Since this holds for every $\rho \in F(\Gamma_A)$, we obtain the left side of (4).

Lemma 4. *Let V_q be a q-dimensional admissible manifold in R^n $(n > q)$ and let $\tilde{A} = R(\tilde{C}_0, \tilde{C}_1)$ be a ring in V_q. Then $M_q^{V_q}(\Gamma_{\tilde{A}}) = 0$ if and and only if $\tilde{C}_0$ or $\tilde{C}_1$ consists of a single point.*

Proof. Let $A = R(C_0, C_1)$ be the ring of $O \subset R^q$, pre-image of $\tilde{A}$ by $\zeta(u): O \to V_q$, where $\zeta(u)$ is the diffeomorphism of Definiton 3. In virtue of Lemma 3, we have to show that $M_q(\Gamma_A) = 0$ if and only if C_0 or C_1 consists of a single point. But this is true for Theorem 11.10 of [6].

Proof of Theorem 2. Choose in R^n a neighbourhood U of $\bar{\zeta}$ such that $U \cap \partial V_q = \{\bar{\zeta}\}$ and that $U \cup V_q$ is a ring $A = R(C_0, C_1)$, where $C_0 = \{\bar{\zeta}\}$, $C_1 = \partial U \cap V_q$. By Lemma 4, $M_q^{V_q}(\Gamma_A) = 0$. Let now $A' = R(C_0', C_1')$ be the image of A by the diffeomorphism $X(\zeta)$, with $C_1' = X(C_1)$, $C_0' = C(X(\zeta), \bar{\zeta}; V_q)$, where $C(X(\zeta), \bar{\zeta}; V_q)$ is the set of all points $X \in R^n$ such that there exists a sequence $\{\zeta_i\}$, such that $\zeta_i \in V_q$, $\lim_{i\to\infty} \bar{\zeta}_i = \bar{\zeta}$, and $\lim_{i\to\infty} X(\zeta_i) = X$. By Theorem 1 we have $M_q^{V_q'}(\Gamma_{A'}) = 0$, hence, from Lemma 4, C_0' consists of a single point. The above reasonings can be used to prove the second part of this theorem. We need only to make use of (1) instead of (2).

Let's now give a theorem about the behavior of an (r, q)-pc diffeomorphism on the non isolated points of the boundary of V_q.

Theorem 3. *Let $X(\zeta): D \to D'$ be an $(\alpha; r, q)$-pc diffeomorphism and let $V_q \subset D$ with $V_q \in N(r, q)$. If V_q and V_q' $(= X(V_q))$ are admissible manifolds in the sense of Definition 3. but with O equal to the unit ball B^q of R^q, and if they are locally connected on the boundary, then, $X(\zeta): V_q \rightleftarrows V_q'$ can be extended to a homeomorphism of $V_q \cup \partial V$ onto $V_q' \cup \partial V_q'$.*

For the proof we need to show two lemmas.

Lemma 5. *Let $O \subset R^q$, $V_q \subset R^n$ $(n > q \geqslant 2)$, $\zeta(u): O \to V_q$ be like in Definition 3. Let $X(\zeta)$; $D \to D'$ $(D, D' \subset R^n)$ be an $(\alpha'; r, q)$-pc diffeomorphism such that $V_q \subset D$, $V_q \in N(r, q)$. Then $X = X[\zeta(u)]: O \to V_q'$ $(V_q' = X(V_q))$ is an $(\alpha\alpha'; 1, q)$-pc diffeomorphism.*

Proof. Let $\Gamma_1 \subset O$ be a family of curves locally rectifiables, and $\Gamma_2 = \zeta(\Gamma_1)$. By Theorem 15.1 of [6] we have $M_q(\Gamma_1) \leqslant \alpha M_q^{V_q}(\Gamma_2)$. If $\Gamma_3 = X(\Gamma_2)$, for (1) we have $M_q^{V_q}(\Gamma_2) \leqslant \alpha' M_q^{V_q'}(\Gamma_3)$.

Hence $M_q((\Gamma_1) \leqslant \alpha\alpha' M_q^{V_q'}(\Gamma_3)$. Since this holds for every curve family Γ_1, from Theorem 15.2 of [6], the lemma follows.

Remark. In the proof of Lemma 5 we made use of Theorems 15.1 and 15.2 of [6] (pp. 46, 47). Indeed. such Theorems are proved in [6] for a diffeomorphism, of a domain $O \subset R^q$ onto a domain $V_q \subset R^q$; yet there are not difficulties (see also Theorems 1 and 2 of [5]) to prove such theorems when V_q is a smooth q-dimensional manifold of R^n $(n > q)$.

Lemma 6. *Let $\zeta(u): B^q \to V_q$ be a $(1, q)$-pc diffeomorphism of the unit ball $B^q \subset R^q$ onto a C^1 q-dimensional manifold $V_q \subset R^n$ $(n > q)$, with V_q locally connected on the boundary. Then, $\zeta(u)$ can be extended to a homeomorphism of $B^q \cup \partial B^q$ onto $V_q \cup \partial V_q$.*

Proof. The proof can be obtained like that of Theorem 1 of [1] (p. 379).

