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SYNTHESIS OF TWO-TERMINAL NETWORKS WITH RECTIFIERS AND A MINIMUM NUMBER OF RELAY-CONTACTS

BY

JUHANI NIEMINEN

1. Introduction. This paper considers a classical problem of realizing a given Boolean function, that describes the chains of relay-contacts in a contact network, into a two-terminal contact network N satisfying the contact demands of this function with a minimum number of relay-contacts. An abstract formulation of the problem is given in terms of graph theory and the base for constructing a desired network N is a directed graph. This directed graph is translated into N and chains of forbidden relay-contacts are excluded by using rectifiers.

There is a large number of roumanian works concerning the realization problem of this paper; a list of those papers is given at end of the monograph of Moisil [4]. As stated in [3], the problem is interesting but hard to solve. As expected, the abstract formulation of this paper does not give any practical solution method, but it offers a base for constructing reasonable approximative realization methods applicable to problems in practice. An approximation is considered in the last section.

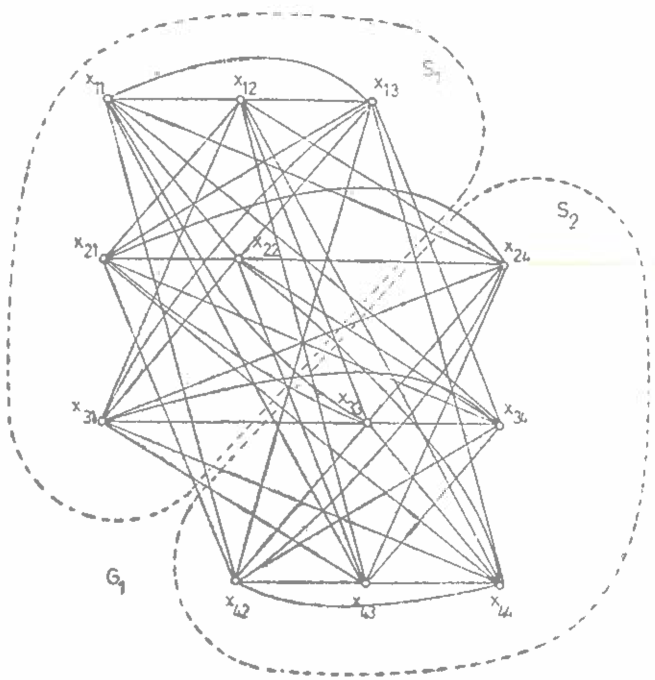
The terminology of graph theory follows that of Harary in [2]. By a Boolean function $f(x_1^a,\dots, x_n^a)$ we shall always mean a function in its disjunctive normal form. The notation $\bar{x}$ stands for the negation of x, and x^a — means x, if $a=1$, and $\bar{x}$, if $a=0$.

2. A graphtheoretic formulation. Let $f(x_1^a \dots, x_n^a)$ be a given Boolean function. A two-terminal network N with terminal points α and β and with labels $x_1^a,\dots,x_n^a$ on lines, realizes $f(x_1^a,\dots, x_n^a)$, if there is a path, possibly directed, from α to β in N corresponding to any conjunction of $f(x_1^a,\dots, x_n^a)$ and containing only those lines with labels forming a conjunction in question; N contains no other paths from α to β. Let $x_{i_1}^a$, $x_{i_2}^a,\dots, x_{i_s}^a$ be a path from α to β in N with line labels $x_{i_r}^a$, $r=1,\dots, s$, given in that order in which they appear in the path. Clearly the paths define a set of orders, a set of chains, among the symbols $x_1^a,\dots, x_n^a$ and these chains form the subject of our interest here. The chains shall be detected by using the labels $x_1^a,\dots, x_n^a$ on points of graphs defined below. A graph with labels on points can be easily translated into a graph with labels on lines.

