Démonstration. Si $\mathcal{S}$ est une séparation Wallace, alors la fonction $k: 2^E \to 2^E$, définie dans (S_8) est une t-fonction expansive (voir th. 1 de [5]). En posant $C = \{x;\ x\mathcal{S}kcB\}$, on déduit $cB \subset kcB$ et $C \subset \{x;\ x\mathcal{S}cB\}$; par suite la condition (S_8) implique $\{x;\ x\mathcal{S}cB\} \subset \subset \{x;\ x\mathcal{S}kcB\} = C$, donc $C = \{x;\ x\mathcal{S}cB\}$. Si $x \in cC$, alors $x\overline{\mathcal{S}}cB$, d'où il résulte que $x \in \in kcB$, donc $cC \subset kcB$. Puis, si $z \in C$, alors on obtient $z \notin kcB$, qui, avec l'inclusion ci-dessus, implique que $z <_{\mathcal{S}} C$; comme l'ordre $<_{\mathcal{S}}$ est parfait, on a $C <_{\mathcal{S}} C$. La relation $A <_{\mathcal{S}} B$ entraîne les implications ; $x \in A \Rightarrow x <_{\mathcal{S}} B$; $x <_{\mathcal{S}} B \Rightarrow x\mathcal{S}cB$; $x\mathcal{S}cB \Rightarrow x \in C$, donc $A \subseteq C$. Enfin, de $x \in C$ il résulte que $x\mathcal{S}cB$, d'où $x \in B$, donc $C \subset B$. Les relations $A \subset C <_{\mathcal{S}} C \subset B$ entraînent $A <_{\mathcal{S}} C <_{\mathcal{S}} B$, donc $<_{\mathcal{S}}$ est idempotent. □

Réciproquement, on a le théorème suivant.

Théorème 3.6. *Si $<$ est un ordre semi-topogène parfait idempotent T_1-séparé, alors la séparation associée $\mathcal{S}_<$ vérifie la condition (S_8).*

Démonstration. Si $x\mathcal{S}_< X$, alors il suit que $x(< \cap <^c)\,cX$, d'où il résulte que $x < cX$ et parce que $<$ est idempotent, on déduit qu'il existe $C \subset E$ tel que $x < C < cX$. Si $y \in \in cC$, on a $y \neq x$ et en vertu de la définition (3.1) il résulte que $y < E - x$, donc $cC < E - x$, c'est-à-dire $x <^c C$ et par suite on a $x(< \cap <^c)C$. De $z \in C < cX$, il s'ensuit que pour tout $u \in X$ on a $u \neq z$ et donc $u < E - z$; puisque $<$ est parfait, il résulte que $X < E - z$, ou que $z <^c cX$. Cette relation et $z < cX$ impliquent que $z(< \cap <^c)cX$, d'où on a $C \subset \{x;\ x(< \cap <^c)\,cX\}$. De $x(< \cap <^c)C$ et $C \subset \{x;\ x(< \cap <^c)\,cX\}$ il résulte que $x(< \cap <^c)\,\{x;\ x(< \cap <^c)cX\}$ et de la définition de la séparation $\mathcal{S}_<$, on déduit que $\{x;\ x(< \cap <^c)\,cX\} = \{x;\ x\mathcal{S}_< X\} = ckX$, donc $x(< \cap <^c)ck_< X$ et par suite $x\mathcal{S}_< k_< X$, c'est-à-dire que $\mathcal{S}_<$ vérifie la condition (S_8). □

Corollaire 3.7. *Les ordres semi-topogènes parfaits idempotents T_1-séparés se correspondent par l'isomorphisme λ avec les séparations Wallace qui satisfont aux conditions* (S_5), (S_6) *et* (S_8). □

Des théorèmes ci-dessus on déduit le théorème suivant de Wallace de caractérisation d'une topologie au moyen des séparations.