Proof of Theorem 3. For Lemmas 5 and 6, it is possible to extend $\zeta(u)$ to a homeomorphism of $B^q \cup \partial B^q$ onto $V_q \cup \partial V_q$, and $X[\zeta(u)]$ to a homeomorphism of $B^q \cup \partial B^q$ onto $V_q' \cup \partial V_q'$. Hence, we can extend $X(\zeta)$ to a homeomorphism of $V_q \cup \partial V_q$ onto $V_q' \cup \partial V_q'$.

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SOME PROPERTIES OF THE REDUCED SUSPENSION AND A GENERALIZATION OF CW-COMPLEXES

BY

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The first part of this paper sets forth some properties of the reduced suspension. In the second part we define generalized *CW*-complexes and using the properties of the reduced suspension, we note their elementary topological properties. In [1] there is also a generalization of *CW*-complexes using mappingtorus to obtain n-skeleton from $(n-1)$-skeleton.

1. Properties of the Reduced Suspension. 1.1. Throughout this section if Y is a pointed topological space, with base point y_0, SY will denote its reduced suspension and define $S^kY = S(S^{k-1}Y)$, $k \geqslant 1$, with elements $[x, t_1, \ldots, t_k]$ for $x \in Y$ and $t_i \in I$. It is very easy to prove that for Y compact, S^kY is compact, $k \geqslant 1$ (SY is Hausdorff space because $Y \times 0 \cup y_0 \times I \cup Y \times 1$ is closed in $Y \times I$ normal, hence any two equivalence classes can be included in disjoint open classes).

Proposition 1.2. *Let T be a locally compact space, $Y \subset T$ a compact set and ∂Y the boundary of Y with respect to T. Let be $y_0 \in \partial Y$ and regard Y and ∂Y as pointed spaces (with the same base point y_0). Then $S^k\,\partial Y$ is homeomorphic to ∂S^kY.*

Proof. Assume $k = 1$. We will prove that the quotient map $\Pi: Y \times I \to SY$ restricts to a homeomorphism $(Y - \partial Y) \times (I - \dot{I}) \to \operatorname{Int} SY$. Since Π is open, $\Pi((Y - \partial Y) \times (I - \dot{I})) \subset \operatorname{Int} SY$. If $[y, t] \in \operatorname{Int} SY$, there is an open neighborhood V of $[y, t]$ such that $[y, t] \in V \subset SY$, and because $(y, t) \in \Pi^{-1}(V) \subset Y \times I$ with $\Pi^{-1}(V)$ open, $(y, t) \in \operatorname{Int}(Y \times I) = (Y - \partial Y) \times (I - \dot{I})$. Thus $\Pi/(Y - \partial Y) \times (I - \dot{I})$: $(Y - \partial Y) \times (I - \dot{I}) \to \operatorname{Int} SY$ is surjective and since $y_0 \notin Y - \partial Y$, it is also injective. Quite analogously we may prove that there exists a homeomorphism $(Y - \partial Y) \times (I - \dot{I}) \approx SY - S\partial Y$, hence $S\partial Y \approx \partial SY$. Suppose $S^i\,\partial Y = \partial S^iY$ for $i < k$ and $k > 1$. Then $S^k\partial Y = S(S^{k-1}\,\partial Y) = S(\partial S^{k-1}Y) = \partial S^kY$, and so the proof is ended.

Let $Y \dot{\times} Y'$ be the smash product of the pointed spaces (Y, y_0) and (Y', y_0').

Proposition 1.3. *$S^nY \dot{\times} S^mY' = S^{n+m}(Y \dot{\times} Y')$, for $n, m \geqslant 1$.*

Proof: For $n = m = 1$ we have

$$SY \dot{\times} SY' = \frac{Y \times I}{Y \times 0 \cup Y \times 1 \cup y_0 \times I} \dot{\times} \frac{Y' \times I}{Y' \times 0 \cup Y' \times 1 \cup y_0' \times I} =$$

$$= \frac{Y \times Y' \times I \times I}{Y \times Y' \times \dot{I} \times I \cup Y \times Y' \times I \times \dot{I} \cup y_0 \times Y' \times I \times I \cup Y \times y_0' \times I \times I} =$$

$$= \frac{Y \times Y' \times I \times I}{Y \times Y' \times \dot{I}^2 \cup Y \times y_0' \times I^2 \cup y_0 \times Y' \times I^2} = S^2(Y \dot{\times} Y').$$

Assume inductively that the property is true for k, k', $1 \leqslant k < n$ and $1 \leqslant k' < m$. Then $S^nY \dot{\times} S^mY' = S^{n-1}(SY) \dot{\times} S^{m-1}(SY') = S^{n+m-2}(S^2(Y \dot{\times} Y')) = S^{n+m}(Y \dot{\times} Y')$.

Remark 1.4. For $n \geqslant 0$, we have also $(S^nY) \dot{\times} Y' = S^n(Y \dot{\times} Y')$. Thus $S^nY \dot{\times} S^mY' = S^{n+m}(Y \dot{\times} Y')$, $n, m \geqslant 0$.

Proposition 1.5. *If (Y, Y_0) is a pair of topological pointed spaces with the same base point y_0 and Y_0 is closed in Y, then there is a homeomorphism $S^kY - S^kY_0 \approx (Y - Y_0) \times \times (I - \dot{I})^{k-1}$.*

Proof. Let $\Pi : Y \times I^{k-1} \to S^kY$ be the quotient map. For $(Y - Y_0) \times (I - \dot{I})^{k-1}$, an equivalence class consists of a single point, hence $\Pi/(Y - Y_0) \times (I - \dot{I})^{k-1}$ is bijective and because $(Y - Y_0) \times (I - \dot{I})^{k-1}$ is open, $\Pi/(Y - Y_0) \times (I - \dot{I})^{k-1}$ is homeomorphism.