The determination process of N contains the following main steps. Let $f(x_1^a,\dots, x_n^a)$ contain m disjoint conjunctions. At first we construct an undirected graph $G_1=(V(G_1), E(G_1))$, $V(G_1)=\{x_{ij}^a \mid i=1,\dots, n,\ j=1,\dots, m, \text{ and } a=0, 1\}$. This graph will be decomposed into a set of specified induced subgraphs of G_1. The decomposition as the base, G_1 will be mapped onto an undirected graph G_2 satisfying a minimality condition. This graph determines the chains, or the order relations, under interest as follows: a line $(y, z)\in E(G_2)$ implies that $y>z$ or $z>y$. These order relations shall be translated into a directed graph $\vec{G}=(V(\vec{G}), E(\vec{G}))$, where $V(\vec{G})=V(G_2)$ and a directed path from a point y to a point z, $y, z\in V(\vec{G})$, implies that $y>z$. The graph $\vec{G}$ can be directly translated into a network N realizing $f(x_1^a,\dots, x_n^a)$ with a minimum number of relay-contacts.

Definition 1. *Let $f(x_1^a, \dots, x_n^a)$ be a given Boolean function of m conjunctions. $G_1=(V(G_1), E(G_1))$ is an undirected graph whose set of points is $V(G_1)=\{x_{ij}^a \mid x_{ij}^a$ is a symbol in $f(x_1^a, \dots, x_n^a)$, $i=1, \dots, n$, $j=1, \dots, m$ and $a=0, 1\}$ and a line $(x_{ij}^k, x_{rs}^h)\in E(G_1)$, if $k=h$, $i=r$ and $j\neq s$, or if $j\neq s$ and x_{is}^k or x_{rj}^h is a symbol of $f(x_1^a, \dots, x_n^a)$.*

A complete t-partite graph is an undirected graph $S=(V(S), E(S))$ where the set of points $V(S)$ is divided into t non-empty point-disjoint sets $A_1, \dots, A_t$, the union of which is $V(S)$, and there are no lines between two points in a set A_u and a line joins any two points z and y, when $z\in A_u$ and $y\in A_v$, $u\neq v$, $u, v=1, \dots, t$. Now we are ready to define the mapping determined the graph G_2.



Definition 2. *Let $D=\{S_1, \dots, S_q, \dots, S_p\}$ be a decomposition of G_1 into p complete $t(q)$-partite graphs such that any point of G_1 belongs to exactly one of the graphs S_q in the decomposition D. D determines a mapping g from G_1 onto a graph $g(G_1)$ as follows:*

(i) If S_q is not a complete graph of constant values of i and k, then g maps all the points x_{ij}^k with constant values of k and i, respectively, into a point ${}_q x_i^k$ in $g(G_1)$.

(ii) If S_q is a complete graph of constant values of i and k, then g maps any point x_{ij}^k of S_q onto a point ${}_q x_{ij}^k$ of $g(G_1)$.

(iii) A line joins any two points ${}_{q_1}x_r^h$ and ${}_{q_2}x_i^k$ of $g(G_1)$ if g has mapped into those points the points x_{rs}^h and x_{ij}^k, respectively, with the same value of s and j, i. e. $s=j$.

Definition 3. *Let $g(G_1)$ be the family of graphs obtained from G_1 by a mapping g defined above. G_2 is a graph for which holds: $|V(G_2)|\leq|V(G)|$ for any graph $G\in g(G_1)$.*

