REFERENCES

1. Borş, C. I. — *Saint-Venant's problem for orthotropic almost cylindrical beams*, Rev. Roum. Sci. Techn.-Mec. Appl., 16, 6 (1971) p. 1275—1284.
2. Borş, C. I. — *Teoria elasticităţii corpurilor anizotrope*, Edit. Acad., Bucureşti, 1970.
3. Borş, C. I. — *On the general problem of orthotropic beams loaded on the lateral surface and on the ends*, An. şt. Univ. Iaşi, ser. nouă, sect. I Mat. XIX, (1973).
4. Borş, C. I. — *Solution du problème des poutres anisotropes chargées sur la surface laterale et sur les extrémités par la méthode des fonctions de contrainte*, C. R. Acad. Sci. Paris, Ser. A, t 278, (1974) p. 101—104.
5. Khatiaşvili, G. M. — *Saint-Venant's problem for homogeneous orthotropic bodies similar to prismatic bodies* (in Russian), Trudy Tbilis. Gosud. Univ., 110, (1964) p. 89—101.
6. Khatiaşvili, G. M., Borş, C. I. — *On the deformation of the anisotropic slightly curved beams*, An. şt. Univ. Iaşi, ser. nouă, sect. I Mat. XVIII, f. 2(1972) p. 447—452.
7. Gorgidze, A. Ya. — *Torsion and bending of composite bars which are similar to prismatic bars* (in Russian), Trudy Tbilis. Inst., 15, (1948) p. 117—141.
8. Panov, D. Yu. — *On torsion of almost prismatic bars* (In Russian), Prikl. Math. and Mech., 2,2, (1938).
9. Rukhadze, A. K. — *The bending of almost prismatic beams* (in Russian), Soob. Gruz. Akad. Nauk. SSR, 8, (1948) p. 23—27.

Received 17.V.1974

Faculty of Mathematics
University of Iaşi
Romania

PROBLEMA GENERALĂ A BARELOR ORTOTROPE APROAPE CILINDRICE

Rezumat

În acest articol se rezolvă problema echilibrului elastic al barelor elastice aproape cilindrice omogene şi ortotrope, pentru care suprafaţa laterală este dată de ecuaţia (1.1). Încărcările la care este supusă bara studiată sînt date de (1.3) şi (1.4).

Analele ştiinţifice ale Universităţii „AL. I. CUZA" din Iaşi
Tomul XXII, s. I a, 1976, f. 1.

AN UNIQUENESS THEOREM FOR COMPRESSIBLE MICROPOLAR FLUIDS

BY

VALERIU A. SAVA

In recent years many papers have presented various aspects of the theory of micropolar fluids introduced by Eringen [1]. In this note we examine the problem of uniqueness of a solution for the theory of homogeneous, compressible micropolar fluids. For the incompressible micropolar fluids the problem of existence and uniqueness of the solution has been studied by Förste [2], [3] and Sava [4].

The equations of motions of a homogeneous, compressible micropolar fluid take the form [1]

$$(\lambda + 2\mu + k)\nabla\nabla \cdot v - (\mu + k)\nabla\times\nabla\times v + k\nabla\times\varphi - \nabla\pi + \rho f = \rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right), \tag{1}$$

$$(\alpha + \beta + \gamma)\nabla\nabla \cdot \varphi - \gamma\nabla\times\nabla\times\varphi + k\nabla\times v - 2k\varphi + \rho l = \rho j\left(\frac{\partial \varphi}{\partial t} + v \cdot \nabla\varphi\right), \tag{2}$$

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{3}$$

in which v, φ, f, l represent respectively the velocity, microrotation, body force, and body couple vectors. ρ, π and j are the density, pressure and gyration respectively. Equations (1), (2), (3) which must hold throughout the region Ω of space occupied by the fluid and at any moment $t \in [0, T]$ are subject to the initial conditions

$$v(x, 0) = V_0(x) \quad \text{in} \quad \Omega, \tag{4}$$

$$\varphi(x, 0) = \varphi_0(x) \quad \text{in} \quad \Omega, \tag{5}$$

$$\rho(x, 0) = \rho_0(x) \quad \text{in} \quad \Omega, \tag{6}$$

and the boundary conditions

$$v(x, t) = V(x, t) \quad \text{on} \quad S, \tag{7}$$

$$\varphi(x, t) = \Phi(x, t) \quad \text{on} \quad S; \tag{8}$$

in the points where $Vn \leqq 0$, we assume

$$\rho(x, t) = U(x, t) \quad \text{on} \quad S, \tag{9}$$

where S refers to the boundary of Ω (n = unit exterior direct normal to S).