Théorème 3.8. *Les topologies classiques T_1-séparées sont en correspondance bijective avec les séparations qui satisfont aux axiomes $(S_5)-(S_8)$. Par cette bijection, à une fonction fermeture Kuratowski h déterminant une topologie T_1-séparée correspond la séparation $\mathcal{S}$ définie par: $X\mathcal{S}Y \Leftrightarrow X \cap hY = \emptyset = hX \cap Y$, et réciproquement, à une séparation vérifiant les axiomes $(S_5)-(S_8)$ correspond la fonction fermeture Kuratowski h définie par $hX = \{x;\ x\overline{\mathcal{S}}X\}$.*

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PSEUDOCONFORMAL MAPPINGS IN SPACE. II

BY

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§ 1. **Introduction and definitions.** In a previous work [5] (see also [2]) we defined the $(\alpha;\ r,\ q)$-pseudoconformal mappings. We recall:

Definition 1. *Let $X = X(\zeta)$ be a diffeomorphism of a domain $D \subset R^n$ onto $D' \subset R^m$ $(m \geq n \geqslant 2)$ and let $\lambda_1^2 \geqslant \ldots \geqslant \lambda_n^2$ be the proper values of J^*J, where J is the Jacobian matrix of $X(\xi)$. If $\alpha \geqslant 1$ is a real constant and if r and q are integers with $1 \leqslant r \leqslant n-1$, $2 \leqslant q \leqslant n - r + 1$, $X(\zeta)$ is said $(\alpha;\ r,\ q)$-pseudoconformal, (abbreviated $(\alpha;\ r,\ q)$-pc) when the inequality:*

$$\lambda_2^q \leqslant \alpha(\lambda_r \ldots \lambda_{r+q-1})$$

holds for every $\zeta \in D$. $X(\zeta)$ is $(r,\ q)$-pc if it is $(\alpha;\ r,\ q)$-pc for some α finite.

Remark. In [5] D and D' were both domains in R^n. Now D' can be a suitable manifold n-dimensional in R^m $(m \geqslant n)$.

For $r = 1$ and $q = n$ we obtain the class of quasiconformal diffeomorphisms. For the other values of r and q we obtain wider classes, whose mappings haven't in general, the typical properties of the quasiconformal mappings. Most of these properties are still true if we take the restriction of the mapping $X(\zeta)$ $(\alpha;\ r,\ q)$-pc to suitable manifolds in D belonging to a family $N(r,\ q)$ defined in [5]. We also recall

Definition 2. *The C^1 q-dimensional manifold $V_q \subset D$ is said to belong to the family $N(r,\ q)$ if $V_q' = X(V_q)$ is at every point orthogonal to the semiaxes $\lambda_1 \geqslant \ldots \geqslant \lambda_{r-1} \geqslant \lambda_{r+q} \geqslant \ldots \geqslant \lambda_n$ of $E(\zeta)$, where $E(\zeta)$ is the image of the unit ball B^n under the linear mapping $X'(\zeta)$.*
In this work we deal with manifolds that satisfy the following

Definition 3. *A C^1 q-dimensional manifold V_q is said to be admissible if there exists a domain $O \subset R^q$ and an $(\alpha;\ 1,\ q)$-pc diffeomorphism $\zeta(u)$ of O onto V_q.*

If $M_q^{V_q}(\Gamma)$ is the modulus of the curve family $\Gamma \subset V_q$ with respect to V_q, then we can show ([5]) that if $X(\zeta)$ is an $(\alpha;\ r,\ q)$-pc diffeomorphism, and if $V_q \in N(r,\ q)$ we have:

$$\sup_{\Gamma \subset V_q} [M_q^{V_q}(\Gamma)/M_q^{V_q'}(\Gamma')] \leqslant \alpha, \tag{1}$$

where $\Gamma' = X(\Gamma)$ and the supremum is taken over all curve families Γ in V_q such that $M_q^{V_q}(\Gamma)$ and $M_q^{V_q'}(\Gamma')$ are not both zero or infinite.

In this work, at section § 2, we show that if $X(\zeta)$ is an $(\alpha;\ r,\ q)$-pc mapping and if $V_q \in N(r,\ q)$ we have also

$$\sup_{\Gamma \subset V_q} [M_q^{V_q'}(\Gamma')/M_q^{V_q}(\Gamma)] \leqslant \alpha^{q-1}; \tag{2}$$

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at section § 3, making use of (1), (2) and of some theorems contained in [1] and in [6] we find some results about the behavior of an (r, q) - pc diffeomorphism on the boundary of the domain D.

§ 2. (r, q) - pc mappings and modulus of curve families

Theorem 1. *Let $X = X(\zeta)$ be an $(\alpha ; r, q)$* - pc *diffeomorphism of D onto $D' \subset R^n$, and let $N(r, q)$ be the family introduced in Definition 2. Then we have:*

$$\sup_{V_q \in N(r,q)} \{\sup_{\Gamma \subset V_q} [M_q^{V'_q}(\Gamma')/M_q^{V_q}(\Gamma)]\} \leqslant \alpha^{q-1}, \tag{2}$$

where the curve families Γ are such that $M_q^{V_q}(\Gamma)$ and $M_q^{V'_q}(\Gamma')$, $(\Gamma' = X(\Gamma))$, are not both zero or infinite.