Before continuing, let us consider an example illuminating the definitions above. Let

$$f(x_1^a, x_2^a, x_3^a, x_4^a)=x_1 x_2 x_3 \vee x_1 x_2 x_4 \vee x_1 x_3 x_4 \vee x_2 x_3 x_4, \tag{1}$$

where $n=m=4$ and $a=1$. Fig. 1 illustrates the graph G_1 and Fig. 2 shows a graph G_2 formed by two complete t-partite graphs S_1 and S_2, where $V(S_1)=\{x_{11}, x_{21}, x_{31}, x_{12}, x_{22}, x_{13}\}$ and $V(S_2)=\{x_{24}, x_{34}, x_{44}, x_{33}, x_{43}, x_{42}\}$. In the mapping g determined by S_1 and S_2 defined above, the points in S_1 and S_2 are mapped into points of G_2 as follows: $\{x_{11}, x_{12}, x_{13}\}\to{}_1x_1$, $\{x_{21}, x_{22}\}\to{}_1x_2$, $\{x_{31}\}\to{}_1x_3$, $\{x_{42}, x_{43}, x_{44}\}\to{}_2x_4$, $\{x_{33}, x_{34}\}\to{}_2x_3$, and $\{x_{24}\}\to{}_2x_2$. $({}_1x_1, {}_1x_3)\in E(G_2)$ as x_{11} and x_{31} are mapped by g into ${}_1x_1$ and ${}_1x_3$, respectively; the existence of the other lines in G_2 can be seen similarly.

Definition 4. *Directed graph $\vec{G}=(V(\vec{G}), E(\vec{G}))$ is a graph derived from the graph G_2 such that $V(\vec{G})=V(G_2)$ and if two points y and z of G_2 are joined in G_2 by a line, there is a directed path from y to z, or from z to y, in $\vec{G}$ but not both of them, i. e. $\vec{G}$ determines an order relation among its points such that if $(y, z)\in E(G_2)$, then $y>z$ or $z>y$.*

The existence of a graph $\vec{G}$ determining the order relation of Definition 4 follows from results of Thiel (see Jung [5]).

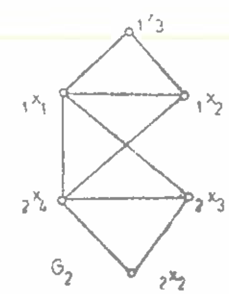


Fig. 2

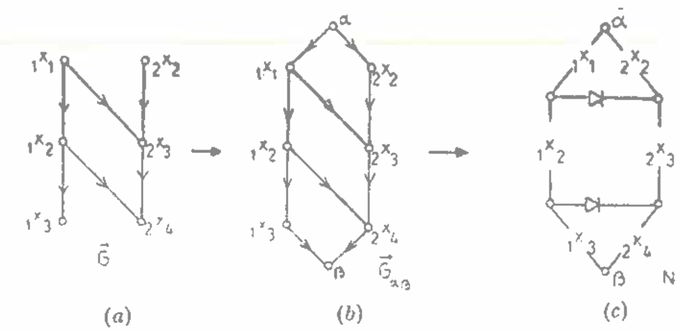


Fig. 3.

Figure 3(*a*) shows the directed graph $\vec{G}$ derived from the graph G_2 of Fig. 2. We connect now the point α to any maximal point of $\vec{G}$ and β to any minimal point of $\vec{G}$ (see Fig. 3 (*b*)); this graph is denoted by $\vec{G}_{\alpha\beta}$. Fig. 3 (*c*) illustrates a graph derived from the graph $\vec{G}_{\alpha\beta}$ of Fig. 3(*b*) by substituting the points, except α and β of $\vec{G}_{\alpha\beta}$ by lines and maintaining the proper order relation by using a set of rectifiers (the directed lines $\overrightarrow{({}_1x_1, {}_2x_3)}$ and $({}_1x_2, {}_2x_4)$ of $\vec{G}_{\alpha\beta}$). By removing the pre-subscripts „1" and „2" from the line labels in N, one can easily see that N realizes the Boolean function (1). Further, N is a realization with a minimum number of relay-contacts as mentioned in [1, p. 386].

We shall call a network N with rectifiers and relay-contacts, realizing a given Boolean function $f(x_1^a, \dots, x_n^a)$, minimal, if after removing any labelled line from N, N no more realizes $f(x_1^a, \dots, x_n^a)$. The following two lemmas illuminate the background of the concepts defined above.