We shall assume that the fluid is barotropic so that

$$\pi = \pi(\rho).$$

The viscosity μ, vortex viscosity k and the material coefficients λ, α, β, γ are all constant, and conform to the appropriate conditions which will be given below. The body force and the body couple are assumed to be continuous vector functions of ρ, x, t and have bounded derivatives with respect to ρ for all ρ between the minimum and maximum values of ρ in $\Omega\times[0,\ T]$ and $(x,\ t)\in\Omega\times[0,\ T]$.

We assume further that the function π is twice differentiable and the second derivative is bounded for all ρ between the minimum and the maximum values of ρ in $\Omega\times[0,\ T]$.

We shall consider for the equations (1)—(3) the regular solution only, i. e. v, φ and their first derivative with respect to t and second derivatives with respect to x are continuous in $\Omega\times[0,\ T]$; v, φ and their first derivative with respect to x are continuous in $\bar\Omega\times[0,\ T]$ $(\bar\Omega=\Omega\cup S)$; ρ is continuous in $\bar\Omega\times[0,\ T]$ and its first derivatives with respect to x and t are continuous in $\Omega\times[0,\ T]$.

Theorem. *If $\{\rho,\ v,\ \varphi\}$ is a regular solution of (1)—(3) in a regular region (in the sense that the Green-Gauss theorem is valid) Ω, bounded by a closed surface S and if the first derivatives of v, φ with respect to t are bounded in $\Omega\times[0,\ T]$, and the first derivatives of ρ are bounded in $\Omega\times[0,\ T]$, then there is no other regular solution of* (1)—(3) *in* $\Omega\times[0,\ T]$ *which assumes the same initial and boundary data.*

Proof. Let $\{\rho,\ v,\ \varphi\}$ and $\{\rho^*,\ v^*,\ \varphi^*\}$ be two possible solutions of (1)—(3) subject to conditions (4)—(9) and satisfying the condition of the theorem and set

$$\rho^*-\rho=g,\qquad v^*-v=w,\qquad \varphi^*-\varphi=\psi.$$

By substracting the two equations satisfied by the two solutions $\{\rho,\ v,\ \varphi\}$, $\{\rho^*,\ v^*,\ \varphi^*\}$ we obtain

$$(\lambda+2\mu+k)\nabla\nabla\cdot w-(\mu+k)\nabla\times\nabla\times w+k\nabla\times\psi-[\pi'(\rho^*)-\pi'(\rho)]\nabla\rho-$$
$$-\pi'(\rho^*)\nabla g+[\rho^*f(\rho^*,\ x,\ t)-\rho f(\rho,\ x,\ t)]=$$
$$=\rho\frac{\partial w}{\partial t}+g\frac{\partial v^*}{\partial t}+gv^*\cdot\Delta v^*+\rho w.\Delta v^*+\rho v.\Delta w.$$

(10) $$(\alpha+\beta+\gamma)\nabla\nabla\cdot\psi-\gamma\nabla\times\nabla\times\psi+k\nabla\times w-2k\psi+[\rho^*l(\rho^*,\ x,\ t)-\rho l(\rho,\ x,\ t)]=$$
$$=\rho j\frac{\partial\psi}{\partial t}+gj\frac{\partial\varphi^*}{\partial t}+gjv^*\cdot\nabla\varphi^*+\rho jw.\nabla\varphi^*+\rho jv\cdot\nabla\psi,$$
$$\frac{\partial g}{\partial t}+\nabla\cdot(gv)+\nabla\cdot(\rho^*w)=0,$$

where $\pi'(\rho)=d\pi/d\rho$.
Let us consider the functional

$$\mathcal{A}=-\int_\Omega g\frac{\partial v^*}{\partial t}\cdot w dx-j\int_\Omega g\frac{\partial\varphi^*}{\partial t}\cdot\psi dx-\int_\Omega g(v^*.\nabla v^*).wdx-$$
$$-j\int_\Omega g(v^*.\nabla\varphi^*).\ \psi dx-\int_\Omega \rho(w.\nabla v^*).wdx-j\int_\Omega\rho(w.\nabla\varphi^*).\ \psi dx-$$