For the proof of this theorem first we prove two lemmas.

Lemma 1. *If $X(\zeta): D \to D'$ is an $(\alpha ; r, q)$* - pc *diffeomorphism, the inverse $\xi(X): D' \to D$ is an $(\alpha' ; r', q)$* - pc *diffeomorphism, with $\alpha' = \alpha^{q-1}$, $r' = n - r - q + 2$.*

Proof. If $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ are the functions considered in Definition 1, we have: $\lambda_r^q \leqslant \alpha(\lambda_r \ldots \lambda_{r+q-1})$ from which we obtain

$$\left(\frac{1}{\lambda_{r+q-1}}\right)^q \leqslant \alpha^{q-1}\left(\frac{1}{\lambda_{r+q-1}} \cdots \frac{1}{\lambda_r}\right).$$

If J is the Jacobian matrix of $X(\xi)$, J^{-1} will be the Jacobian matrix of $\xi(X)$. If $\mu_1^2 \geqslant \ldots \geqslant \mu_n^2$ are the proper values of $(J^{-1})^*(J^{-1}) = (JJ^*)^{-1}$, we have $\mu_i = 1/\lambda_{n-i+1}$, $i = 1, \ldots, n$. So we have, for each $X \in D'$: $\mu^q_{n-r-q+2} \leqslant \alpha^{q-1}(\mu_{n-r-q+2} \cdots \cdot_{n-r+1})$ therefore $\xi(X)$ is (α', r', q) - pc.

Lemma 2. *Let $X(\xi): D \to D'$ be a diffeomorphism and let $V_q \subset D$ belong to $N(r, q)$. Then, $V'_q (= X(V_q))$ with respect to the inverse diffeomorphism $\xi(X): D' \to D$, belongs to $N(r', q)$ with $r' = n - r - q + 2$.*

Proof. Let $\zeta_i = \zeta_i(\mu_1, \ldots, \mu_q)$, $i = 1, \ldots, n$, be the equations of V_q and let $\bar{\zeta}_i = \zeta_i(\bar{\mu}_1, \ldots, \bar{\mu}_q)$ be the coordinates of a point of V_q. The image of $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$ will be $(\bar{X}_1, \ldots, \bar{X}_n) \in V'_q$. By performing a preliminary translation and rotation followed by symmetries, we may suppose $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n) = (0, \ldots 0) = (\bar{X}_1, \ldots, \bar{X}_n)$, and that the linear mapping $X'(0)$ is $x_i = \lambda_i \zeta_i$, $(i = 1, \ldots, n)$ where $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ are the numbers considered in the Introduction. Then the equations of the tangent hyperplane of V'_q at 0 will be, as $V_q \in N(r, q)$:

$$x_i = 0 \quad i = 1, \ldots, r-1, \quad r+q, \ldots, n.$$

The equations of the linear mapping $\zeta'(0)$ will be $\zeta_i = (1/\lambda_i)x_i$ $(i = 1, \ldots, n)$ with $(1/\lambda_i) = \mu_{n-i+1}$, where μ_i are as in Lemma 1. The equations of the tangent hyperplane to V_q at 0 will be $\zeta_i = 0$ $(i = 1, \ldots, r-1, r+q, \ldots, n)$. Therefore, if $E(0)$ is the ellipsoid image of the unit ball B^n under the linear mapping $\zeta'(0)$, and if $\mu_1 \geqslant \ldots \geqslant \mu_n$ are its semiaxes, we see that V_q is in 0 orthogonal to the semiaxes $\mu_1, \ldots, \mu_{n-r-q+1}, \mu_{n-r+2}, \ldots, \mu_n$. Since this argument can be made for every point of V_q, V'_q belongs to $N(r', q)$.

Proof of Theorem 1. It follows immediatily from Lemma 1, Lemma 2 and from the (1).

Remark. For $r = 1$ and $q = n$, we don't find the same result of [6] (p. 46), where there is α instead of α^{n-1}. This depends on the fact that an $(\alpha ; 1, n)$ - pc diffeomorphism is not, in general, α-quasiconformal (see [6], p. 48), but it is α^{n-1}-quasiconformal, while an α-quasiconformal diffeomorphism is $(\alpha ; 1, n)$ - pc.