Proposition 1.6. *If Y is a compact metrizable space, then SY is metrizable.*

Proof. If Y is metrizable then $Y \times I$ is metrizable and let ρ be a metric on $Y \times I$. Define $\rho' : SY \times SY \to R$ by the formula:

$$\rho'([y, t], [y', t']) = \begin{cases} \rho((y, t), (y', t')) & (y, t), (y', t') \in (Y - y_0) \times (I - \dot{I}) \\ \inf\limits_{(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I} \{\rho((y, t), (a, \tau))\} & (y, t) \in (Y - y_0) \times (I - \dot{I}) \text{ and} \\ & (y', t') \in Y \times 0 \cup Y \times 1 \cup y_0 \times I \\ 0 & (y, t), (y', t') \in Y \times 0 \cup Y \times \\ & \times 1 \cup y_0 \times I \end{cases}$$

Obviously $\rho'([y, t], [y', t']) \geqslant 0$, $\rho'([y, t], [y', t']) = 0$ if and only if $[y, t] = [y', t']$ and $\rho'([y, t], [y', t',]) = \rho'([y', t'], [y, t])$. We must prove that $\rho'([y, t], [y'', t'']) \leqslant \rho'([y, t], [y', t']) + \rho'([y', t'], [y'', t''])$. Since excepting the following case, all the others are very easy to prove, we suppose $(y, t), (y'', t'') \in (Y - y_0) \times (I - \dot{I})$ and $(y', t') \in Y \times 0 \cup Y \times 1 \cup \cup y_0 \times I$. Then we have:

$$\rho'([y, t], [y', t']) + \rho'([y', t'], (y'', t'']) = \inf \rho((y, t), (a, \tau)) + \inf \rho((y'', t''), (a, \tau)) \geqslant$$
$$\geqslant \inf [\rho((y, t), (a, \tau)) + \rho((y'', t''), (a, \tau)] \geqslant \rho((y, t), (y'', t'')) = \rho'([y, t], [y'', t'']) \quad \text{for each}$$
$$(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I.$$

Let $p : Y \to SY$ be the quotient map. We must prove that the metric topology is equivalent to the quotient topology on SY. Let $\tilde{y}_0$ be the base point of SY, that is the equivalence class obtained by collapsing $Y \times 0 \cup Y \times 1 \cup y_0 \times I$ to a single point.

Assume $[y, t] \neq \tilde{y}_0$ and U is an open neighborhood of $[y, t]$ in the quotient topology such that $\tilde{y}_0 \notin U$ (possible because SY is a Hausdorff space by 1.1). Then $(y, t) \in p^{-1}(U)$, $p^{-1}(U)$ is open and $p^{-1}(U) \subset (Y - y_0) \times (I - \dot{I})$. Because $p^{-1}(U)$ has the metric topology induced by ρ, U is metric space and is open in SY with the metric topology $(p/p^{-1}(U) : :p^{-1}(U) \to U$ is homeomorphism). Choose now U an open neighborhood of $\tilde{y}_0$ in SY (SY with the quotient topology). Since $\tilde{y}_0 \in U$, $Y \times 0 \cup Y \times 1 \cup y_0 \times I \subset p^{-1}(U)$ and $p^{-1}(U)$ is open in $Y \times I$. Then $(Y \times I) - p^{-1}(U)$ is compact and $\mu = \inf \{\rho((a, \tau), (x, t'))\} > 0$ for each) $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I$ and $(x, t') \in (Y \times I) - p^{-1}(U)$. Define $S_{\rho'}(\tilde{y}_0, \mu/2) = \{[y, t] \in' \in SY/\rho'(\tilde{y}_0, [y, t]) < \mu/2\}$. We will prove that $S_{\rho'}(\tilde{y}_0, \mu/2) \subset U$. Suppose there is $[y, t] \in S_\rho (\tilde{y}_0, \mu/2)$ and $[y, t] \notin U$. Then $(y, t) \notin p^{-1}(U)$ and $\mu \leqslant \inf \{\rho((y, t), (a, \tau))\} = \rho'([y, t], \tilde{y}_0)$ for $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I$. This contradicts the hypothesis that $\rho'(\tilde{y}_0, [y, t]) < \mu/2$. Suppose again $[y, t] \neq \tilde{y}_0$. If SY has the ρ'-metric topology, we choose a sphere $S_{\rho'}([x, t], r)$ such that $\tilde{y}_0 \notin S_{\rho'}([y, t], r)$. Then $p^{-1}(S_{\rho'}([y, t], r)) = \{(y', t') \in (Y - y_0) \times (I - \dot{I})/\rho((y, t), (y', t')) < r\}$ is open, hence $S_{\rho'}([y, t], r))$ is an open neighborhood of $[y, t]$ in the quotient topology on SY. Now suppose that $S_{\rho'}(\tilde{y}_0, r)$ is an open sphere in SY with the ρ'-metric topology. Then $p^{-1}(S_{\rho'}(\tilde{y}_0, r)) = \{(y, t)/\inf \{\rho((y, t), (a, \tau)) < r\}$ for $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup \cup y_0 \times I$ and we will prove that $p^{-1}(S_{\rho'}(\tilde{y}_0, r))$ is open in $Y \times I$.