(11) $$-\int_\Omega\rho(v.\nabla w).wdx-j\int_\Omega\rho(v\ .\nabla\psi).\ \psi dx-\int_\Omega[\pi'(\rho^*)-\pi'(\rho)]\nabla\rho\ .\ wdx+$$
$$+\int_\Omega[\rho^*f(\rho^*,\ x,\ t)-\rho f(\rho,\ x,\ t)]\ .\ wdx+\int_\Omega[\rho^*l(\rho^*,\ x,\ t)-\rho l(\rho,\ x,\ t)].\psi dx+$$
$$+\int_\Omega(H\ .\ w+K\ .\psi)dx+\frac{1}{2}\int_\Omega\frac{\partial\rho}{\partial t}w\ .\ wdx+\frac{1}{2}j\int_\Omega\frac{\partial\rho}{\partial t}\psi.\psi dx,$$

where

$$H=(\lambda+2\mu+k)\nabla\nabla.w-(\mu+k)\nabla\times\nabla\times w+k\nabla\times\psi,$$
$$K=(\alpha+\beta+\gamma)\nabla\nabla\cdot\psi-\gamma\nabla\times\nabla\times\psi+k\nabla\times w-2k\psi.$$

If we denote by $2N_1$, $2N_2$, $2M_1$, $2M_2$ the upper bounds of

$\left|\frac{\partial v^*}{\partial t}\right|$, $\left|v^*\ .\nabla v^*\right|$, $\left|\frac{\partial\varphi}{\partial t}\right|$ and $\left|v^*\ .\nabla\varphi^*\right|$ respectively, then the first four terms in $\mathcal{A}$ satisfy:

$$-\int_\Omega g\frac{\partial v^*}{\partial t}\ .\ wdx-j\int_\Omega g\frac{\partial\varphi^*}{\partial t}\ .\psi dx-\int_\Omega g(v^*\ .\nabla v^*)\ .\ wdx-j\int_\Omega g(v^*.\nabla\varphi^*).\psi dx\leqq$$

(12) $$\leqq[N_1+N_2+j(M_1+M_2)]\int_\Omega g^2dx+(N_1+N_2)\int_\Omega w^2dx+j(M_1+M_2)\int_\Omega\psi^2dx.$$

Using the Green-Gauss theorem in the next two terms in A we get

(13) $$-\int_\Omega\rho(v\ .\ \nabla w).wdx-\int_\Omega\rho(v\ .\nabla\psi).\psi dx\leqq N_3\left(\int_\Omega w^2dx+j\int_\Omega\psi^2dx\right),$$

where $2N_3$ is the upper bound of $|\nabla.(\rho\rho)|$ in $\Omega\times[0,\ T]$. Now let N_4 and M_3 the upper bounds of $\rho|\nabla v^*|$ and $\rho|\nabla\varphi^*|$ in $\Omega\times[0,\ T]$ respectively; then

(14) $$-\int_\Omega\rho(v.\nabla w).wdx-j\int_\Omega\rho(v\ .\nabla\psi)\ .\psi dx\leqq N_4\int_\Omega w^2dx+M_3j\int_\Omega\psi^2dx.$$

From a mean value theorem we obtain

$$|(\pi'(\rho^*)-\pi'(\rho))\nabla\rho.w|\leqq N''g|\nabla\rho.w|\leqq N''N'g|w|\leqq N_5(g^2+w^2).$$

where $N_5=2N'N''$, $|\nabla\rho|\leqq N'$ and N'' is the upper bound of the $\pi''(\rho)$. Hence we can write

(15) $$-\int_\Omega(\pi'(\rho^*)-\pi'(\rho))\nabla\rho.wdx\leqq N_5\int_\Omega(g^2+w^2)dx.$$