Lemma 1. *Let $f(x_1^a,\dots,x_n^a)$ be a given Boolean function and G_1 a graph derived from it. If N is a minimal two-terminal network with rectifiers and relay-contacts realizing $f(x_1^a,\dots,x_n^a)$, then the relay-contacts of N imply a decomposition of G_1 into p point-disjoint complete $t(q)$-partite subgraphs $S_1,\dots,S_q,\dots,S_p$ such that $\bigcup_q V(S_q)=V(G_1)$.*

Proof. Evidently the relay-contact lines of N can be decomposed into p paths in a family P of paths with the following properties: Let $\{1', 2',\dots,c'\}\cup\{1, 2,\dots,n\}$ and if a path $(x_{1'}^a, x_{2'}^a,\dots,x_{c'}^a)\in P$, then it is a path from α to β in N realizing a conjunction of $f(x_1^a,\dots,x_n^a)$ or it forms a connected part of such a path; any relay-contact of N belongs to exactly one of the paths in P. Consider two relay contacts $x_{i'}^a$, and $x_{r'}^a$, of a path $(x_{1'}^a,\dots,x_{c'}^a)\in$ $\in P$ and assume that $x_{i'}^a$ corresponds to the symbols $x_{i'j_1}^a, x_{i'j_2}^a,\dots,x_{i'j_u}^a$ in the conjunctions indexed with $j_1,\dots,j_u$ in $f(x_1^a,\dots,x_n^a)$ and $x_{r'}^a$, to the symbols $x_{r's_1}^a, x_{r's_2}^a,\dots,x_{r's_v}^a$ in the conjunctions with the indices $s_1,\dots,s_v$. Clearly the assignement of the symbols $x_{r's_1}^a,\dots,x_{r's_v}^a$ to a line label $x_{r'}^a$ need not be unique; as N is minimal, in any assignement each line label is associated with at least one symbol x_{ij}^a of $f(x_1^a,\dots,x_n^a)$. We suppose that in going from α to β in N along $(x_1^{a'},\dots,x_{c'}^a)$ one must first go through $x_{i'}^a$ and thereafter through $x_{r'}^a$. As N realizes $f(x_{1'}^a,\dots,x_n^a)$ without forbidden paths and as it is possible to go from $x_{i'}^a$ to $x_{r'}^a$ in N, any of the conjunctions $j_1,\dots,j_u$ must contain the symbol $x_{r'}^a$. Similarly, as it is possible to go from $x_{i'}^a$ to $x_{r'}^a$, then any of the conjunctions $s_1,\dots,s_v$ must contain the symbol $x_{i'}^a$. It follows now directly from Definition 1 that $(x_{1'}^a,\dots,x_{c'}^a)$ generate a complete t-partite subgraph of G_1. The validity of the other assertions of the is obvious.

The path decomposition P of N is not the only way of finding a decomposition of G_1 into subgraphs $S_1,\dots,S_p$. Note that if $(x_{1'}^a,\dots,x_{b'}^a)$ is a bridge in N, i. e. one can go through it from $x_{1'}^a$ to $x_{b'}^a$ and also from $x_{b'}^a$ to $x_{1'}^a$ in going from α to β, then also the tree structure of a part $(x_{1'}^a,\dots,x_{c'}^a)$ of a path from α to β and of the bridge $(x_{1'}^a,\dots,x_{b'}^a)$ with a common point generate a subgraph S of G_1.

Clearly, if any decomposition P of N into paths or into paths and tree structures of paths and bridges, contains at least one set $\{x_r^a\}$ with a single element corresponding to the symbols $x_{rj_1}^a,\dots,x_{rj_u}^a$ in the conjunctions $j_1,\dots,j_u$ of $f(x_1^a,\dots,x_n^a)$, $u\geq 2$, then $f(x_1^a,\dots,x_n^a)$ contains at least two conjunctions one contained by the other. Hence there is for any N a decomposition P such that the number of relay-contacts and the number of points in the graph $g(G_1)$ are equal, where g is the mapping determined by the subgraphs $S_1,\dots,S_p$ of G_1 generated by the decomposition P of N.