§ 3. (r, q) - pc mappings and boundary correspondence. It is known (see [6], p. 52) that if $X(\zeta): D \to D'$ is a quasiconformal diffeomorphism, and if $\bar{\xi}$ is an isolated point of ∂D, then there exists the $\lim_{\bar{\xi} \to p} X(\zeta)$; moreover, if $\bar{X}$ is an isolated point of $\partial D'$, then there exists the $\lim_{X \to \bar{X}} \zeta(X)$ (where $\zeta(X): D' \to D$ is the inverse of $X(\zeta)$). This result is not true in general, for (r, q) — *pc* mappings with $r > 1$ or $q < n$. Take, for example, the diffeomorphism:

$$(\rho' = \rho + 1 \quad O'_i = O_i \quad i = 1, \ldots, n-1; \quad \rho > 0), \tag{3}$$

where polar coordinates are used. If J denotes the Jacobian matrix of (3) at the point $(\rho, O_1, \ldots, O_{n-1})$, the proper values of J^*J are: $\lambda_1^2 = \ldots = \lambda_{n-1}^2 = (1 + 1/\rho)^2$, $\lambda_n^2 = 1$. Therefore the (3) is $(1 ; r, q)$ - pc with $r = 1, \ldots, n-1$ and $r + q \leqslant n$. For this mapping $D = \{\zeta \in R^n : \| \zeta \| > 0\}$ and $D' = \{X \in R^n : \| X \| > 1\}$. The origin O is an isolated point of ∂D, but there does not exist the limit of (3) for $\zeta \to 0$. Moreover the inverse mapping of (3), is $(1 ; r', q')$ - pc with $r' = 2, \ldots, n-1$ and $r' + q' \leqslant n + 1$, and also for it, the previous result is not true. We can say something about the behavior of an $(\alpha ; r, q)$ - pc diffeomorphism on the boundary of its domain if we consider q-dimensional manifolds V_q that satisfy Definitions 2 and 3. In fact we have the following

Theorem 2. *Let $X(\zeta): D \to D'$ be a (r, q)* - pc *diffeomorphism and let $V_q \subset D$ be an admissible manifold belonging to $N(r, q)$. If $\bar{\zeta}$ is an isolated point of the border of V_q, then there exists the $\lim_{\substack{\bar{\xi} \to \xi \\ q \in V_q}} X(\zeta)$; moreover, if $\bar{X}$ is an isolated point of the border of V'_q, then there exists the $\lim_{\substack{X \to \bar{X} \\ X \in V'_q}} \zeta(X)$, where $V'_q = X(V_q)$ and $\zeta(X)$ is the inverse mapping of $X(\zeta)$.*

In order to prove this theorem, we give two preliminary lemmas.

Lemma 3. *Let $A = R(C_0, C_1)$ be a ring of $O \subset R^q$ and let $\tilde{A} = R(\tilde{C}_0, \tilde{C}_1)$ be the image of A by $\zeta(u): O \to V_q$, where V_q is a q-dimensional admissible manifold in $R^n (n > q)$. If Γ_A is the family of curves that join C_0 to C_1 and are contained in A, and $\Gamma_{\tilde{A}}$ is the image of Γ_A by $\zeta(u)$, then we have:*

$$\alpha^{1-q} M_q^{V_q}(\Gamma_{\tilde{A}}) \leqslant M_q(\Gamma_A) \leqslant \alpha M_q^{V_q}(\Gamma_{\tilde{A}}), \tag{4}$$

where $\alpha \geqslant 1$ is a constant that does not depend on A neither on $\tilde{A}$.

