^{(1)}_k\,H_1+a^{(2)}_k\,H_2,$$

$$D_t=-\iint_S(\xi_1\,H_{,1}+\xi_2\,H_{,2})\,d\sigma=\iint_S[C_{44}\,\xi^2_1+C_{55}\,\xi^2_2+C_{44}\,\xi_1\varphi_{,2}-C_{55}\,\xi_2\,\varphi_{,1}]\,d\sigma>0.$$

Everywhere in this paper we must consider that

(2.22) $$\Omega_{n+5+p}=0;\quad \Phi_{n+3+p}=0;\quad \psi_{n+4+p}=0;\quad \chi_{n+5+p}=0;\quad \theta_{n+5+p}=0.\quad (p=1,\ 2,\dots)$$

From the end conditions (2.7) we obtain

(2.23) $$\begin{cases} I_{\beta\beta}\,a^{(\alpha)}_1=R^0_\alpha+(-1)^\beta\displaystyle\iint_S [\psi_0+(-1)^\alpha\Omega_1]_{,\beta}\,d\sigma,\ (\alpha\neq\beta),\\[2ex] SA_0=-R^0_3-2\displaystyle\iint_S \Omega_{0,12}\,d\sigma,\\[2ex] c_1\,D_t=-M^0_3+\displaystyle\iint_S[\xi_1(\psi_0+\tilde\chi_1-\theta_1+\Omega_1)_{,1}+(\psi_0+\tilde\chi_1+\theta_1-\Omega_1)_{,2}]d\sigma,\\[2ex] I_{\beta\beta}\,a^{(\alpha)}_0=(-1)^\alpha M^0_\beta-2\displaystyle\iint_S \xi_\alpha\Omega_{0,12}\,d\sigma,\quad(\alpha\neq\beta),\end{cases}$$

where by S we have denoted also the area of the cross-section S. The constant C_0 may be taken as zero.

The calculus of the constants and of the unknown functions can be made in the same way as in [4].

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Faculty of Mathematics
University of Iaşi
Romania

PROBLEMA GENERALĂ A BARELOR ORTOTROPE APROAPE CILINDRICE

Rezumat

În acest articol se rezolvă problema echilibrului elastic al barelor elastice aproape cilindrice omogene şi ortotrope, pentru care suprafaţa laterală este dată de ecuaţia (1.1). Încărcările la care este supusă bara studiată sînt date de (1.3) şi (1.4).

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AN UNIQUENESS THEOREM FOR COMPRESSIBLE MICROPOLAR FLUIDS

BY

VALERIU A. SAVA

In recent years many papers have presented various aspects of the theory of micropolar fluids introduced by Eringen [1]. In this note we examine the problem of uniqueness of a solution for the theory of homogeneous, compressible micropolar fluids. For the incompressible micropolar fluids the problem of existence and uniqueness of the solution has been studied by Förste [2], [3] and Sava [4].

The equations of motions of a homogeneous, compressible micropolar fluid take the form [1]

$$(\lambda + 2\mu + k)\nabla\nabla \cdot v - (\mu + k)\nabla\times\nabla\times v + k\nabla\times\varphi - \nabla\pi + \rho f = \rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right), \tag{1}$$

$$(\alpha + \beta + \gamma)\nabla\nabla \cdot \varphi - \gamma\nabla\times\nabla\times\varphi + k\nabla\times v - 2k\varphi + \rho l = \rho j\left(\frac{\partial \varphi}{\partial t} + v \cdot \nabla\varphi\right), \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{3}$$

in which v, φ, f, l represent respectively the velocity, microrotation, body force, and body couple vectors. ρ, π and j are the density, pressure and gyration respectively. Equations (1), (2), (3) which must hold throughout the region Ω of space occupied by the fluid and at any moment $t \in [0, T]$ are subject to the initial conditions

$$v(x, 0) = V_0(x) \quad \text{in } \Omega, \tag{4}$$

$$\varphi(x, 0) = \varphi_0(x) \quad \text{in } \Omega, \tag{5}$$

$$\rho(x, 0) = \rho_0(x) \quad \text{in } \Omega, \tag{6}$$

and the boundary conditions

$$v(x, t) = V(x, t) \quad \text{on } S, \tag{7}$$

$$\varphi(x, t) = \Phi(x, t) \quad \text{on } S; \tag{8}$$

in the points where $Vn \leqq 0$, we assume

$$\rho(x, t) = U(x, t) \quad \text{on } S, \tag{9}$$

where S refers to the boundary of Ω (n = unit exterior direct normal to S).

We shall assume that the fluid is barotropic so that

$$\pi = \pi(\rho).$$

The viscosity μ, vortex viscosity k and the material coefficients λ, α, β, γ are all constant. The body force and the body couple are assumed to be continuous vector functions of ρ, x, t and have bounded