In a similar way we can obtain

$$\int_\Omega(\rho^*f(\rho^*,\ x,\ t)-\rho f(\rho,\ x,\ t)).wdx+\int_\Omega(\rho^*l(\rho^*,\ x,\ t)-\rho l(\rho,\ x,\ t)).\psi\leqq$$

(16) $$\leqq(N_6+M_4)\int_\Omega g^2dx+N_6\int_\Omega w^2dx+M_4\int_\Omega\psi^2dx,$$

where $\max\limits_\rho\left|\frac{\partial}{\partial\rho}(\rho f(\rho,\ x,\ t)\right|\leqq 2N_6$ and $\max\limits_\rho\left|\frac{\partial}{\partial\rho}(\rho l(\rho,\ x,\ t)\right|\leqq 2M_4$. By partial integration we get

$$\int_\Omega(H\ .\ w+K.\psi)dx=-\int_\Omega\{(\lambda+2\mu+k)(\nabla.w)^2+\left(\mu+\frac{k}{2}\right)(\nabla\times w)^2+(\alpha+\beta+\gamma)(\nabla.\psi)^2+$$
$$+\gamma(\nabla\times\psi)^2+2k[\psi-\frac{1}{2}(\nabla\times w)]^2\}dx+\int_S\{(\lambda+2\mu+k)(\nabla.w)w.n-$$
$$-(\mu+k)n.[(\nabla\times w)\times w]+kn.(\psi\times w)+(\alpha+\beta+\gamma)(\nabla.\psi)\psi.n-\gamma n.[(\nabla\times\psi)\times\psi]\}ds.$$

But w and ψ are zero on S so that the surface integral is zero. If

(17) $$\lambda + 2\mu + k \geqq 0, \quad \mu + k/2 \geqq 0, \quad \gamma \geqq 0, \quad k \geqq 0, \ \alpha + \beta + \gamma \geqq 0,$$

then

(18) $$\int_\Omega \{(\lambda + 2\mu + k)(\nabla . w)^2 + \left(\mu + \frac{k}{2}\right)(\nabla \times w)^2 + (\alpha + \beta + \gamma)(\nabla . \psi)^2 + \\ + \gamma(\nabla \times \psi)^2 + 2k \ \psi - \frac{1}{2}(\nabla \times w)]^2\} dx \geqq 0,$$

so that

(19) $$\int_\Omega (H . w + K . \psi) dx \leqq 0.$$

Remark. The conditions (17) on the material coefficients are more relaxed than those established by Eringen [1] for positivness of internal energy (18).

Finaly if $\left|\dfrac{\partial \rho}{\partial t}\right| \leqq 2N_7$ in $\Omega \times [0, T]$ then

(20) $$\frac{1}{2}\int_\Omega \frac{\partial \rho}{\partial t}(w^2 + j\psi^2) dx \leqq N_7 \left(\int_\Omega w^2 dx + j \int_\Omega \psi^2 dx\right).$$

From (12)–(16) and (19)–(20), we obtain for the functional $\mathscr{A}$ the inequality:

(21) $$\mathscr{A} \leqq A \int_\Omega g^2 dx + B \int_\Omega w^2 dx - C \int_\Omega \psi^2 dx,$$

where

$$A = N_1 + N_2 + j(M_1 + M_2) + N_5 + N_6 + M_4,$$
$$B = N_1 + N_2 + N_3 + N_4 + N_5 + N_6,$$
$$C = N_3 + M_1 + M_2 + M_3 + jM_4.$$

Now let us consider

(22) $$\pi'(\rho^*) w . \nabla g = \nabla . [\pi'(\rho^*) g w\} - g(\nabla \pi'(\rho^*)) . w - \pi'(\rho^*)) . w - \pi'(\rho^*) g \nabla . w.$$

From the continuity equation (3) yields

$$\nabla . w = -\frac{1}{\rho^*}\left[\frac{\partial g}{\partial t} + \nabla . (gv) + \nabla \rho^* . w\right].$$