If $(y, t) \in p^{-1}(S_{\rho'}(\tilde{y}_0, r))$ then $\inf \{\rho((y, t), (a, \tau))\} < r$, for $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I$ Since $Y \times 0 \cup Y \times 1 \cup y_0 \times I$ is compact, there is $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times I$ such that $\inf \{\rho((y, t), (a, \tau))\} = \rho((y, t), (a, \tau))$, for $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup y_0 \times 1$. Denote $\rho((y, t), (a, \tau)) = b$ and choose $\eta < r - b$. If $(y', t') \in S_\rho((y, t), \eta)$ then for each $(a, \tau) \in Y \times 0 \cup$

$\cup\ Y \times 1 \cup y_0 \times I$ we have $\rho((y', t'), (a, \tau)) \leqslant \rho((y, t), (y', t')) + \rho((y, t), (a, \tau))$. Thus $\inf \rho((y', t'), (a, \tau)) \leqslant \rho((y, t), (y', t')) + \inf \{\rho((y, t), (a, \tau))\} < \eta + b < r$, for $(a, \tau) \in Y \times 0 \cup Y \times 1 \cup \cup y_0 \times \mathrm{I}$.

It follows $S_\rho((y, t), \eta) \subset p^{-1}(S_{\rho'}(\tilde{y}_0, r))$ and then $p^{-1}(S_{\rho'}, (\tilde{y}_0, r))$ is open in $Y \times I$ and $S_{\rho'}(\tilde{y}_0, r)$ is open in SY.

Corollary 1.7. If Y is a compact metrizable space, then S^nY is metrizable, $n \geqslant 0$.

2. A Generalization of CW-Complexes. Let Y be a compact subset of a locally compact space T. Let ∂Y denote the boundary of Y in T and $\partial S^{k-1}Y$ is the boundary of the compact set $S^{k-1}Y$ with respect to the locally compact space $S^{k-1}T$.

Definition 2.1. *Let X be a Hausdorff space and let A be a closed subset of X. X is said to be obtained from A by adjoining (T, Y, n)-cells if there is a family of continuous maps $f_i : S^{n-1} \partial Y \to A$ such that $X = V(S^{n-1} Y)_i V_{(f_i)} A$, where $(S^{n-1} Y)_i = S^{n-1}Y$ for each i. f_i is called an attaching map. Then the characteristic maps $f_i' : S^{n-1}(Y, \partial Y) \to (X, A)$ are well defined and $e_n^i = f_i'(S^{n-1} Y)$ is called a n-cell.*

For us it is convenient to consider that $S^{-1}Y$ is a one-point space if $Y \neq \partial Y$ and $S^{-1}Y = \emptyset$ if $Y = \partial Y$. Then $S^{-1}\partial Y = \emptyset$ for $n = 0$, X is the topological sum of A and a discrete space.

Definition 2.2. *Let X be a Hausdorff space and let A be a closed subset of X. A sequence of skeleta of X relative to A is a sequence of closed subspaces $(X, A)^n$, $n = 0, 1, \ldots$ of X such that:*

1) $(X, A)^0$ is obtained from A by adjoining 0-cells,
2) $(X, A)^n$ is obtained from $(X, A)^{n-1}$ by adjoining n-cells.

Definition 2.3. *A generalized CW-complex or a Y-CW-complex (X, A) consists of a Hausdorff space X, a closed subspace A and a sequence of skeleta $(X, A)^n$ of X relative to A such that:*

(1) $X = \bigcup\limits_{n \in N} (X, A)^n$,
(2) *X has the weak topology with respect to the subsets $(X, A)^n \subset X$.*

Example 2.4. The usual CW-complexes are obtained for $Y = I \approx E^1$ and $\partial Y = \partial I = \dot{I} \approx S^0$. It suffices to show that $SE^n \approx E^{n+1}$ where $E^n = \{x \in R^n / \| x \| \leqslant 1\}$. Since $S(S^{n-1}) \approx \approx S^n$, it only remains to verify that $S(\mathrm{int}\, E^n) \approx \mathrm{int}\, E^{n+1}$. In [5] the following homeomorphism is defined $f : S(S^n) \approx S^{n+1}$ by

$$f([z, t]) = \begin{cases} p_-^{-1}(2tz + (1 - 2t)p_0') & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ p_+^{-1}((2 - 2t)z + (2t - 1)p_0') & \dfrac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where $p_- : H_- \to D^{n+1}$ and $p_+ : H_+ \to D^{n+1}$ are projection maps with

$$H_- = \{z = (z_1, \ldots, z_{n+2}) \in S^{n+1}/z_{n+2} \leqslant 0\}, \quad H_+ = \{z = (z_1, \ldots, z_{n+2})/z_{n+2} \geqslant 0\}$$

and $p_0' = (1, 0, \ldots, 0) \in S^{n+1}$.

It is well-known that $\mathrm{int}\, E^n \approx S^n - p_0$ with $p_0 = (1, 0, \ldots, 0) \in S^n$ and $S(S^n) - - S(S^n - p_0) = [p_0, 1/2]$($p_0$ is assumed to be the base point of S^n). $f/S(S^n - p_0)) : S(S^n - p_0) \to \to S^{n+1}$ is a homeomorphism on its image, hence we only must prove that $f(S(S^n - p_0)) = = S^{n+1} - p_0'$, or because f is bijective $f[p_0, 1/2]) = p_0'$. The last fact is very easy to prove.

Example 2.5. If the compact space Y is such that $Y - \partial Y \approx (\partial Y - y_0) \times (I - \dot{I})$, then $S^{n-1}Y$ is an absolute Y-CW-complex with

$$(S^{n-1} Y)^p = \begin{cases} y_0 & p < n - 1, \\ S^{n-1} \partial Y & p = n - 1, \\ S^{n-1} Y & p \geqslant n, \end{cases}$$

by identifying y_0 with the base point of $S^{n-1} Y$. For $p = n - 1$ let $f : S^{n-2} \partial Y \to y_0$ be the attaching map and using Propositions 1.2. and 1.5. we can prove $(S^{n-1} Y)^{n-1} \approx S^{n-1} \partial Y$. If $p = n$ let $1_{S^{n-1}\partial Y}$ be the attaching map and clearly $(S^{n-1}Y)^n = S^{n-1} Y$.

Example 2.6. Each compact space Y with discrete boundary ∂Y is an absolute Y-
CW-complex of dimension one. Y is obtained from ∂Y by attaching a single 1-cell by the
map $1_{\partial Y}$.

Example 2.7. If (X_1, A_1) is a pointed Y_1-CW-complex and (X_2, A_2) is a pointed Y_2 –
– CW-complex and either X_1 or X_2 is locally compact, then $X_1 \dot{\times} X_2$ is a $(Y_1 \dot{\times} Y_2)$ – CW-
complex. If f'_{i_1}: $(S^{n-1} Y_1, S^{n-1} \partial Y_1) \rightarrow (e^n_{i_1}, \dot{e}^n_{i_1})$ and $g'_{i_2}: (S^{m-1} Y_2, S^{m-1} \partial Y_2) \rightarrow (\varepsilon^m_{i_2}, \dot{\varepsilon}^m_{i_2})$ are
characteristic maps for (X_1, A_1) and (X_2, A_2) respectively, then for $X_1 \dot{\times} X_2$ we have the
following characteristic maps $f'_{i_1} \times g'_{i_2}: (S^{n-1} Y_1, S^{n-1} \partial Y_1) \dot{\times} (S^{m-1} Y_2, S^{m-1} \partial Y_2) \rightarrow (e^n_{i_1}, \dot{e}^n_{i_1}) \times (\varepsilon^m_{i_2}, \dot{\varepsilon}^m_{i_2})$
It follows from Propositios 1.2 and 1.3 that $(S^{n-1} Y_1, S^{n-1} \partial Y_1) \dot{\times} (S^{m-1} Y_2, S^{m-1} \partial Y_2) =$
$= (S^{n-1} Y_1 \dot{\times} S^{n-1} Y_2, S^{n-1} Y_1 \dot{\times} S^{m-1} \partial Y_2 \cup S^{n-1} \partial Y_1 \dot{\times} S^{m-1} Y_2) = (S^{n+m-2} (Y_1 \dot{\times} Y_2)$, $(S^{n-1} Y_1 \times$
$\times \partial S^{m-1} Y_2) \cup (\partial S^{n-1} Y_1 \dot{\times} S^{m-1} Y_2)) = (S^{n+m-2} (Y_1 \times Y_2), \partial (S^{n-1} Y_1 \dot{\times} S^{m-1} Y_2)) = (S^{n+m-2} (Y_1 \dot{\times} Y_2)$,
$(S^{n+m-2} (\partial (Y_1 \dot{\times} Y_2)))$. Since Y_1 and Y_2 are compact spaces, $Y_1 \dot{\times} Y_2$ is compact, thus the new
characteristic maps are well defined. We have $(X_1 \dot{\times} X_2)^n = \bigcup_{i+j=n} (X^i_1 \dot{\times} X^j_2)$. In the weak topology
with respect to the new characteristic maps, $X_1 \dot{\times} X_2$ is a generalized CW-complex and be-
cause Y_1 or Y_2 is a locally compact space, the weak topology and the quotient topology
on $X_1 \dot{\times} X_2$ coincide.

Example 2.8. In 2.7 let $(X_1, A_1) = (X, A)$ be a Y – CW-complex and let $(X_2, A_2) =$
$= (Z, B)$ be a $\dot{I}$ – CW-complex. Then $(X \dot{\times} Z, A \dot{\times} B)$ is a Y – CW-complex.

Remark 2.9. It follows from the definition of the generalized CW-complexe that the
characteristic maps are continuous and also that the inclusion $A \subset X$ is continuous. Clearly
$X = A \cup (\bigcup_{i \in I} (e^n_i - \dot{e}^n_i))$ and this is a disjoint union. By Proposition 1.5, int $e^n_i = e^n_i - \dot{e}^n_i$ is
homeomorphic to $(Y - \partial Y) \times (I - \dot{I})^{n-2}$.

Remark 2.10. By proposition 1.1 and Remark 2.9 each e^n_i is a compact set, thus
closed in X.