Proof. Let $\tilde{\rho} \in F(\Gamma_{\tilde{A}})$, that is a non-negative Borel function $\tilde{\rho}: V_q \to \dot{R}^1$ such that $\int_{\gamma} \tilde{\rho}\, ds \geqslant 1$ for every locally rectifiable $\tilde{\gamma} \in \Gamma_{\tilde{A}}$. Define $\tilde{\rho}: O \to \dot{R}^1$ by $\rho(\zeta) = \tilde{\rho}[\zeta(u)]\mu_1(u)$, where $\mu_1^2 \geqslant \ldots \geqslant \mu_q^2$ are the proper values of W^*W, with $W = W(u)$ the Jacobian matrix, of $\zeta(u)$. If $\gamma \in \Gamma_A$ is the pre-image of $\tilde{\gamma}$ (that is $\tilde{\gamma} = \zeta(\gamma)$), it will be locally rectifiable (as $\zeta(u)$ is a diffeomorphism) and we have, by Corollary 5.4 of [6]

$$\int_{\gamma} \rho\, ds = \int_{\gamma} \tilde{\rho}[\zeta(u)]\mu_1(u)\, ds \geqslant \int_{\tilde{\gamma}} \tilde{\rho}\, ds \geqslant 1.$$

so $\rho \in F(\Gamma_A)$. Moreover, as $\mu_1^q \leqslant \alpha(\mu_1, \ldots, \mu_q)$ (since V_q is admissible) and $dV_q = |W^*W|^{\frac{1}{2}} dm_q = (\mu_1^2 \ldots \mu_n^r)^{1/2} dm_q$, it follows:

$$M_q(\Gamma_A) \leqslant \int_{O} \rho^q\, dm_q = \int_{O} \tilde{\rho}[\zeta(u)]^q \mu_1^q(u)\, dm_q \leqslant \alpha \int_{V_q} \tilde{\rho}^q\, dV_q.$$

Since this holds for every $\tilde{\rho} \in F(\Gamma_{\tilde{A}})$, we obtain the right side of (4). Let now $\rho \in F(\Gamma_A)$. Define $\tilde{\rho} = \rho[u(\zeta)].[\mu_q(u(\zeta)]^{-1}$ where $u(\zeta): V_q \to O$ is the inverse mapping of $\zeta(u)$. If $\gamma \in \Gamma_{\tilde{A}}$ is locally rectifiable, so will be $\tilde{\gamma} = \zeta(\gamma) \in \Gamma_{\tilde{A}}$ and we have

$$\int_{\tilde{\gamma}} \tilde{\rho} ds = \int_{\tilde{\gamma}} \rho[u(\zeta)] . [\mu_q(u(\zeta))]^{-1} ds \geqslant \int_{\gamma} \rho(u) ds \geqslant 1,$$

so $\tilde{\rho} \in F(\Gamma_{\tilde{A}})$. As $\mu_1^q \leqslant \alpha(\mu_1 \ldots \mu_q)$ will be $\left(\frac{1}{\mu_q}\right)^q \leqslant \alpha^{q-1}\left(\frac{1}{\mu_1} \cdots \frac{1}{\mu_q}\right)$ and we also have

$$M_q^{V_q}(\Gamma_{\tilde{A}}) \leqslant \int_{V_q} \tilde{\rho}^q dV_q = \int_{V_q} \rho[u(\zeta)]^q . [\mu_q(u(\zeta))]^{-q} dV_q \leqslant \alpha^{q-1} \int_{O} \rho^q dm_q.$$

Since this holds for every $\rho \in F(\Gamma_A)$, we obtain the left side of (4).

Lemma 4. *Let V_q be a q-dimensional admissible manifold in R^n $(n > q)$ and let $\tilde{A} = R(\tilde{C}_0, \tilde{C}_1)$ be a ring in V_q. Then $M_q^{V_q}(\Gamma_{\tilde{A}}) = 0$ if and and only if $\tilde{C}_0$ or $\tilde{C}_1$ consists of a single point.*

Proof. Let $A = R(C_0, C_1)$ be the ring of $O \subset R^q$, pre-image of $\tilde{A}$ by $\zeta(u): O \to V_q$, where $\zeta(u)$ is the diffeomorphism of Definiton 3. In virtue of Lemma 3, we have to show that $M_q(\Gamma_A) = 0$ if and only if C_0 or C_1 consists of a single point. But this is true for Theorem 11.10 of [6].