Substituting this in (22) we obtain

$$\pi'(\rho^*) w . \nabla g = \nabla . [\pi'(\rho^*) g w] - g\left\{\nabla \pi'(\rho^*) - \frac{\pi'(\rho^*)}{\rho^*}\nabla \rho^*\right\} . w + \frac{1}{2}\frac{\pi'(\rho^*)}{\rho^*}\frac{\partial (g^2)}{\partial t} + \\ + \frac{\pi'(\rho^*)}{\rho^*} g^2 \nabla . w + \frac{\pi'(\rho^*)}{\rho^*} g w . \nabla g.$$

After some obvious calculations we get

$$\pi'(\rho^*) w . \nabla g = \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\pi'(\rho^*)}{\rho^*} g^2\right] + \left\{-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\pi'(\rho^*)}{\rho^*}\right] - \frac{1}{2}\frac{\pi'(\rho^*)}{\rho^*}\nabla . w - \right.$$

$$\left. - \frac{1}{2} w . \nabla \left(\frac{\pi'(\rho^*)}{\rho^*}\right)\right\} g^2 + \nabla . (\pi'(\rho^*) g w) - g\left\{\nabla \pi'(\rho^*) - \frac{\pi'(\rho^*)}{\rho^*}\nabla \rho^*\right\} . w + \frac{1}{2}\nabla . \left[\frac{\pi'(\rho^*)}{\rho^*} g^2 w\right].$$

If

$$\left|\nabla (\pi'(\rho^*) - \frac{\pi'(\rho^*)}{\rho^*}\nabla \rho^*\right| \leqq 2N_8, \text{ and}$$

$$\left|\frac{\partial}{\partial t}\left(\frac{\pi'(\rho^*)}{\rho^*}\right) + \frac{\pi'(\rho^*)}{\rho^*}\nabla . w + w . \nabla \left(\frac{\pi'(\rho^*)}{\rho^*}\right)\right| \leqq 2N_9 \text{ in } \Omega \times [0, T],$$

then

$$-\int_\Omega \pi'(\rho^*) w . \nabla g dx \leqq -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\pi'(\rho^*)}{\rho^*} g^2\right) dx + N_8 \left(\int_\Omega g^2 dx + \int_\Omega w^2 dx\right) + \\ + N_9 \int_\Omega g^2 dx - \int_S \pi'(\rho^*) g w . n ds - \frac{1}{2}\int_S \frac{\pi'(\rho^*)}{\rho^*} g^2 w . n ds.$$

But the surface integrals are zero since w is zero on S and in the points of S where $w . n \leqslant \leqslant 0$, g is zero on S. Hence it follows

(23) $$-\int_\Omega \pi'(\rho^*) w . \nabla g dx \leqq -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\pi'(\rho^*)}{\rho^*} g^2\right) dx + (N_8 + N_9)\int_\Omega g^2 dx + N_8 \int_\Omega w^2 dx.$$

If we consider the scalar product of the first member of (10) by w and ψ and integrate over Ω we obtain

$$\int_\Omega \left(\rho \frac{\partial w}{\partial t} . w + \rho j \frac{\partial \psi}{\partial t} . \psi\right) dx = ci - \int_\Omega \pi'(\rho^*) w . \nabla g dx - \frac{1}{2}\int_\Omega \frac{\partial \rho}{\partial t}(w^2 + j\psi^2) dx.$$

After some calculations, taking into account (21), (23), we get

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left[\rho(w^2 + j\psi^2) + \frac{\pi'(\rho^*)}{\rho^*} g^2\right] dx \leqq A \int_\Omega g^2 dx + B \int_\Omega w^2 dx + C \int_\Omega \psi^2 dx.$$

From physical considerations it results that $\rho > 0$. Moreover, there exists $m > 0$ such that $\rho \geqq m$.
Let n be such that $\pi'(\rho^*)/\rho^* \geqq n$ and let us denote

$$2q = \min\{m, n\}, \quad Q = \max\{A, B, C\}.$$

Then the above inequality becomes

$$q \int_\Omega \frac{\partial}{\partial t}(w^2 + j\psi^2 + g^2) dx \leqq \int_\Omega (w^2 + j\psi^2 + g^2) dx,$$

from which it follows

$$\int_\Omega (w^2 + j\psi^2 + g^2) dx \leqq 0$$

for all $t \in [0, T]$ and hence

$$w = 0, \quad \psi = 0, \quad g = 0 \text{ in } \Omega \times [0, T].$$

This gives the stated uniqueness result.