Lemma 2. *Let $g(G_1)$ be a graph determined by the mapping g of Definition 2 and $\vec{gG}$ a directed graph derived from $g(G_1)$ satisfying the demands: $V(\vec{gG})=V(g(G_1))$ and there is a directed path from y to z or z to y if and only if $(y, z)\in E(g(G_1))$. Let $\vec{gG}_{\alpha\beta}$ be a graph obtained from $\vec{gG}$ by joining α to any point of $\vec{gG}$ with zero indegree and β to any point with zero outdegree. The graph $\vec{N}$ obtained from $\vec{gG}_{\alpha\beta}$ by substituting any point of $\vec{gG}$ by an undirected line with the label of the point in question realizes $f(x_1^a,\dots,x_n^a)$.*

Proof. According to Definition 2, two points $y, z\in V(g(G_1))$ are joined by a line in $g(G_1)$ only if into y and z are mapped the points x_{rs}^a and x_{tj}^a of G_1, respectively, such that $s=j$. According to the definition of $\vec{gG}$ and as such a graph always exists, there is for any conjunction of $f(x_1^a,\dots,x_n^a)$ a path from α to β in $\vec{gG}_{\alpha\beta}$, and also in $\vec{N}$, containing only the points which correspond to the symbols of the conjunction. On the other hand, if $\vec{gG}_{\alpha\beta}$ contains a path from α to β corresponding to a conjunction not in $f(x_1^a,\dots,x_n^a)$, then the points of this path, except α and β, form a complete subgraph of $g(G_1)$, which is possible only if there is a conjunction of $f(x_1^a,\dots,x_n^a)$ with the symbols of the points of this complete subgraph. This is a contradiction, and the lemma follows.

Clearly any directed line of $\vec{N}$ corresponds to a rectifier; the final realization N is obtained by removing the unnecessary rectifiers from $\vec{N}$.

According to the lemmas above and as for any minimal N realizing $f(x_1^a,\dots,x_n^a)$ there is a graph $g(G_1)$ such that $|V(g(G_1))|$ equals the number of relay-contacts in N, we can write the following theorem.

Theorem. *Let $f(x_1^a,\dots,x_n^a)$ be a given Boolean function. Any realization N of $f(x_1^a,\dots,x_n^a)$ with rectifiers and with a minimum number of relay-contacts is derived from a graph G_2 and any graph G_2 determines a realization N of $f(x_1^a,\dots,x_n^a)$ with rectifiers and with a minimum number of relay-contacts.*

3. Some observations on approximative methods. According to our knowledge, there are no methods for determining such a partition $D=\{S_1,\dots,S_p\}$ of G_1 which determines a graph G_2. On the other hand exact methods for finding a graph G_2 seem to be fairly tedious. Hence, we shall in the following look for a way of determining a reasonable approximation G_2^0 for G_2 of a given function $f(x_1^a,\dots,x_n^a)$. The method for finding $\vec{G}$ from G_2 or from G_2^0 must be exact. This method ought be such that the resulted graph $\vec{G}$ contains as few directed lines as possible.

