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INJECTIVE PROPERTIES OF TORSION-COMPLETE MIXED GROUPS

BY

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1. Introduction. In what follows only additive abelian groups will be considered.

We know that algebraically compact groups can be described as groups A for which $\mathrm{Ext}(Q, A) = 0 = \mathrm{Pext}(Q/Z, A)$. The first equality defines the class of cotorsion groups while groups defined by the second equality, called P_T-groups, were studied in [3] and [4]. All torsion free groups are P_T-groups and if we restrict ourselves to p-groups then the P_T-groups are exactly the torsion-complete groups. Only the mixed case is therefore of interest. Prof. L. Fuchs suggested that these groups be called torsion-complete mixed groups, and this terminology will be used henceforth.

Definition: *A is called a torsion-complete mixed group if* $\mathrm{Pext}\,(Q/Z, A) = 0$.

Considering that for every group A there exists a pure-exact sequence $0 \to A \to B \to C \to 0$ with B algebraically compact, as well as an exact sequence $0 \to A \to C \to D \to 0$ with C cotorsion and D torsion-free divisible, the problem of embedding in torsion-complete mixed groups arises naturally. It turns out to be possible to embed an arbitrary mixed group A as a pure subgroup in a torsion-complete mixed group V such that V/A is torsion and divisible. Moreover, there exists a minimal such V which is unique up to isomorphism over A.

It is known that the class of algebraically compact groups is exactly the class of groups having the injective property relative to the class of all pure-exact sequences, while the class of cotorsion groups fits into this scheme with respect to the so called torsion-splitting exact sequences. It follows that the torsion-complete mixed groups can be characterized in a similar way (cf. F u c h s [1], problem 46).

The notation and terminology will be the same as in [1]. In particular, we emphasize that a cotorsion group is a group A for which $\mathrm{Ext}\,(Q, A) = 0$, and hence need not be reduced.

2. Embedding properties. We begin this section with a lemma which is very usefull for our purposes.

Lemma 1. *Let A be a torsion-complete mixed group. Then A^1 coincides with the maximal divisible subgroup of A.*

Proof: If A is reduced, then A can be embedded in $\mathrm{Ext}\,(Q/Z, A)$ ([1] p. 247). But $(\mathrm{Ext}\,(Q/Z, A))^1 = \mathrm{Pext}\,(Q/Z, A) = 0$ whenever A is a torsion-complete mixed group. Hence $A^1 = 0$. On the other hand, if $A = D \oplus R$ with D divisible and R reduced then $A^1 = D \oplus (R \cap A^1)$. Since R contains no element of infinite height in A it follows that $R \cap A^1 = 0$. Hence $A^1 = D$, as stated.

Corollary 2. *If A is a torsion-complete mixed group then so is A/A^1.*

Theorem 3. *For every mixed group A there exists a pure-exact sequence $0 \to A \to V \to W \to 0$ with V a torsion-complete mixed group and W torsion and divisible.*

Proof: If X is the minimal algebraically compact group containing A as pure subgroup then we have a pure-exact sequence $0 \to A \to X \to X/A \to 0$ with X/A divisible ([1] p. 173). Let V be the subgroup of X such that $V/A = T(X/A)$. Then A is also pure in V and we have the pure-exact sequence $0 \to A \to V \to V/A \to 0$ with V/A torsion and divisible.

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It remains to prove that V is a torsion-complete mixed group. Since $X/V \cong (X/A)/(V/A) \cong$
$\cong (X/A)/T(X/A)$ we have an exact sequence $0 \to V \to X \to J \to 0$ with $J = (X/A)/T(X/A)$ torsion-free. Hence the sequence is pure-exact and so $0 = \mathrm{Hom}(Q/Z, J) \to \mathrm{Pext}(Q/Z, V) \to$
$\to \mathrm{Pext}(Q/Z, X) = 0$ is exact. Thus $\mathrm{Pext}(Q/Z, V) = 0$ and the theorem is proved.

In the notation of the previous theorem we have the next two corollaries:

Corollary 4. *If A is a torsion group then so is V.*

Proof: V coincides with the maximal torsion subgroup of X.

Corollary 5. *V is a reduced group if and only if $A^1 = 0$.*

Proof: If $A^1 = 0$ than X is reduced ([1] p. 173). The converse is obvious.

3. Injective property. Call a group A T-injective if every diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] & W \arrow[r] & 0 \\ & A & & & \end{tikzcd}$$

with pure-exact row and W torsion and divisible can be embedded in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[dl, dashed] & W \arrow[r] & 0 \\ & A & & & \end{tikzcd}$$

Theorem 6. *A is a torsion-complete mixed group if and only if A is T-injective.*

Proof: Let A be a torsion-complete mixed group and $0 \to U \to V \to W \to 0$ a pure-exact sequence with W torsion and divisible. Then $\mathrm{Hom}(V, A) \to \mathrm{Hom}(U, A) \to \mathrm{Pext}(W, A) = 0$ is exact. Thus every homomorphism $U \to A$ is induced by a homomorphism $V \to A$, proving that A is T-injective.

Conversely, let A be T-injective. Theorem 3 implies the existence of a pure-exact sequence $0 \to A \to V \to W \to 0$ with V a torsion-complete mixed group and W a torsion divisible group. Since A is T-injective it follows that the sequence is splitting. Hence, A being a direct summand of V, is a torsion-complete mixed group.

4. Minimality and uniqueness. In this section we show that for a given group A, there exists a minimal torsion-complete mixed group which contains A as pure subgroup such that the factor group is torsion and divisible, and which is unique up to isomorphism over A.

First of all we need a few definitions. Let A be a pure subgroup of X and denote by $K(A, X)$ the set of all subgroups H of X such that

(i) $A \cap H = 0$ and
(ii) $(A + H)/H$ is pure in X/H.

If $K(A, X) = \{0\}$ then X is called a pure-essential extension of A. Among the pure-essential extensions of A there is a maximal one, which is exactly the pure-injective hull of A (See [1] § 41 for these definitions and results).

Theorem 7. *For every group A there exists a minimal torsion-complete mixed group V containing A as pure subgroup such that V/A is torsion and divisible.*

Proof: Let X be the pure-injective hull of A and let V be the torsion-complete mixed group constructed in theorem 3. We shall prove that V is a minimal torsion-complete mixed group containing A as pure subgroup such that V/A is torsion and divisible. Suppose that V is not minimal, and let Y be a proper subgroup of V satisfying the stated conditions. Since Y/A is divisible and A is pure in V it follows that Y is pure in V ([1] lemma 26.1), and since $V/Y \cong (V/A)/(Y/A)$ is torsion it follows that Y is a direct summand of V. If $V = Y \oplus R$ then the theorem will be proved if we can show that $R = 0$. To prove this, we merely show that $R \in K(A, X)$, for X is a pure-essential extension of A (cf. [1] p. 172).

That $A \cap R = 0$ is clear. Next we prove that $A + R$ is pure in X. Let $nx = a +$
$+ r \in A + R$, $x \in X$ and n a natural number. Since $A + R \leqslant V = Y \oplus R$ (which is pure in X) there exists an element $y + \bar{r} \in Y \oplus R$ such that $n(y + \bar{r}) = a + r$. It follows that

$n\bar{r} = r$ and $ny = a$, and since A is pure in Y we have $n\bar{a} = a$ for some $\bar{a} \in A$. But then $n(\bar{a} + \bar{r}) = a + r$ so that $A + R$ is pure in X, and the theorem follows.

Before proving a uniqueness theorem we need the following

Lemma 8. *If X is a pure-essential extension of A, then every homomorphism on X which induces a monomorphism on A such that φA is pure n φX, is itself a monomorphism.*

Proof: Let X be a pure-essential extension of A, and let φ be a homomorphism on X such that $\varphi|A$ is one-to-one. To prove that φ is a monomorphism we show that $K =$
$= \mathrm{Ker}\,\varphi \in K(A, X) = \{0\}$. That $A \cap K = 0$ is clear. To show that $(A + K)/K$ is pure in X/K, let $n(x + K) = a + k + K \in (A + K)/K$, $x + K \in X/K$ and n a natural number. Then $n\varphi x = \varphi a$ and since A is pure in X it follows that φA is pure in φX. Hence $n\varphi\bar{a} =$
$= \varphi a$ for some $\varphi\bar{a} \in \varphi A$. But then $n\bar{a} - a = k \in K$. Thus $n\bar{a} = a + k$ and $n(\bar{a} + k) = a +$
$+ k + nk$, so that $n(\bar{a} + k + K) = a + k + K$. The lemma is proved.

Theorem 9. *Let V and V' be two minimal torsion-complete mixed groups each containing A as pure subgroup such that V/A and V'/A are torsion and divisible. Then the identity map on A can be extended to an isomorphism of V onto V'.*

Proof: We may assume that one of the groups, say V, is contained in the pure-injective hull X of A. Theorem 6 implies the existence of a homomorphism φ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal, "I_A"] & V \arrow[d, "\varphi"] \\ 0 \arrow[r] & A \arrow[r] & V' \end{tikzcd}$$

Since A is pure in V which is pure in X, and X is a pure-essential extension of A, we have that V is a pure-essential extension of A ([2] lemma 3). Thus lemma 8 implies that φ is one-to-one and the minimality of V' implies that φ is epic.

The minimal torsion-complete mixed group containing A as pure subgroup such that the factor group is torsion divisible will be denoted by V_A.

Corollary 10. *If A is cotorsion then V_A is algebraically compact, and the cotorsion hull of a torsion-complete mixed group is algebraically compact.*

Proof: From the pure-exact sequence $0 \to A \to V_A \to T \to 0$, with $T = \bigoplus Z(p^\infty)$ we have the exactness of $0 = \mathrm{Pext}(Q, A) \to \mathrm{Pext}(Q, V_A) \to \mathrm{Pext}(Q, T) = 0$. Thus V_A is cotorsion and hence algebraically compact (see [4]).

Let A be a torsion-complete mixed group and M the minimal cotorsion group such that $0 \to A \to M \to D \to 0$ is pure-exact with D a torsion-free divisible group (cf. [1] § 58).

Then $0 = \mathrm{Pext}(Q/Z, A) \to \mathrm{Pext}(Q/Z, M) \to \mathrm{Pext}(Q/Z, D) = 0$ is exact. Thus M is algebraically compact.

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