Proof of Theorem 2. Choose in R^n a neighbourhood U of $\bar{\zeta}$ such that $U \cap \partial V_q = \{\bar{\zeta}\}$ and that $U \cup V_q$ is a ring $A = R(C_0, C_1)$, where $C_0 = \{\bar{\zeta}\}$, $C_1 = \partial U \cap V_q$. By Lemma 4, $M_q^{V_q}(\Gamma_A) = 0$. Let now $A' = R(C'_0, C'_1)$ be the image of A by the diffeomorphism $X(\zeta)$, with $C'_1 = X(C_1)$, $C'_0 = C(X(\zeta), \bar{\zeta}; V_q)$, where $C(X(\zeta), \bar{\zeta}; V_q)$ is the set of all points $X \in R^n$ such that there exists a sequence $\{\zeta_i\}$, such that $\zeta_i \in V_q$, $\lim\limits_{i\to\infty} \zeta_i = \bar{\zeta}$, and $\lim\limits_{i\to\infty} X(\zeta_i) = X$. By Theorem 1 we have $M_q^{V'_q}(\Gamma_{A'}) = 0$, hence, from Lemma 4, C'_0 consists of a single point. The above reasonings can be used to prove the second part of this theorem. We need only to make use of (1) instead of (2).

Let's now give a theorem about the behavior of an (r, q)-pc diffeomorphism on the non isolated points of the boundary of V_q.

Theorem 3. *Let $X(\zeta): D \to D'$ be an $(\alpha; r, q)$-pc diffeomorphism and let $V_q \subset D$ with $V_q \in N(r, q)$. If V_q and V'_q $(= X(V_q))$ are admissible manifolds in the sense of Definition 3. but with O equal to the unit ball B^q of R^q, and if they are locally connected on the boundary, then, $X(\zeta): V_q \rightleftarrows V'_q$ can be extended to a homeomorphism of $V_q \cup \partial V$ onto $V'_q \cup \partial V'_q$.*

For the proof we need to show two lemmas.

Lemma 5. *Let $O \subset R^q$, $V_q \subset R^n$ $(n > q \geqslant 2)$, $\zeta(u): O \to V_q$ be like in Definition 3. Let $X(\zeta)$; $D \to D'$ $(D, D' \subset R^n)$ be an $(\alpha'; r, q)$-pc diffeomorphism such that $V_q \subset D$, $V_q \in N(r, q)$. Then $X = X[\zeta(u)]: O \to V'_q$ $(V'_q = X(V_q))$ is an $(\alpha\alpha'; 1, q)$-pc diffeomorphism.*

Proof. Let $\Gamma_1 \subset O$ be a family of curves locally rectifiables, and $\Gamma_2 = \zeta(\Gamma_1)$. By Theorem 15.1 of [6] we have $M_q(\Gamma_1) \leqslant \alpha M_q^{V_q}(\Gamma_2)$. If $\Gamma_3 = X(\Gamma_2)$, for (1) we have $M_q^{V_q}(\Gamma_2) \leqslant \alpha' M_q^{V'_q}(\Gamma_3)$.

Hence $M_q(\Gamma_1) \leqslant \alpha\alpha' M_q^{V'_q}(\Gamma_3)$. Since this holds for every curve family Γ_1, from Theorem 15.2 of [6], the lemma follows.

Remark. In the proof of Lemma 5 we made use of Theorems 15.1 and 15.2 of [6] (pp. 46, 47). Indeed, such Theorems are proved in [6] for a diffeomorphism, of a domain $O \subset R^q$ onto a domain $V_q \subset R^q$; yet there are not difficulties (see also Theorems 1 and 2 of [5]) to prove such theorems when V_q is a smooth q-dimensional manifold of R^n $(n > q)$.

Lemma 6. *Let $\zeta(u): B^q \to V_q$ be a $(1, q)$-pc diffeomorphism of the unit ball $B^q \subset R^q$ onto a C^1 q-dimensional manifold $V_q \subset R^n$ $(n > q)$, with V_q locally connected on the boundary. Then, $\zeta(u)$ can be extended to a homeomorphism of $B^q \cup \partial B^q$ onto $V_q \cup \partial V_q$.*

Proof. The proof can be obtained like that of Theorem 1 of [1] (p. 379).

Proof of Theorem 3. For Lemmas 5 and 6, it is possible to extend $\zeta(u)$ to a homeomorphism of $B^q \cup \partial B^q$ onto $V_q \cup \partial V_q$, and $X[\zeta(u)]$ to a homeomorphism of $B^q \cup \partial B^q$ onto $V'_q \cup \partial V'_q$. Hence, we can extend $X(\zeta)$ to a homeomorphism of $V_q \cup \partial V_q$ onto $V'_q \cup \partial V'_q$.