Remark 2.11. If $f'_i: S^{n-1}(Y, \partial Y) \rightarrow (X, A)$ is the characteristic map for e^n_i, then f'_i is
a closed map and thus $f'_i|(S^{n-1} Y - S^{n-1} \partial Y): S^{n-1} Y - S^{n-1} \partial Y \rightarrow e^n_i - \dot{e}^n_i$ is a homeomor-
phism.

There is the following generalization of a well-known result concerning the usual
CW-complexes.

Proposition 2.12. *Let (X, A) be a topological pair such that X is a Hausdorff space
and A is closed in X. Suppose that there is a sequence of skeletons $(X, A)^n$ of X such that
$\bigcup (X, A)^n = X$. (X, A) is a generalized CW-complex if and only if X has the weak topology
with respect to the subsets A and each e^n_i.*

Proof. The first part is obvious. Conversely assume that $F \subset X$ is such that $F \cap A$
is a closed subset of A and also $F \cap e^n_i$ is closed in e^n_i for each e^n_i. We show by induction
that $F \cap (X, A)^n$ is closed in $(X, A)^n$, for all $n \geqslant 0$. If $n = 0$, $(X, A)^0 = A \vee B$ where B
is a discrete set and the result is clear. Assume the result for $k < n$ where $n \geqslant 1$. Since
$(X, A)^n = V(S^{n-1} Y)_i V_{(f_i)} (X, A)^{n-1}$, $F \cap (X, A)^n$ is closed in $(X, A)^n$ if and only if $\Pi^{-1}_n (F \cap$
$\cap (X, A)^n)$ is closed in $V(S^{n-1} Y)_i V(X, A)^{n-1}$ where $\Pi_n: V(S^{n-1} Y)_i V(X, A)^{n-1} \rightarrow (X, A)^n$ is
the quotient map. Because $\Pi^{-1}_n (F \cap (X, A)^n) \cap (S^{n-1} Y)_i = f'^{-1}_i (F \cap e^n_i)$ and the characteristic
map f'_i is continuous, $\Pi^{-1}_n (F \cap (X, A)^n) \cap (S^{n-1} Y)_i$ is closed in $(S^{n-1} Y)_i$. By the inductive
assumption $\Pi^{-1}_n (F \cap (X, A)^n) \cap (X, A)^{n-1} = F \cap (X, A)^{n-1}$ is closed in $(X, A)^{n-1}$.

Corollary 2.13. *If A is a compact set, the generalized CW-complex X is a compactly
generated space.*

Corollary 2.14. *If X is a generalized CW-complex, if Y is a topological space, and if
$f: X \rightarrow Y$ is a function, then f is continuos if and only if $f|A$ is continuous and $f|e^n_i$ is con-
tinous for each cell of X. One can prove by the same technique as in the case of the usual
CW-complexes:*

Proposition 2.15. *If (X, A) is a generalized CW-complex, then each n-cell e^n_j, disjoint
of A, meets only a finite number of open cells of dimension $< n$.*

Definition 2.16. *Given the generalized CW-complexes (X, A) and (Z, B), a map f: $(X, A) \rightarrow (Z, B)$ is said to be cellular if $f((X, A)^n) \subset (Y, B)^n$. Therefore the (T, Y) – CW-complexes and cellular maps define a category.*

Definiton 2.17. *Let (X, A) be a generalized CW-complex, Then $(Z, B) \subset (X, A)$ is a subcomplex of (X, A) provided the following condition is satisfied: if $(X, A)^n$ is a sequence of skeletons of (X, A) there is a sequence of skeletons $(Z, B)^n$ of (Z, B) such that $(Z, B)^n = (X, A)^n \cap Z$, for all $n \geqslant 0$.*

Remark 2.18. From Proposition 2.12 it follows that if (Z, B) is a subcomplex of the generalized CW-complex (X, A) and if B is closed in A, then Z is closed in X. Because the union of subcomplexes is a subcomplex, each union of subcomplexes is a closed set.

Analogous to the usual CW-complexes we can state the following result:

Proposition 2.19. *If (Z, B) is a subcomplex of the generalized CW-complex (X, A), then in the induced topology, Z is a generalized CW-complex.*

Example 2.20. Let (X', A') and (Z, B) be Y – CW-complexes, let (X, A) be a subcomplex of (X', A') and suppose f: $(X, A) \rightarrow (Z, B)$ is a continuous cellular map. Then $(X' V_f Z, A' V_f B)$ is a generalized CW-complex. If $\Pi: (X' V Z, A' V Z) \rightarrow (X' V_f Z, A' V_f Z)$ is the quotient map, then for $(X' V_f Z, A' V_f Z)$ we have the following characteristic maps:

a) $(S^{n-1} Y, S^{n-1} \partial Y) \xrightarrow{f'_i} (X', A') \xrightarrow{\Pi} (X' V_f Z, A' V_f B)$ where $f'_i: (S^{n-1} Y, S^{n-1} \partial Y) \rightarrow (X', A')$ is the characteristic map of (X', A') and not of (X, A).

b) $(S^{n-1} Y, S^{n-1} \partial Y) \xrightarrow{g_i} (Z, B) \xrightarrow{\Pi} (X' V_f Z, A' V_f B)$ where g_i is a characteristic map of (Z, B).

Lemma 2.21. *Let (X, A) be a generalized CW-complex and $M \subset X$ such that $M - A \neq$ $\neq \emptyset$. Then M contains a discrete set.*

Proof. $M - A \subset \bigcup_{i \in I_1} (e^n_i - \dot{e}^n_i)$ and it is a disjoint union. For each $i \in I_1$, let $z_i \in (e^n_i -$ $- \dot{e}^n_i) \cap (M - A)$ and let $F \subset \{z_i\}$. Clearly $F \cap A \neq \emptyset$. If e^n_i is a cell of X, then $F \cap e^n_i = (F \cap \dot{e}^n_i) \cup (F \cap (e^n_i - \dot{e}^n_i))$. $F \cap \dot{e}^n_i \subset \bigcup (e^m_i - \dot{e}^m_i)$ where $m < n$ and the union must be finite. Thus $F \cap e^n_i$ is a finite set. Obviously $F \cap (e^n_i - \dot{e}^n_i) = \emptyset$ $i \notin I_1$ and z_i $i \in I_1$. It follows that $F \cap e^n_i$ is a closed subset of e^n_i. Now we can apply Proposition 2.12.

Proposition 2.22. *A subcomplex (Z, B) of a generalized CW-complex (X, A) is finite if and only if Z is a compact set of X.*

Proof. If Z is finite, then it is compact by Remark 2.10. The converse follows from Lemma 2.12 and the fact that a compact set can contain at most a finite discrete subset.

Definition 2.23. *If (X, A) is a generalized CW-complex and if $M \subset X$, then the set $\sigma(M) = \cap \{(Z_i, B_i)/M \subset Z_i$ and (Z_i, B_i) is a subcomplex of $(X, A)\}$ is called the subcomplex generated by M.*

Remark 2.24. $\sigma(M)$ is really a subcomplex of (X, A) and if $M \subset M'$, then $\sigma(M) \subset$ $\subset \sigma(M')$. Thus σ is a covariant functor from the category of all subsets of X to the category of subcomplexes of (X, A).

Lemma 2.24. *$\sigma(e^n_i)$ is a finite subcomplex of the generalized complex (X, A).*

Proof. We first remark that $\sigma(e^n_i) = \sigma(\dot{e}^n_i) \cup (e^n_i - \dot{e}^n_i)$ and $\sigma(\dot{e}^n_i) \subset (X, A)^{n-1}$. Now by induction on n: if $n = 0$, then e^n_i is a point. Suppose the result holds for $0 \leqslant k \leqslant n - 1$ and $n > 1$. If $\dot{e}^n_i \subset A$, since $\sigma(e^n_i) = \sigma(\dot{e}^n_i) \cup (e^n_i - \dot{e}^n_i)$ the proof is ended; if $\dot{e}^n_i - A \neq \emptyset$, then $\dot{e}^n_i \subset (\dot{e}^n_i \cap A) \cup (\bigcup_{j \in J} (e^m_j - \dot{e}^m_j)$ and J is finite. It follows that $\dot{e}^n_i \subset A \cup (\bigcup_{j \in J} \sigma(\dot{e}^m_j))$ and by the inductive hypothesis the right part is a finite subcomplex. Then $\sigma(e^n_i)$ is a finite subcomplex and we obtain the desired result.

Proposition 2.25. *If K is a compact set of a generalized CW-complex (X, A), then $\sigma(K)$ is a finite subcomplex.*

Proof. If $K \subset A$, then $\sigma(K) = A$. If $K - A \neq \emptyset$, then $K \subset A \cup (\bigcup_{j \in J} (e^n_j - \dot{e}^n_j))$ where J is finite. It results that $\sigma(K) \subset A \bigcup_{j \in J} \sigma(e^n_j)$ and we can apply Lemma 2.24.

Proposition 2.26. *If (X, A) is a generalized CW-complex with A compact, then X is paracompact.*

Proof. The proof is similar to that used for the usual *CW*-complexes showing that for each open cover of X there is a partition of unity subordinated to it. We essentially use Proposition 2.25.

The following are some generalizations of the Definitions 2.1, 2.2 and 2.3.

Definition 2.27. *Let Y be a compact space and let Y_0 be a closed subspace of Y. We say that a Hausdorff space X is obtained from A by adjoining (Y, Y_0)—cells if there is family of continuous maps $f_i : S^{n-1} Y_0 \to A$ such that $X = V(S^{n-1}Y)_i V f_i A$. We define a sequence of skeletons of X relative to A like in Definition 2.2.*

A generalized CW-complex or (Y, Y_0)-CW-complex (X, A) is a pair of a Hausdorff space X, a closed subspace A of X together with a sequence of skeletons $(X, A)^n$ of X relative to A such that are verified the conditions 1) and 2) of Definition 2.3.

Remark 2.28. By Proposition 1.5, $\operatorname{Int} e_i^n$ is homeomorphic to $(Y - Y_0) \times (I - \dot{I})^{n-2}$ (e_i^n is a cell defined in an obvious sense with respect to Definition 2.27).

Proposition 2.29. *Each (T, Y)-CW-complex is a (Y, Y_0)-complex.*

Proof. It is sufficient to put $Y_0 = \partial Y$ and we obtain the generalized *CW*-complexes in the sense of Definition 2.3.

Proposition 2.30. *There exists a generalized CW-complex in the sense of Definition 2.27 that is not an usual CW-complex.*

Proof. Let A be a compact space and let Y_0 be a closed subset of A such that Y_0 is not path connected and has more than two points. Choose Y a compact space such that Y_0 is a subset of Y (in particular we can have $Y = A$). Let $X = Y V_f A$ where f is the inclusion map $f : Y_0 \subset A$. Then corresponding to f we have a characteristic map $f' : (Y, Y_0) \to (X, A)$ and let e be the closed cell $f'(Y)$. If we give on X the weak topology with respect to the inclusion maps $A \subset X$ and $e \subset X$, than it is clear that (X, A) is a (Y, Y_0) — *CW*-complex. Now we prove that (X, A) with the given weak topology is not an usual relative *CW*-complex. We first observe that (X, A) can have at most a single cell (in the sense of usual *CW*-complexes), namely e. In this case there is a continuous map $g : (E^n, S^{n-1}) \to (X, A)$ such that $g(E^n) = e$ and $g(S^{n-1}) = e \cap A = Y_0$. This fact contradicts the properties of Y_0.

The following property is a generalization of a result conserning the usual *CW*-complexes.

Proposition 2.31. *Let (Y, Y_0) be a pair such that Y is compact, Y_0 is closed in Y and the inclusion map $Y_0 \subset Y$ has the property that $S^k Y_0 \subset S^k Y$ is a cofibration for each $k \geq 0$. Then for every (Y, Y_0) — CW-complex (X, A), the inclusion map $A \subset X$ is a cofibration.*

The proof is similar to that used in [2].

Definition 2.32. *Let $\{(Y_\alpha, Y'_\alpha)\}_{\alpha \in A}$ be a family of pairs such that for each $\alpha \in A$, Y_α is a compact space and Y_α is closed in Y'_α. (X, A) is a $\{(Y_\alpha, Y'_\alpha)\}_{\alpha \in A}$ — CW-complex if X is a Hausdorff space, A is a closed subspace of X and there is a sequence $(X, A)^n$ of skeletons of (X, A) ($(X, A)^n$ is obtained from $(X, A)^{n-1}$ in the sense of Definition 2.27, but with more than one compact pair) so that all the conditions of the Definition 2.3 are verified.*

Proposition 2.33. The attachement of handles to a closed manifold (in the sense of Morse theory) is obtained by adjoining of cells in the sense of Definition 2.32.

Proof. Let $\{(E^{k+1}, S^{k-1} \times E^1)\}_{(k,l)}$ be the family considered in the Definition 2.32 and let $f_i : (S^{k-1} \times E^1) \to f_i (S^{k-1} \times E^1)$ be the attaching map. Then $f'_i / (\operatorname{Int} E^k) \times E^1$ (where f'_i is the corresponding to f_i characteristic map) is homeomorphism.

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AGAIN ON THE SURFACES WHICH ALLOW ∞^2 PROJECTIVE TRANSFORMATIONS INTO THEMSELVES

BY

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To the memory of Mendel Haimovici

Introduction. The surfaces of the ordinary space with ∞^2 (and not more) projective transformations (p. t.) into themselves have been studied for the first time by the eminent geometer F. E n r i q u e s in [1] and [2].

In the ordinary space, all algebraic surfaces with only ∞^2 p. t. into themselves belong to the following types, (see [3]):

a. *The algebraic surfaces of Klein-Lie,*

b. *The surfaces containing a pencil of conics which are obtained by intersection with the planes through a straight line,*

c. *The surfaces of the sixth order with elliptic sections.*

Among the last surfaces, E n r i q u e s considered in [2] the surface

$$k(x_1x_3 - x_2^2)^3 - (x_1x_4 + 2x_2^3 - 3x_1x_2x_3)^2 = 0, \tag{1.1}$$

where k is a parameter.

If we exclude the values $0, -4, \infty$ of k one obtains non-ruled surfaces with ∞^2 p. t. into themselves.

In [2] *Enriques stressed that the asymptotic curves of the surfaces* (1.1) *are twisted cubics.*

In our paper [4] we have given some properties which characterize the surfaces with ∞^2 p. t. into themselves from the point of view of Fubini's projective differential geometry [5]. So we have shown that:

1. *The Klein-Lie algebraic surfaces are the coincidence minimal projective surfaces;*

2. *The surfaces which possess a pencil of conics obtained by the intersection of the surfaces with a pencil of planes, are determined by:*

$$\beta = \sqrt{\frac{2}{|K|}}(u - v)^{-1}; \quad \gamma = -\frac{2}{|K|}(u - v)^{-1}; \quad K(K + 2) \neq 0,$$

$$L = M = -\frac{3}{|K|}(u - v)^{-2}, \tag{1.2}$$

K constant < 0 being the *curvature of the first differential form of Fubini*, and β, γ, L, M the well known functions of the theory of surfaces, u, v, the asymptotic parameters. They are *projective-rotation surfaces projectively indeformable*, and *the conjugate lines to the conics form with these a double net of Koenigs.* (see [4]).

3. *The surfaces of the sixth order with elliptic sections admitting ∞^3 p. t. into themselves are determined by*

$$\beta = \frac{1}{\alpha}(u - v)^{-1}; \ \gamma = -\alpha(u - v)^{-1}; \ L = M = \frac{3}{2}(u - v)^{-2}. \tag{$\mathcal{J}$}$$