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ON LOOPS WITH LEFT INVERSE PROPERTY

BY

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1. Introduction. One of the problems of varieties is to define some class of varieties by a single identity. This problem has been solved for groups, Abelian groups, inverse loops and WIP loops in [3], [1], [6], [5] and [2]. The object of this paper is to give characterization of the variety of left inverse property (LIP) loops as a subvariety of quasigroup with a single identity. An interesting corollary is that Bol loops can be defined by means of a single identity.

Let us recall that a quasigroup is a groupoid $< Q, . >$ in which the equations $a . x = b$ and $y . a = b$ are soluble uniquely for x and y respectively. *A loop* $< Q, . , e >$ *is a quasigroup with a two sided identity. A loop* $< Q, . , e >$ with identity e is called a LIP loop if to each element x in Q there corresponds an element x^{-1} in Q such that

(1.1) $$x^{-1} . xy = x . x^{-1}y = y \text{ for all } y \in Q.$$

From (1.1) we can easily deduce that

(1.2) $$x . x^{-1} = x^{-1} . x = e, \ (x^{-1})^{-1} = x \text{ for all } x \in Q.$$

2. Let $< Q, ^* >$ be a groupoid. We say that the groupoid $< Q, ^* >$ is an iso-LIP loop if there is a LIP loop $< Q, \circ >$ which is a principal isotope of $< Q, ^* >$ such that * and $\circ$ are connected by the relation $x ^* y = x \circ y^{-1}$ for all x, y in Q.

Theorem I. *A quasigroup* $< Q, ^* >$ *is an iso-LIP loop if and only if the identity*

(2.1) $$(w ^* w)^* x = ((u ^* u) ^* y) ^* ((v ^* v)^* (y ^* x))$$

holds for all $x, y, u, v, w \in Q$.

Proof: Suppose that the quasigroup $< Q, ^* >$ satisfies the identity (2.1). Keeping x, y, u and v the same and changing w to t in (2.1),

$$(w ^* w)^* x = (t ^* t)^* x.$$

Using the right cancellativity of $(^*)$,

(2.2) $$w ^* w = t ^* t = \text{constant} = e$$

(a fixed element of Q) for all $t, w \in Q$.
On using (2.2) in (2.1),

(2.3) $$e ^* x = (e ^* y)^* (e ^* (y^* x)).$$

Putting $y = e$ in (2.3), and using the left cancellativity of $(^*)$, we have

(2.4) $$x = e^* (e ^* x).$$

Further substituting y by x in (2.3) and using (2.4), we get

(2.5) $$x = x ^* e.$$

Let us define $x^{-1} = e * x$ and

$$x \circ y = x * y^{-1}. \tag{2.6}$$

Now we show that $< Q, \circ >$ is a LIP loop.

$$x \circ e = x * (e * e) = x \text{ (by (2.2) and (2.5)) for all } x \in Q. \tag{2.7}$$

$$e \circ x = e * x^{-1} = e * (e * x) = x \text{(by (2.4)) for all } x \in Q. \tag{2.8}$$

From (2.7) and (2.8) we see that e is the identity of $< Q, \circ >$.

$$(x^{-1})^{-1} = (e * x)^{-1} = e * (e * x) = x \text{ (by (2.4)) for all } x \in Q, \tag{2.9}$$

$$\begin{aligned} y^{-1} \circ (y \circ z) &= (e * y) * (y \circ z)^{-1} = (e * y) * (e * (y \circ z)) = \\ &= (e * y) * (e * (y * (e * z))) = e * (e * z) \text{ (by (2.3))} = z \text{(by (2.4))} \end{aligned} \tag{2.10}$$

for $y, z \in Q$.

Finally the equations $a * x = b$ and $y * a = b$ have unique solutions in the quasi-group $< Q, * >$. Thus the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions in the system $< Q, e >$. We have shown that $< Q, e >$ is a LIP loop. In other words, the quasi-group $< Q, * >$ is an iso $-$ LIP loop.

Conversely, let the quasigroup $< Q, * >$ be an iso-LIP loop. Let $< Q, e >$ be the corresponding LIP loop with the identity e. Then the (*) and (o) are connected by the relation.

$$x * y = x * y^{-1}. \text{ for all } x, y \in Q. \tag{2.11}$$

Since $(y^{-1})^{-1} = y$ for all $y \in < Q, \circ >$, so (2.11) takes the form

$$x * y^{-1} = x \circ y. \tag{2.12}$$

Putting $x = y$ in (2.11),

$$x * x = e \text{ for all } x \in Q. \tag{2.5}$$

Putting $x = e$ in (2.10),

$$y^{-1} = e * y \text{ for all } y \in Q. \tag{2.13}$$

From (2.12) and (2.13),

$$x \circ y = x * (e * y) \text{ for all } x, y \in Q. \tag{2.14}$$

Further the loop $< Q, \circ >$ satisfies the left inverse property, that is,

$$y^{-1} \circ (y \circ x^{-1}) = x^{-1} \text{ for all } x, y \in Q. \tag{2.15}$$

Hence, using (2.5) and (2.14), (2.15) becomes

$$(w * w) * x = ((u * u) * y) * ((v * v) * (y * x)). \tag{2.1}$$

This completes the proof of the theorem.

3. Let $w = w(x_1, \dots, x_n)$ be some word in the variables $x_1, \dots, x_n$ in the groupoid $< Q, * >$.

Theorem 2. *The quasigroup $< Q, * >$ is an iso-LIP loop in which the law $w(x_1, \dots, x_n) = e$ holds if and only if it satisfies the law*

$$((t * t) * w) * x = ((u * u) * y * ((v * v) * (y * x)) \tag{3.1}$$

for all $x, y, u, v, t \in Q$.

Proof: The „only if“ part is an easy consequence of theorem I and the hypothesis $w = e$. We need only to prove 'if' part. As in theorem I, from (3.1), we have

$$(t * t) * w = (s \,.\, * s) * w \text{ for all } s, t \in Q,$$

which in turn implies that

$$t * t = s * s = e \text{ for all } s, t \in Q. \tag{2.5}$$

Putting $x = y$ and using (2.5) in (3.1), it gives,

$$e * w = e. \tag{2.6}$$

The given identity (3.1) reduces to the identity (2.1) of theorem 1 and so the quasigroup $< Q, * >$ is an iso-LIP loop. Hence, for all $x \in Q$, we have

$$e * (e * x) = x \text{ and in particular } w = e * (e * w) = e * e \text{ by (2.6)} = e \text{ by (2.5).}$$

Thus the quasigroup $< Q, * >$ is an iso-LIP loop in which the identity

$$w(x_1, \dots x_n) = e$$

holds. This completes the proof of the theorem.

Corollary : *Bol loops can be defined by a single law as a sub-variety of quasigroups.*

Proof: In [4], it is given that Bol loops can be characterized among left inverse property loops by the identity $y(z \,.\, yx) = (y \,.\, zy)x$.

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