Consider a way of finding G_2^0. By $w(x_i^a)$ we denote the number of occurences of the variable x_i^a in a function $f(x_1^a,\dots,x_n^a)=f(X)$. Let $x_{1'}^a$ be a variable of maximum weight $w(x_{1'}^a)$ in $f(X)$ and let us decompose $f(X)$ with respect to this variable into the form $f(X)=$ $f_{11}(X^{11})x_{1'}^a\vee f_{12}(X^{12})$; the subformulas $f_{11}(X^{11})$ and $f_{12}(X^{12})$ are called residuals. Choose from $f_{11}(X^{11})$ a variable $x_{2'}^a$, with a maximum weight $w(x_{2'}^a)$ in the function $f_{11}(X^{11})$. Suppose that $f_{k1}(X^{k1})$ and $f_{k2}(X^{k2})$ are the residuals obtained from $f_{(k-1)1}(X^{(k-1)1})$ at step k. Then, at step $k+1$, we choose a variable $x_{k'+1}^a$ with a maximum weight in the function $f_{k1}(X^{k1})$. The process stops in the following cases: (I) Let each variable of $f_{k1}(X^{k1})$ have the weight one in $f_{k1}(X^{k1})$. We choose an arbitrary variable $x_{k'+1}^a$, remove it from $f_{k1}(X^{k1})$ and denote the residual thus obtained by $f_{(k+1)2}(X^{(k+1)2})$. (II) Let $f_{k1}(X_k^{k1})=x_{k'+1}^a\vee f'_{k1}(X^{k1})$, where $w(x_{k'+1}^a)=2$ in $f_{k1}(X^{k1})$. Then $f_{k1}(X^{k1})=x_{k'+1}^a(\emptyset\vee f'_{k1}(X^{k1}-x_{k'+1}^a))=x_{k'+1}^a f_{(k+1)1}(X^{(k+1)1})$. From $f_{(k+1)1}(X^{(k+1)1})$ we choose an arbitrary variable $x_{k'+2}^a$, remove it from $f_{(k+1)1}(X^{(k+1)1})$ and denote the residual thus obtained by $f_{(k+1)2}(X^{(k+1)2})$.

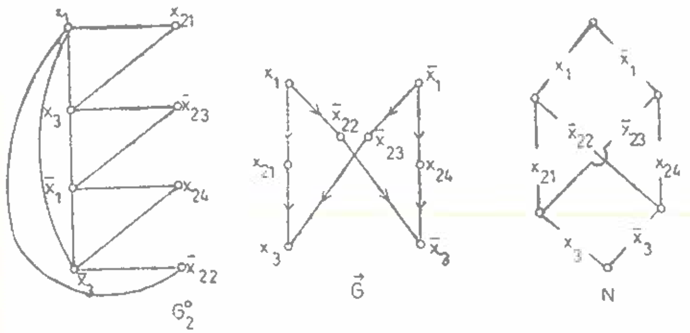
Let $f_{k1}(X^{k1})$ contain a part of the conjunctions of $f(X)$ with indices $j_{1(k)},\dots,j_{r(k)}$. In case (I) the corresponding complete t-partite graph S_1 is determined by the point sets $\{x_{1'j_{1(1)}}^a,\dots,x_{1'j_{r(1)}}^a\},\{x_{2'j_{1(2)}}^a,\dots,x_{2'j_{r(2)}}^a\},\dots,\{x_{(k'+1)j_{(k+1)}}^a\}$ and the new function to which the process will be applied is $f_{12}(X^{12})\vee f_{22}(X^{22})\vee\dots\vee f_{(k+1)2}(X^{(k+1)2})$. In case (II) the new function is the same as above, but S_1 is determined by the point sets

$$\{x_{1'j_{1(1)}}^a,\dots,x_{1'j_{r(1)}}^a\},\dots,\{x_{(k'+1)j_{1(k+1)}}^a,\dots,x_{(k'+1)j_{r(k+1)}}^a\},\quad\{x_{(k'+2)j_{1(k+2)}}^a\}.$$

As an example consider the function

$$f(x_1^a, x_2^a, x_3^a)=x_1 x_2 x_3\vee x_1\bar{x}_2\bar{x}_3\vee\bar{x}_1\bar{x}_2 x_3\vee\bar{x}_1 x_2\bar{x}_3. \tag{2}$$