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SOME PROPERTIES OF THE REDUCED SUSPENSION AND A GENERALIZATION OF *CW*-COMPLEXES

BY

D. MOTREANU

The first part of this paper sets forth some properties of the reduced suspension. In the second part we define generalized *CW*-complexes and using the properties of the reduced suspension, we note their elementary topological properties. In [1] there is also a generalization of *CW*-complexes using mappingtorus to obtain n-skeleton from $(n-1)$-skeleton.

1. Properties of the Reduced Suspension. **1.1.** Throughout this section if Y is a pointed topological space, with base point y_0, SY will denote its reduced suspension and define $S^kY = S(S^{k-1}Y)$, $k \geqslant 1$, with elements $[x, t_1, \ldots, t_k]$ for $x \in Y$ and $t_i \in I$. It is very easy to prove that for Y compact, S^kY is compact, $k \geqslant 1$ (SY is Hausdorff space because $Y \times 0 \cup y_0 \times I \cup Y \times 1$ is closed in $Y \times I$ normal, hence any two equivalence classes can be included in disjoint open classes).

Proposition 1.2. *Let T be a locally compact space. $Y \subset T$ a compact set and ∂Y the boundary of Y with respect to T. Let be $y_0 \in \partial Y$ and regard Y and ∂Y as pointed spaces (with the same base point y_0). Then $S^k\,\partial Y$ is homeomorphic to ∂S^kY.*

Proof. Assume $k = 1$. We will prove that the quotient map $\Pi: Y \times I \to SY$ restricts to a homeomorphism $(Y - \partial Y) \times (I - \dot{I}) \to \operatorname{Int} SY$. Since Π is open, $\Pi((Y - \partial Y) \times (I - \dot{I})) \subset \operatorname{Int} SY$. If $[y, t] \in \operatorname{Int} SY$, there is an open neighborhood V of $[y, t]$ such that $[y, t] \in V \subset SY$, and because $(y, t) \in \Pi^{-1}(V) \subset Y \times I$ with $\Pi^{-1}(V)$ open, $(y, t) \in \operatorname{Int}(Y \times I) = (Y - \partial Y) \times (I - \dot{I})$. Thus $\Pi/(Y - \partial Y) \times (I - \dot{I})$: $(Y - \partial Y) \times (I - \dot{I}) \to \operatorname{Int} SY$ is surjective and since $y_0 \notin Y - \partial Y$, it is also injective. Quite analogously we may prove that there exists a homeomorphism $(Y - \partial Y) \times (I - \dot{I}) \approx SY - S\partial Y$, hence $S\partial Y \approx \partial SY$. Suppose $S^i\,\partial Y = \partial S^iY$ for $i < k$ and $k > 1$. Then $S^k\partial Y = S(S^{k-1}\,\partial Y) = S(\partial S^{k-1}Y) = \partial S^kY$, and so the proof is ended.

Let $Y \dot{\times} Y'$ be the smash product of the pointed spaces (Y, y_0) and (Y', y'_0).

Proposition 1.3. *$S^nY \dot{\times} S^mY' = S^{n+m}(Y \dot{\times} Y')$, for n, $m \geqslant 1$.*

Proof: For $n = m = 1$ we have

$$SY \dot{\times} SY' = \frac{Y \times I}{Y \times 0 \cup Y \times 1 \cup y_0 \times I} \dot{\times} \frac{Y' \times I}{Y' \times 0 \cup Y' \times 1 \cup y'_0 \times I} =$$

$$= \frac{Y \times Y' \times I \times I}{Y \times Y' \times \dot{I} \times I \cup Y \times Y' \times I \times \dot{I} \cup y_0 \times Y' \times I \times I \cup Y \times y'_0 \times I \times I} =$$

$$= \frac{Y \times Y' \times I \times I}{Y \times Y' \times \dot{I}^2 \cup Y \times y'_0 \times I^2 \cup y_0 \times Y' \times I^2} = S^2(Y \dot{\times} Y').$$

Assume inductively that the property is true for k, k', $1 \leqslant k < n$ and $1 \leqslant k' < m$. Then $S^nY \dot{\times} S^mY' = S^{n-1}(SY) \dot{\times} S^{m-1}(SY') = S^{n+m-2}(S^2(Y \dot{\times} Y')) = S^{n+m}(Y \dot{\times} Y')$.

Remark 1.4. For $n \geqslant 0$, we have also $(S^nY) \dot{\times} Y' = S^n(Y \dot{\times} Y')$. Thus $S^nY \dot{\times} S^mY' = S^{n+m}(Y \dot{\times} Y')$, n, $m \geqslant 0$.