REFERENCES

1. Eringen, A. C. — *Theory of Micropolar Fluids*, J. Math. Mech., 16 (1966), pp. 1—18.
2. Förste, J. — *Instationäre Strömung einer inkompressiblen mikropolaren Flüssigkeit*, Monatsb. Deutsch. Akad. Wiss. Berlin, 13 (1971). pp. 741—748.
3. Förste, J. — *Zur Existenz stationärer Strömungen bei inkompressiblen mikropolaren Flüssigkeiten*, Monatsb. Deutsch. Akad. Wiss, Berlin, 13 (1971), pp. 1—8.
4. Sava, V. A. — *On the existence and uniqueness of the solution of a boundary-value problem in the theory of incompressible micropolar fluids*. An. şt. Univ. Iaşi Secţ. Ia, Mat. XIX, (1973), pp. 449—460.

Received 4. XI. 1974

Institute of Mathematics
University of Iaşi
Iaşi, Romania

Analele ştiinţifice ale Universităţii „AL. I. CUZA" din Iaşi
Tomul XXII, s. I a, 1976, f. 1.

ON STABILITY IN THE THEORY OF ELASTIC CONTINUA WITH DIRECTORS

BY

GH. CHIORESCU

1. Introduction. The study of stability of continuous systems is the subject of a large number of works. At the end of this paper a selective bibliography is given [1]—[5], in which a more comprehensive reference on this subject can be found.

In extending the ideas of Liapunov, Movchan [1] has given a useful definition of stability and has proved analogous theorems (his main results are also presented in [2]). In the present paper the stability of elastic directed bodies according to Movchan's definition, is investigated.

2. Basic equations of thermoelastic directed continuum. The motion of a continuous medium with directors is refered to a single fixed rectangular cartesian coordinate system ox^i. The material point **X** determines the position in ox^i of each particle in the medium at reference time $\tau = 0$. With each material point **X** a noncoplanar triad field of director vectors $D_\alpha(\mathbf{X})$ and a temperature θ_0 are associated. Then at time τ the spatial position **x** of a material point **X**, as well as the associated triad of directors d_α and the temperature θ are given by

$$x_i(\tau) = x_i(X_K, \tau), \; d_{\alpha i}(\tau) = d_{\alpha i}(X_K, \tau), \; \theta(\tau) = \theta(X_K, \tau). \tag{2.1}$$

For $\tau = 0$ we have

$$\mathbf{X} = \mathbf{x}(X_K, 0), \quad D_\alpha = d_\alpha(X_K, 0), \quad \theta_0 = \theta(X_K, 0). \tag{2.2}$$

The current position of the particle is denoted by $x_i = x_i(t)$, and as a consequence, Eq. (2.1) may be rewritten as

$$x_i(\tau) = x_i(x_j, t, \tau), \; d_{\alpha i}(\tau) = d_{\alpha i}(x_j, t, \tau), \; \theta(\tau) = \theta(x_j, t, \tau), \tag{2.3}$$

provided that

$$\det\left(\partial x_i(\tau)/\partial X_I\right) > 0.$$

The components of velocity at the point x_i are given by $\dot{x}_i$ and $\dot{d}_{\alpha i}$, the material derivative being taken with respect to τ and keeping x_j and t fixed in Eq. (2.3). The second order material time derivatives of x_i and $d_{\alpha i}$ are similarly defined.

Let the material volume Ω of the body be bounded by the surface Γ which is sufficiently regular to ensure the common laws of transformation of surface integrals and Friedrichs' inequality. We assume the energy balance at time t in the form [6]

$$\int_\Omega \rho \ddot{x}_i \dot{x}_i \, d\Omega + \int_\Omega \rho A^{\alpha\beta\gamma\delta} \dot{w}_{\alpha\beta} w_{\gamma\delta} \, d\Omega + \int_\Omega \rho \dot{e} \, d\Omega = \int_\Omega \rho (Q + F_i \dot{x}_i + B_i^\alpha \dot{d}_{\alpha i}) d\Omega + \int_\Gamma (t_i \dot{x}_i + p_i^\alpha \dot{d}_{\alpha i} - q) d\Gamma, \tag{2.4}$$