Now, $f(X)=x_1(x_2x_3\vee \bar{x}_2\bar{x}_3)\vee \bar{x}_1 x_2 x_3\vee \bar{x}_1 x_2 \bar{x}_3$ and $f_{11}(X^{11})=x_2x_3\vee \bar{x}_2\bar{x}_3 f_{11}(X^{11})$ is the function of the type reported in (I) and we choose x_2 from $f_{11}(X^{11})$; the corresponding residual is $f_{22}(X^{22})=x_3\vee \bar{x}_2\bar{x}_3$. Thus $V(S_1)=\{\{x_{11}, x_{12}\}, \{x_{21}\}\}$ and the new function is $x_3\vee \bar{x}_2 x_3\vee \bar{x}_1 x_2 x_3\vee \bar{x}_1 x_2 \bar{x}_3=x_3(\varnothing\vee \bar{x}_1 \bar{x}_2)\vee \bar{x}_2 \bar{x}_3\vee \bar{x}_1 x_2 \bar{x}_3$. According to the case (II) we choose from $\bar{x}_1 x_2$ the variable x_2. Hence the new function is $\bar{x}_1\vee \bar{x}_2 \bar{x}_3\vee \bar{x}_1 x_2 \bar{x}_3$ and $V(S_2)=\{\{x_{31}, x_{33}\}, \{\bar{x}_{23}\}\}$. $\bar{x}_1\vee x_2\bar{x}_3\vee \bar{x}_1 x_2 \bar{x}_3=\bar{x}_1(\varnothing\vee x_2\bar{x}_3)\vee \bar{x}_2\bar{x}_3$ and from $x_2 x_3$ we choose x_2. $V(S_3)=\{\{\bar{x}_{13}, \bar{x}_{14}\}, \{x_{24}\}\}$ and the new function is $\bar{x}_2 \bar{x}_3\vee \bar{x}_3=\bar{x}_3(\varnothing\vee \bar{x}_2)$. After choosing $\bar{x}_2$ there are no variables and hence we have decomposed G_1; $V(S_4)=\{\{\bar{x}_{32}, \bar{x}_{34}\}, \{\bar{x}_{22}\}\}$. By denoting the sets $\{x_{11}, x_{12}\}$, $\{x_{31}, x_{33}\}$, $\{\bar{x}_{13}, \bar{x}_{14}\}$, $\{\bar{x}_{32}, \bar{x}_{34}\}$ briefly by x_1, x_3, $\bar{x}_1$, $\bar{x}_3$, respectively, we can directly draw the graph G_2^0 of Fig. 4 (*a*). Fig. 4 (*b*) shows the corresponding graph $\vec{G}$ and Fig. 4 (*c*) illustrates the network N realizing the function (2).



(*a*) (*b*) (*c*)

Fig. 4

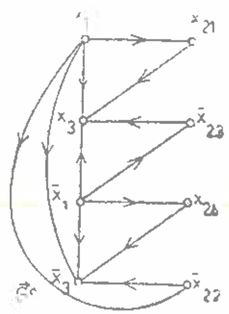


Fig. 5

We have no efficient algorithm for determining a graph $\vec{G}$ from the graph G_2 (or from G_2^0) with a minimum number of lines. A labelling algorithm giving a graph $\vec{G}$ such that $|E(\vec{G})| = |E(G_2)|$ (or $|E(\vec{G})| = |E(G_2^0)|$), can be constructed, but such a graph is clearly not a desired graph $\vec{G}$. Figure 5 shows such a graph $\vec{G}$ derived from G_2^0 of Figure 4(*a*). For further information on directing the lines of G_2 and G_2^0 and for constructing suitable algorithms the reader is referred to the papers [6] and [7] of Jung.

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ON THE THEORY OF REDUCTION OF SEMILATTICIAL AUTOMATA

BY

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I. Introduction. We intend to study in this paper several aspects of the reduction theory of semilatticial automata. This new class of automata is an extension of the class of linear Boolean automata [3], which includes as special cases Leticevskij automata [2] and macroautomata related to nondeterministic automata (see [1], for instance).

A semilatticial automaton (SLA) is a 5-uple $\mathbf{A} = (I, L, K, \{f_i \mid i \in I\}, \{g_i \mid i \in I\})$ where I is the input alphabet, $L = (S, \cup)$ is the state semilattice of te automaton, $K = (0, \cup)$ is the semilattice of output signals, $\{f_i \mid i \in I\}$ is the set of transition mappings which are endomorphisms of the lattice L and $\{g_i \mid i \in I\}$ is the set of output mapping which consists of lattice-homomorphisms from L to K.

When an input signal is presented at the input wire of the automaton, its interior state being $s \in S$ we obtain $g_i(s)$ at the output wire of the semiautomaton, the next state being $f_i(s)$. The transition mappings and the output mappings can be extended for words belonging to I^* by induction on the lenght of the words. Namely, for every p from I^* we shall take

$$f_{pi} = f_i \circ f_p \text{ and } g_{pi} = g_i \circ f_p.$$

It is clear that $f_p \in \mathrm{Hom}(L, L)$ and $g_p \in \mathrm{Hom}(L, K)$. In the sequel we shall present some classes of automata which appear to be special cases of semilatticial automata.

Let $\mathbf{A} = (B_2^{(m,1)}, B_2^{(n,1)}, B_2^{(l,1)}, f, g) = \langle A, B, C, D \rangle$ be a linear Boolean automaton [3]. It is clear that $B_2^{(m,1)}$, $B_2^{(n,1)}$, $B_2^{(l,1)}$ are disjunctive semilattices. The mappings f_i and g_i are given by $f_i(s) = As \cup Bi$ and $g_i(s) = Cs \cup Di$. We remark that they are homomorphisms since

$$f_i(\cup (s_j \mid j \in J)) = A(\cup (s_j \mid j \in J)) \cup Bi =$$
$$= \cup (As_j \mid j \in J) \cup Bi = \cup (As_j \cup Bi \mid j \in J) = \cup (f_i(s_j) \mid j \in J)$$

and

$$g_i(\cup (s_j \mid j \in J)) = C(\cup (s_j \mid j \in J)) \cup Di =$$
$$= \cup (Cs_j \mid j \in J) \cup Di = \cup (Cs_j \cup Di \mid j \in J) = \cup (g_i(s_j) \mid j \in J).$$

Let $\mathbf{A} = (I, S, O, f, g)$ be a nondeterministic automaton. The *macroautomaton* related to $\mathbf{A}$ is defined as $\mathbf{M(A)} = (I, P(S), P(O), \bar{F}, \bar{G})$, where $\bar{F} = \{\bar{f}_i \mid \bar{f}_i : P(S) \to P(S),\ i \in I,\ \bar{f}_i(S_1) = \cup (f_i(s) \mid s \in S_1),\ \forall S_1 \in P(S)\}$ and $\bar{G} = \{\bar{g}_i \mid \bar{g}_i : P(S) \to P(O),\ i \in I,\ \bar{g}_i(S_1) = \cup (g_i(s) \mid s \in S_1),\ \forall S_1 \in P(S)\}$. It is immediat that $\mathbf{M(A)}$ is a SLA for, $\bar{f}_i(\cup (S_j \mid j \in J)) = \cup (\bar{f}_i(S_j) \mid j \in J))$ and $\bar{g}_i(\cup (S_j \mid j \in J)) = \cup (g_i(S_j) \mid j \in J)$.

A Leticevskij - like automaton is an infinite semiautomaton $\mathbf{A} = (I, P_f(Z^*), F, E_0)$ where I is the input alphabet, $P_f(Z^*)$ the set of finite subsets of the free semigroup Z^* generated by Z, F is the state set, $E_0 \in P_f(Z^*)$ is the subset from Z which is the initial state for $\mathbf{A}$ and the family F satisfies the following conditions: