Proposition 2.26. *If (X, A) is a generalized CW-complex with A compact, then X is paracompact.*

Proof. The proof is similar to that used for the usual CW-complexes showing that for each open cover of X there is a partition of unity subordinated to it. We essentially use Proposition 2.25.

The following are some generalizations of the Definitions 2.1, 2.2 and 2.3.

Definition 2.27. *Let Y be a compact space and let Y_0 be a closed subspace of Y. We say that a Hausdorff space X is obtained from A by adjoining (Y, Y_0)–cells if there is family of continuous maps $f_i : S^{n-1} Y_0 \to A$ such that $X = V(S^{n-1}Y)_i V f_i A$. We define a sequence of skeletons of X relative to A like in Definition 2.2.*

A generalized CW-complex or (Y, Y_0)-CW-complex (X, A) is a pair of a Hausdorff space X, a closed subspace A of X together with a sequence of skeletons $(X, A)^n$ of X relative to A such that are verified the conditions 1) and 2) of Definition 2.3.

Remark 2.28. By Proposition 1.5, $\operatorname{Int} e_i^n$ is homeomorphic to $(Y - Y_0) \times (I - \dot{I})^{n-2}$ (e_i^n is a cell defined in an obvious sense with respect to Definition 2.27).

Proposition 2.29. *Each (T, Y)-CW-complex is a (Y, Y_0)-complex.*

Proof. It is sufficient to put $Y_0 = \partial Y$ and we obtain the generalized CW-complexes in the sense of Definition 2.3.

Proposition 2.30. *There exists a generalized CW-complex in the sense of Definition 2.27 that is not an usual CW-complex.*

Proof. Let A be a compact space and let Y_0 be a closed subset of A such that Y_0 is not path connected and has more than two points. Choose Y a compact space such that Y_0 is a subset of Y (in particular we can have $Y = A$). Let $X = YV_f A$ where f is the inclusion map $f : Y_0 \subset A$. Then corresponding to f we have a characteristic map $f' : (Y, Y_0) \to (X, A)$ and let e be the closed cell $f'(Y)$. If we give on X the weak topology with respect to the inclusion maps $A \subset X$ and $e \subset X$, than it is clear that (X, A) is a $(Y, Y_0) - CW$-complex. Now we prove that (X, A) with the given weak topology is not an usual relative CW-complex. We first observe that (X, A) can have at most a single cell (in the sense of usual CW-complexes), namely e. In this case there is a continuous map $g : (E^n, S^{n-1}) \to (X, A)$ such that $g(E^n) = e$ and $g(S^{n-1}) = e \cap A = Y_0$. This fact contradicts the properties of Y_0.

The following property is a generalization of a result conserning the usual CW-complexes.

Proposition 2.31. *Let (Y, Y_0) be a pair such that Y is compact, Y_0 is closed in Y and the inclusion map $Y_0 \subset Y$ has the property that $S^k Y_0 \subset S^k Y$ is a cofibration for each $k \geq 0$. Then for every $(Y, Y_0) - CW$-complex (X, A), the inclusion map $A \subset X$ is a cofibration.*

The proof is similar to that used in [2].

Definition 2.32. *Let $\{(Y_\alpha, Y'_\alpha)\}_{\alpha \in A}$ be a family of pairs such that for each $\alpha \in A$, Y_α is a compact space and Y_α is closed in Y'_α. (X, A) is a $\{(Y_\alpha, Y'_\alpha)\}_{\alpha \in A} - CW$-complex if X is a Hausdorff space, A is a closed subspace of X and there is a sequence $(X, A)^n$ of skeletons of (X, A) ($(X, A)^n$ is obtained from $(X, A)^{n-1}$ in the sense of Definition 2.27, but with more than one compact pair) so that all the conditions of the Definition 2.3 are verified.*

Proposition 2.33. The attachement of handles to a closed manifold (in the sense of Morse theory) is obtained by adjoining of cells in the sense of Definition 2.32.

Proof. Let $\{(E^{k+1}, S^{k-1} \times E^1)\}_{(k,l)}$ be the family considered in the Definition 2.32 and let $f_i : (S^{k-1} \times E^1) \to f_i(S^{k-1} \times E^1)$ be the attaching map. Then $f'_i / (\operatorname{Int} E^k) \times E^1$ (where f'_i is the corresponding to f_i characteristic map) is homeomorphism.

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AGAIN ON THE SURFACES WHICH ALLOW ∞^2 PROJECTIVE TRANSFORMATIONS INTO THEMSELVES

BY

FROIM MARCUS

To the memory of Mendel Haimovici

Introduction. The surfaces of the ordinary space with ∞^2 (and not more) projective transformations (p. t.) into themselves have been studied for the first time by the eminent geometer F. Enriques in [1] and [2].

In the ordinary space, all algebraic surfaces with only ∞^2 p. t. into themselves belong to the following types, (see [3]):

a. *The algebraic surfaces of Klein-Lie,*

b. *The surfaces containing a pencil of conics which are obtained by intersection with the planes through a straight line,*

c. *The surfaces of the sixth order with elliptic sections.*

Among the last surfaces, Enriques considered in [2] the surface

$$k(x_1x_3 - x_2^2)^3 - (x_1x_4 + 2x_2^3 - 3x_1x_2x_3)^2 = 0, \tag{1.1}$$

where k is a parameter.

If we exclude the values $0, -4, \infty$ of k one obtains non-ruled surfaces with ∞^2 p. t. into themselves.

In [2] *Enriques stressed that the asymptotic curves of the surfaces* (1.1) *are twisted cubics.*

In our paper [4] we have given some properties which characterize the surfaces with ∞^2 p. t. into themselves from the point of view of Fubini's projective differential geometry [5]. So we have shown that:

1. *The Klein-Lie algebraic surfaces are the coincidence minimal projective surfaces;*

2. *The surfaces which possess a pencil of conics obtained by the intersection of the surfaces with a pencil of planes, are determined by:*

$$\beta = \sqrt{\frac{2}{|K|}}(u - v)^{-1}; \quad \gamma = -\frac{2}{|K|}(u - v)^{-1}; \quad K(K + 2) \neq 0,$$

$$L = M = -\frac{3}{|K|}(u - v)^{-2}, \tag{1.2}$$

K constant < 0 being the *curvature of the first differential form of Fubini*, and β, γ, L, M the well known functions of the theory of surfaces, u, v, the asymptotic parameters. They are *projective-rotation surfaces projectively indeformable*, and *the conjugate lines to the conics form with these a double net of Koenigs.* (see [4]).

3. *The surfaces of the sixth order with elliptic sections admitting ∞^3 p. t. into themselves are determined by*

$$\beta = \frac{1}{\alpha}(u - v)^{-1}; \; \gamma = -\alpha(u - v)^{-1}; \; L = M = \frac{3}{2}(u - v)^{-2}. \tag{$\mathcal{J}$}$$

Their asymptotic curves *belong to linear complexes and they are minimal projective surfaces.* Consequently, by [4] *they belong to the limiting case of Tzitzeica-Wilczynski's surfaces.* (See also [6]).

The purpose of this paper is to complete the study of these last surfaces and among others to determine them in finite form.

Some other properties of the surface $(\mathcal{J})$. 2. We will prove first the following result.

The asymptotic curves of the surfaces $(\mathcal{J})$ are twisted cubics.

Indeed let be

$$x^{(4)} + 4p_1x^{(3)} + 6p_2x^{(2)} + 4p_3x^{(1)} + p_4x = 0, \tag{2.1}$$

the differential equation of an asymptotic $v =$ const. or $u =$ const. of a surface (x). Suppose that the coordinates x are normalized in Wilczynski's sense; then they are the solutions of the completely integrable system:

$$x_{uu} = \beta x_v + p_{11}x; \quad x_{vv} = \gamma x_u + p_{22}x. \tag{2.2}$$

Taking into account (2.2) it follows that for an asymptote $v =$ const., we have:

$$\begin{gathered} 4p_1 = -2(\log\beta)_u; \quad 6p_2 = (\log\beta)_u^2 - \beta_v - (\log\beta)_{uu} - 2p_{11}; \\ 4p_3 = 2p_{11}(\log\beta)_u - 2p_{11u} - \beta^2\gamma; \\ p_4 = 2(p_{11})_u(\log\beta)_u - p_{11}(\log\beta)_u^2 + p_{11}^2 - p_{22}\beta^2 - \beta(p_{11})_v + (p_{11})_{uu} + p_{11}(\log\beta)_{uu}. \end{gathered} \tag{2.3}$$

Interchanging β, p_{11}, u, with γ, p_{22}, v, we obtain the coefficients of (2.1) from an asymptote $u =$ const. If (x) is a surface $(\mathcal{J})$ then from

$$L = -2p_{11} - \beta_v; \quad M = -2p_{22} - \gamma_u, \tag{2.4}$$

results

$$2p_{11} = \frac{3\alpha - 2}{2\alpha(u-v)^2}; \quad 2p_{22} = \frac{3 - 2\alpha}{2(u-v)^2}. \tag{2.5}$$

Consequently, for an asymptote $v =$ const. and $u =$ const. we get respectively:

$$p_1 = \frac{1}{2(u-v)}; \quad p_2 = -\frac{1}{4(u-v)^2}; \quad p_3 = \frac{3}{8(u-v)^3}; \quad p_4 = -\frac{15}{16(u-v)^4} \tag{2.6}$$

and

$$p_1 = -\frac{1}{2(u-v)}; \quad p_2 = -\frac{1}{4(u-v)^2}; \quad p_3 = -\frac{3}{8(u-v)^3}; \quad p_4 = -\frac{15}{16(u-v)^4}. \tag{2.7}$$

A curve on a surface is a twisted cubic if [7],

$$\theta_3 = \theta_4 = 0, \tag{2.8}$$

where θ_3 and θ_4 are invariants of a linear and homogeneous differential equation of the fourth order [7]. Their expressions are ([7] p. 240);

$$\theta_3 = P_3 - \frac{3}{2}P_2'; \quad \theta_4 = P_4 - 2P_3' + \frac{6}{5}P_2'' - \frac{6}{25}P_2^2, \tag{2.9}$$

where ([7] p. 16),

$$\begin{gathered} P_2 = p_2 - p_1' - p_1^2; \quad P_3 = p_3 - p_1'' - 3p_1p_2 + 2p_1^3, \\ P_4 = p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1^3, \end{gathered} \tag{2.10}$$

and P_i are the half-invariants of the differential equation of the fourth order.

The case $\theta_3 = 0$ shows that the asymptotic curves of the surface belong to linear-complexes ([7] p. 254). From (2.6), (2.7) and (2.10) follows:

$$P_2 = P_3 = P_4 = 0, \tag{2.11}$$

and consequently

$$\theta_3 = \theta_4 = 0. \tag{2.12}$$

Therefore:

The asymptotic curves of the surfaces $(\mathcal{J})$ are twisted cubics.

This result entails as we shall see later, why *the asymptotes of a Wilczynski's surfaces* ([8] and [9]) *with $U(u)$, $V(v)$ being polynomials of the third degree, are twisted cubics.*

From (2.6) and (2.7), *it follows that the coefficients p_i of the differential equation (2.1) of an asymptote of a surface $(\mathcal{J})$ are independent of the parameter α.* Therefore, we have the following result:

The asymptotic curves of the same family on a surface $(\mathcal{J})$ which are obtained when α varies, are projectively equivalent. This is another property of the surfaces $(\mathcal{J})$.

3. In an important paper, W i l c z y n s k i [8] *has determined in finite form the surfaces with undeterminate directrix curves and the asymptotic curves belonging to linear complexes.* They are generated by the points whose coordinates are:

$$x^1 = 4(U + V) - 2(u - v)(U^{10} - V^{01}); \quad x^2 = U^{10} - V^{01}, \quad x^3 = u + v; \quad x_4 = 1, \tag{A}$$

where u, v are asymptotic parameters, U is an arbitrary function of u alone and V of v alone whose derivatives or the third order do not vanish.

In a communication made to the *American Mathematical Society at Madison in 1913, W i l c z y n s k i* [9] *announced and seemingly without later demonstration* (see [10], p. 132) *that in the single case when $U(u)$, $V(v)$ are third degree polynomials, the surfaces (A) are rational of the sixth order, whose asymptotes are twisted cubics.*

A. T e r r a c i n i [10] has determined in finite form the non-ruled surfaces *whose asympto tes belong to linear complexes. They form three classes and Terracini has observed that the surfaces of the third class coincide with the surfaces (A) of Wilczynski.*

In our paper [6] we have proved the following theorems:

1. *Among Terracini's surfaces only the surfaces of the third class are minimal projective.*

2. *They belong to the limiting case of Tzizeica-Wilczynski's surfaces.*

With the formula established in (2.3) we can show that the asymptotes of the surfaces (A) of Wilczynski or of the third class of Terracini's surfaces *are twisted cubics $U(u)$, $V(v)$ are third order polynomials.* Indeed according to O. R o z e t et N. L e g r a i n-P i s s a r d [11] Terracini's surfaces of the third class are given in Wilczynski's coordinates by:

$$x^1 = \frac{u+v}{2\sqrt{\sigma}}; \quad x^2 = \frac{h}{2\sqrt{\sigma}}(U^{10} - V^{01}); \quad x^3 = \frac{1}{2\sqrt{\sigma}}\left[\frac{(U^{10} - V^{01})(u-v)}{2} - (U+V)\right]; \quad x^4 = \frac{1}{\sqrt{\sigma}}, \tag{A'}$$

$U^{30}V^{03} \neq 0$, h const., U^{i0}, V^{0i} the derivatives of order i with regard to u or v of U and V, and $\sigma = U^{20} + V^{02}$. One finds immediately that

$$\beta = -\frac{U^{30}}{\sigma}; \quad \gamma = -\frac{V^{03}}{\sigma}. \tag{3.1}$$

If $U(u)$ and $V(v)$ are polynomials of third degree, then,

$$\beta = -\frac{a}{\sigma}; \; \gamma = -\frac{b}{\sigma}, \tag{3.1'}$$

and

$$p_{11} = \frac{A}{4\sigma^2}; \quad p_{22} = \frac{B}{4\sigma^2}, \tag{3.2}$$

with

$$A = 3a^2 - 2ab\,; \quad B = 3b^2 - 2ab. \tag{3.3}$$

From (2.3) follows now

$$p_1 = \frac{a}{2\sigma}\,; \quad p_2 = -\frac{a^2}{4\sigma^2}\,; \quad p_3 = \frac{3a^3}{8\sigma^3}\,; \quad p_4 = -\frac{15a^4}{16\,\sigma^4}\,, \tag{3.4}$$

and from (2.10)

$$P_2 = P_3 = P_4 = 0. \tag{3.5}$$

According to (2.9) we have

$$\theta_3 = \theta_4 = 0, \tag{3.6}$$

which proves Wilczynski's proposition.

Determination of the surfaces $(\mathcal{T})$. 4. Wilczynski [8] has shown that it is possible to determine in finite form the surfaces whose directrix curves are indeterminate and that the asymptotes belong to linear complexes. We have shown [4] that among the surfaces whose asymptotes belong to linear complexes the only surfaces being minimal-projective, are the Tzitzeica-Wilczynski limiting surfaces. Consequently, Wilczynski's first directrix of a surface $(\mathcal{T})$ passes through a fixed point P and the second directrix lays in a fixed plane passing through the same point P, ([12] first vol. p. 168).

From $(\mathcal{T})$ and (2.7) it results that the coordinates x of a point which generates a surfaces $(\mathcal{T})$ verify the system

$$x_{uu} = \frac{1}{\alpha(u-v)}\,x_v + \frac{(3\alpha-2)}{4\alpha(u-v)^2}\,x\,; \quad x_{vv} = -\frac{\alpha}{u-v}\,x_u + \frac{3-2\alpha}{4(u-v)^2}\,x. \tag{$\mathcal{T}^*$}$$

A straight line congruence to a surface $x(u, v)$, where u, v are the asymptotic parameters is generated by the straight line (x, y) where (see [13])

$$y = x_{uv} - l_1 x_v - l_2 x_u. \tag{4.1}$$

If (x, y) is the first directrix of Wilczynski and supposing $\theta =$ const., then [13]:

$$l_1 = \frac{1}{2}\frac{\gamma_u}{\gamma} = -\frac{1}{2(u-v)}\,; \quad l_2 = \frac{1}{2}\frac{\beta_v}{\beta} = \frac{1}{2(u-v)}\,, \tag{4.2}$$

and the two focal points of (x, y) are given by:

$$\bar{x} = x + R(x_{uv} - l_1 x_v - l_2 x_u), \tag{4.3}$$

if we put

$$d\bar{x} = \lambda x + (\mu + dR)\,(x_{uv} - l_1 x_v - l_2 x_u), \tag{4.3'}$$

where λ and μ arbitrary functions of u, v. Taking into account the system $(\mathcal{T}^*)$ it follows

$$R = 4(u-v)^2. \tag{4.4}$$

Consequently

$$x = x + 4(u-v)^2 x_{uv} + 2(u-v)x_v - 2(u-v)x_u. \tag{4.3''}$$

Differentiating this equation with respect to u and v, and using $(\mathcal{T}^*)$ we obtain

$$(\bar{\mathrm{A}})\qquad \begin{aligned} \bar{x}_u &= \frac{5}{2(u-v)}\,[4(u-v)^2\,x_{uv} + 2(u-v)x_v - 2(u-v)x_u] = \frac{5}{2(u-v)}\,\bar{x}\,, \\ \bar{x}_v &= \frac{5}{2(u-v)}\,[4(u-v)^2\,x_{uv} + 2(u-v)x_v - 2(u-v)x_u] = \frac{5}{2(u-v)}\,\bar{x}\,, \end{aligned}$$

which proves once more that $\bar{x}$ is a fixed point.

Consider the points:

$$(\mathrm{B})\qquad \begin{aligned} x_1 &= x_u - \frac{1}{2}(\log\gamma)_u = x_u + \frac{1}{2(u-v)}\,x\,, \\ x_2 &= x_v - \frac{1}{2}(\log\beta)_v = x_v - \frac{1}{2(u-v)}\,x\,, \end{aligned} \qquad (\theta = \text{const.})$$

which determine Wilczynski's second directrix.

From the first equation of (B) and the system $(\mathcal{T}^*)$ it follows:

$$x_{1u} = \frac{\alpha-2}{4\alpha(u-v)^2}\,x + \frac{x_u}{2(u-v)} + \frac{x_v}{\alpha(u-v)}\,,$$

and taking into account $(\mathcal{T}^*)$ and (B), this equation can be written

$$x_{1u} - \frac{1}{2(u-v)}\,x_1 = \frac{1}{\alpha(u-v)}\left[x_v - \frac{x}{2(u-v)}\right] = \frac{1}{\alpha(u-v)}\,x_2. \tag{4.5}$$

Likewise we obtain

$$x_{1v} - \frac{x_1}{2(u-v)} = \frac{\bar{x}}{4(u-v)^2}\,. \tag{4.6}$$

From x_{2u} and x_{2v} we find, using $(\mathcal{T}^*)$ and (B) that

$$x_{2u} + \frac{1}{2(u-v)}\,x_2 = \frac{\bar{x}}{4(u-v)^2}\,; \quad x_{2v} + \frac{1}{2(u-v)}\,x_2 = -\frac{\alpha}{u-v}\,x_1 \tag{4.7}$$

Integration of $(\bar{\mathrm{A}})$ yields

$$\bar{x} = 8c_1(u-v)^{5/2}. \tag{C}$$

Therefore:

$$x_{1u} - \frac{x_1}{2(u-v)} = \frac{x_2}{\alpha(u-v)}\,; \quad x_{1v} - \frac{x_1}{2(u-v)} = 2c_1(u-v)^{1/2}, \tag{D}$$

and:

$$x_{2u} + \frac{x_2}{2(u-v)} = 2c_1(u-v)^{1/2}\,; \quad x_{2v} + \frac{x_2}{2(u-v)} = -\frac{\alpha x_1}{u-v}\,. \tag{E}$$

From the second equation of (D) and the first of (E) we obtain

$$x_1 = \frac{U_1(u)}{(u-v)^{1/2}} - c_1(u-v)^{3/2}\,; \quad x_2 = \frac{V_1(v)}{(u-v)^{1/2}} + c_1(u-v)^{3/2}. \tag{F}$$

Introducing the expressions of x_2 and x_1 in the first equation of (D) and in the second equation of (E) entails:

$$(\mathrm{G})\qquad \begin{aligned} \frac{U_1'(u)}{(u-v)^{1/2}} - \frac{U_1(u)}{(u-v)^{3/2}} - c_1(u-v)^{1/2} &= \frac{V_1(v)}{\alpha(u-v)^{3/2}} + \frac{c_1(u-v)^{1/2}}{\alpha}\,, \\ \frac{V_1'(v)}{(u-v)^{1/2}} + \frac{V_1(v)}{(u-v)^{3/2}} - c_1(u-v)^{1/2} &= \frac{\alpha U_1(u)}{(u-v)^{3/2}} + c_1\alpha(u-v)^{1/2}. \end{aligned}$$

Then

$$(\mathrm{H})\qquad \begin{aligned} &c_1(\alpha + 1)(u - v)^2 - (u - v)\alpha U_1'(u) + \alpha U_1(u) + V_1(v) = 0,\\ &c_1(1 + \alpha)(u - v)^2 - (u - v)\, V_1'(v) - \alpha U_1(u) - V_1(v) = 0. \end{aligned}$$

Therefore

$$(4.8)\qquad 2c_1(1 + \alpha)(u - v) = V_1'(v) + \alpha U_1'(u).$$

Operating with $\partial/\partial v$ or with $\partial/\partial u$, we find

$$(4.9)\qquad V_1''(v) + 2c_1(1 + \alpha) = 0; \quad \alpha U_1''(u) - 2c_1(1 + \alpha) = 0.$$

Thus

$$(4.10)\qquad \alpha U_1''(u) = - V_1''(v) = 2c_1(1 + \alpha),$$

and consequently

$$(4.11)\qquad \alpha U_1(u) = c_1(1 + \alpha)u^2 + c_2 u + c_3, \quad V_1(v) = - c_1(1 + \alpha)v^2 + \overset{*}{c}_2 v + \overset{*}{c}_3.$$

Introducing the expressions (4.11) in one of the equations (H), it follows:

$$(4.12)\qquad \overset{*}{c}_2 = - c_2; \quad \overset{*}{c}_3 = - c_3.$$

Therefore

$$(\mathrm{L})\qquad \alpha U_1(u) = c_1(1 + \alpha)u^2 + c_2 u + c_3; \quad V_1(v) = - c_1(1 + \alpha)v^2 - c_2 v - c_3.$$

The system (B) becomes now,

$$(\mathrm{B}')\quad x_u + \frac{1}{2(u - v)}\,x = \frac{U_1(u)}{(u - v)^{1/2}} - c_1(u - v)^{3/2}, \quad x_v + \frac{1}{2(u - v)}\,x = \frac{V_1(v)}{(u-v)^{1/2}} + c_1(u-v)^{3/2}.$$

By integration we get

$$(\Omega)\qquad x = \frac{\int U_1(u)du}{(u - v)^{1/2}} + \frac{V_2(v)}{(u - v)^{1/2}} - \frac{c^1}{3}(u - v)^{5/2} = \frac{\int V_1(v)du}{(u-v)^{1/2}} + \frac{U_2(u)}{(u-v)^{1/2}} - \frac{c_1(u-v)^{5/2}}{3}.$$

Consequently

$$U_2(u) - \int U_1(u)du = V_2(v) - \int V_1\, dv = k$$

where k is a constant.

If we denote

$$U(u) = \int U_1(u)du; \quad V(v) = \int V_1(v)dv,$$

we finally have

$$(\mathcal{J}^{**})\qquad x = \frac{U(u) + V(v) + k}{(u - v)^{1/2}} - \frac{c_1}{3}(u - v)^{5/2}.$$

One can prove that the four functions given by:

$$(\mathrm{M})\qquad \begin{aligned} &x^1 = \left(\frac{u}{\alpha} - v\right)(u - v)^{-1/2}; \quad x^2 = \left(\frac{u^2}{\alpha} - v^2\right)(u - v)^{-1/2},\\ &x^3 = (1 + \alpha)\left(\frac{u^3}{\alpha} - v^3\right)(u - v)^{-1/2} - (u - v)^{5/2}; \quad x^4 = \frac{\sqrt{\alpha}}{\alpha}(u - v)^{-1/2}, \end{aligned}$$

satisfy the system ($\mathcal{J}^*$) and the matrix (δ) ($x\, x_u x_v$), is of the third rank.

Therefore we have the folowing theorem:

The functions given by (M) *may be taken as the coordinates of the points generating (by variation of α) the surfaces ($\mathcal{J}$) which allow ∞^2 (and no more) projective transformations into themselves.*

The interpretation of the condition that U(u), V(v) be polynomials of third degree.

5. We now wish to prove the following result:

The surfaces (A) of Wilczynski or Terracini's surfaces of third class, allow a group G_2 of projective deformations into themselves if U and V are polynomials of third degree.

Indeed the coordinates x^i of a surface (A) satisfy the system:

$$(5.1)\qquad x_{uu} = \beta x_v + \theta_u x_u; \quad x_{vv} = \gamma x_u + \theta_v x_v, \qquad (\beta\gamma \neq 0)$$

with

$$(5.2)\qquad \beta = - \frac{U^{30}}{\sigma} = - \theta_u; \quad \gamma = - \frac{V^{03}}{\sigma} = - \theta_v; \quad \sigma = U^{20} + V^{02} + ,U^{30} V^{03} \neq 0.$$

From [18]

$$(\omega)\qquad L = \theta_{uu} - \frac{1}{2}\theta_u^2 - \beta\theta_v - \beta_v; \quad M = \theta_{vv} - \frac{1}{2}\theta_v^2 - \gamma\theta_u - \gamma_u,$$

it follows

$$(5.3)\qquad L = \frac{2\sigma U^{40} - 3(U^{30})^2}{2\sigma^2}; \qquad M = \frac{2\sigma V^{04} - 3(V^{03})^2}{2\sigma^2}.$$

Because $U^{30}V^{03} \neq 0$, it results that a surface (A) cannot be a *coincidence surface.* Suppose that a surface (A) admits a group G_2 of projective deformations into itself and that the group is generated by the infinitesimal transformations [12]

$$(5.4)\qquad X_1 = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}; \quad X_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

From Fubini's projective linear element, the following conditions for β, γ, must hold:

$$(5.5)\qquad \beta_u + \beta_v = \gamma_u + \gamma_v = 0; \quad \beta + u\beta_u + v\beta_v = 0, \quad \gamma + u\gamma_u + v\gamma_v = 0.$$

Taking into account (5.2) it follows that

$$(5.6)\qquad \begin{aligned} &(U^{30})^2 - \sigma U^{40} + U^{30}V^{03} = (V^{03})^2 - \sigma V^{04} + U^{30} V^{03},\\ &u(U^{30})^2 + vU^{30}V^{03} - \gamma(U^{30} + uU^{40}) = 0,\\ &uU^{30} V^{03} + v(V^{03})^2 - \sigma(V^{03} + vV^{04}) = 0. \end{aligned}$$

Observing that $U^{30} V^{03} \neq 0$, one obtains, from (5.6)

$$(5.7)\qquad \sigma + V^{03}(u - v) = 0; \quad \sigma + U^{30}(v - u) = 0.$$

Consequently

$$(\alpha)\qquad U^{30} + V^{03} = 0.$$

This shows that *the functions $U(u)$ and $V(v)$ are polynomials of degree three.* One can show that *the group G_2 is a group of collineations.* Indeed the group is generated by the infinitesimal transformations iff the third diferential form of Fubini $Ldu^2 + Mdv^2$ is invariant under the inf. trans. X_1 and X_2.

By (5) and (α) one obtains:

$$(5.8)\qquad L = -\frac{3}{2}\left(\frac{U^{30}}{\sigma}\right)^2; \quad M = -\frac{3}{2}\left(\frac{V^{03}}{\sigma}\right)^2; \quad U^{30} + V^{03} = 0,$$

from which

$$L_u + L_v = 0\,; \quad M_u + M_v = 0. \tag{5.9}$$

Similarly L and M also satisfy the conditions:

$$2L_u + uL_u + vL_v = 0\,; \quad 2M + uM_u + vM_v = 0, \tag{5.10}$$

because:

$$\begin{aligned} 2L_u + uL_u + vL_v &= -\frac{3(U^{30})^2}{\sigma^3}(\sigma + V^{03}(u-v)), \\ 2M + uM_u + vM_v &= -\frac{3(V^{03})^2}{\sigma^3}(\sigma + U^{30}(v-u)), \end{aligned} \tag{5.11}$$

which are annihilated by virtue of (5.7). *This shows that if $U(u)$ and $V(v)$ are polynomials of degree three, then the surfaces (A) admit a group G_2 of collineations into themselves.*

Reciprocally: Let (A') be a third class surface of Terracini or of Wilczynski with U, V third degree polynomials.

By (3.1), (3.1′), (3.2) and (3.3) putting

$$U^{30} = a\,; \quad V^{03} = b, \tag{5.12}$$

it follows that

$$\begin{aligned} &\beta = -\frac{a}{\sigma}\,; \quad \gamma = -\frac{b}{\sigma}\,; \quad \sigma = au + a_1 + bv + b_1\,, \\ &p_{11} = \frac{A}{4\sigma^2}\,; \quad p_{22} = \frac{B}{4\sigma^2}\,; \quad A = 3a^2 - 2ab\,; \quad B = 3b^2 - 2ab, \end{aligned} \tag{5.13}$$

and

$$L = -\frac{3a^2}{2\sigma^2}; \quad M = -\frac{3b^2}{2\sigma^2}. \tag{5.14}$$

Changing the asymptotic parameters and putting

$$\bar u = au + a_1\,; \quad \bar v = -bv + b_1. \tag{5.15}$$

and indicating by $\bar\beta$, $\bar\gamma$ the new values of β, γ, that results in

$$\begin{aligned} &\bar\beta = \frac{b}{a(\bar u - \bar v)}\,; \quad \bar\gamma = -\frac{a}{b(\bar u - \bar v)}\,; \quad \bar p_{11} = \frac{p_{11}}{a^2} = \frac{3a^2 - 2ab}{4a^2(\bar u - \bar v)^2}\,, \\ &\bar p_{22} = \frac{p_{22}}{b^2} = \frac{3b^2 - 2ab}{4b^2(\bar u - \bar v)^2}\,, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &\bar L = -\bar\beta_{\bar v} - 2\bar p_{11} = \frac{-b}{a(\bar u - \bar v)^2} - \frac{3a^2 - 2ab}{2a^2(\bar u - \bar v)^2} = \frac{-3}{2(\bar u - \bar v)^2}\,, \\ &\bar M = -\bar\gamma_{\bar u} - 2\bar p_{22} = -\frac{3}{2(\bar u - \bar v)^2}\,. \end{aligned} \tag{5.17}$$

Putting

$$\frac{a}{b} = \frac{1}{\alpha}, \tag{5.18}$$

we obtain

$$\bar\beta = \frac{1}{\alpha(\bar u - \bar v)}\,; \quad \bar\gamma = -\frac{\alpha}{\bar u - \bar v}, \quad L = M - \frac{3}{2(\bar u - \bar v)^2}\,, \tag{5.19}$$

which coincides up to the notations, with the formula $(\mathcal{J})$.

Changing the parameters by (5.15) the surfaces (A) or (A′) are given in finite form by (M). We have shown that the asymptotes of the surfaces $(\mathcal{J})$ or equivalently the surfaces determined by (M), are twisted cubics.

Taking into account the results obtained in this paragraph, one arrives at the following consequences:

1. *If the functions U and V, which determine a surface (A) of Wilczynski or a surface (A′) of Terracini, are polynomials of degree three, then it admits ∞^2 projective transformations into itself and reciprocally. It appears that this property was not observed by W i l c z y n s k i in his communication [9].*

2. In [2] F. E n r i q u e s observed that the asymptotes of the surfaces (1.1) admitting ∞^2 (and no more) projective transformations into themselves, are twisted cubics. As we have seen, *this property is a consequence of the fact that his surfaces (1.1) ($k \neq 0, -4, \infty$), admit ∞^2 (and no more) projective transformations into themselves.*

3. *The surfaces (A) or (A′) with $U(u)$, $V(v)$ polynomials of degree three coincide with the surfaces $(\mathcal{J})$ determined in finite form by (M).*

The Enriques surfaces coincide by a great deal with the surfaces $(\mathcal{J})$.

Other properties of the surfaces $(\mathcal{J})$. 6. Let S be a surface generated by the point $x(u, v)$ where u, v are the asymptotic parameters.

The ruled surfaces R_u, R_v [13] are generated by the points:

$$X = x_v + tx\,; \quad Y = x_u + tx. \tag{6.1}$$

The flecnodal points of the tangents of the asymptotes $u =$ const., and $v =$ const. of the R_u and R_v ruled surfaces, are determined by the conditions [13]

$$At^2 + 2Bt + C = 0\,; \quad A't^2 + 2B't + C = 0 \tag{6.2}$$

with [14]:

$$\begin{aligned} &A = \beta; \quad B = \frac{1}{2}(\beta\theta_v + \beta_v)\,; \quad C = \frac{1}{2}(\beta^2 M + \beta_{vv} + \beta_v\theta_v + \frac{1}{2}\beta\theta_v^2)\,; \\ &A' = \gamma; \quad B' = \frac{1}{2}(\gamma\theta_u + \gamma_u)\,; \quad C' = \frac{1}{2}(\gamma^2 L + \gamma_{uu} + \gamma_u\theta_u + \frac{1}{2}\gamma\theta_u^2). \end{aligned} \tag{6.3}$$

Then

$$B^2 - AC = \frac{\omega}{4}\Delta^2\,; \quad B'^2 - A'C' = \frac{\omega'}{4}\Delta'^2, \tag{6.4}$$

where [14]

$$\Delta^2 = -\omega(2\beta^2 M + 2\beta\beta_{vv} - \beta_v^2)\,; \quad \Delta'^2 = -\omega'(2\gamma^2 L + 2\gamma\gamma_{uu} - \gamma_u^2),$$

$$\Delta \geqslant 0\,; \quad \Delta' \geqslant 0 - \omega = \operatorname{sign}(2\beta^2 M + 2\beta\beta_v - \beta_v^2)\,; \quad -\omega' = \operatorname{sign}(2\gamma^2 L + 2\gamma\gamma_u - \gamma_u^2).$$

According to O. M a y e r [14] the *minimal surfaces are characterized by one of the equivalent conditions:*

$$\Delta = U(u)\,; \quad \Delta' = V(v), \tag{6.6}$$

where $U(u)$ and $V(v)$ are two arbitrary functions $\geqslant 0$.

According to D e m o u l i n [15] the flecnodal points of the ruled surfaces R_u, R_v which correspond to the point x, determine on Lie's quadrics $\mathcal{L}$ a left-quadrilateral (Demoulin's quadrilateral), having as vertices the characteristic points, other than x. According to M a y e r [14], these points are called the D (Demoulin) transformations of the point x. The D-points coincide if and only if [14]

$$\Delta = \Delta' = 0. \tag{6.7}$$

In this case S is a surface D of D e m o u l i n [16]. Therefore:

The sixth order surfaces with elliptical sections which admit ∞^2 (and no more) *projective transformations into themselves, the surfaces* $(\mathcal{J})$, enjoy the following properties:

a. *The flecnodal points of the asymptotic ruled surfaces coincide;*

b. *The Demoulin transformations of their points coincide.*

7. The relation between the surfaces $(\mathcal{J})$ and a class of surfaces obtained by O. Boruvka.

It seems that to the present the only known $(\mathcal{J})$ surfaces in finite form which admit ∞^2 projective transformations into themselves are the following:

1. *Enriques surfaces in asymptotic coordinate* given by [17],

$$x^1 = \frac{1+h}{2}u + \frac{1-h}{2}v; \quad x^2 = \frac{1+h}{2}u^2 + \frac{1-h}{2}v^2;$$

$$x^3 = \frac{(1+h)^2}{4h}u^3 - \frac{3(1-h^2)}{4h}u^2v + \frac{3(1-h^2)}{4h}uv^2 - \frac{(1-h)^2}{4h}V^3; \quad x^4 = 1. \tag{7.1}$$

where

$$h^2 = \frac{4}{4+k}; \tag{7.1'}$$

k the parameter from (1.1), u, v the asymptotic parameters.

2. *Wilczynski's or Terracini's surfaces of the third class with U, V third degree polynomials, or the surfaces $(\mathcal{J})$, given by* (M) which contain almost all Enriques surfaces. This follows as a consequence of the results in the present paper.

In one of his papers on projective differential geometry which the author sent to me recently, Otokar Boruvka [18] has determined in finite form a class of surfaces which admit a group of two parameters of projective deformations intro themselves with a projective subgroup. They are given by

$$x^1 = e^{ax+y/a}; \quad x^2 = \left(x - y + a + \frac{1}{a}\right)e^{y/a}; \quad x^3 = \left(x - y - a - \frac{1}{a}\right)e^{ax}, \tag{7.2}$$

where x, y are the asymptotic parameters, and four more classes of surfaces which admit a two parameter group of collineations into themselves, determined by:

$$x_1 = a^2x^3 + 3x^2y - 3xy^2 - \frac{1}{a^2}y^3; \quad x_2 = ax^2 - \frac{1}{a}y^2; \quad x_3 = ax - \frac{1}{a}y, \tag{7.3}$$

(one has corrected the signs in the formulae given in [18]),

$$x_1 = (x-y)^{-A}; \quad x_2 = (x+y)(x-y)^{-A}; \quad x_3 = a(x-y)^B - (a-2)(x+y)^2(x-y)^{-A}, \tag{7.4}$$

where

$$-A = \frac{a+2}{a}; \quad B = \frac{a-2}{2}; \quad a \neq 0, \ \pm 1, \ \pm 2,$$

$$x_1 = \frac{1}{(x-y)^2}; \quad x_2 = \frac{x+y}{(x-y)^2}; \quad x_3 = \log(x-y) - \left(\frac{x+y}{x-y}\right)^2, \tag{7.5}$$

and

$$x_1 = \log(x-y); \quad x_2 = x + y; \quad x_3 = (x-y)^2 - 2(x+y)^2, \tag{7.6}$$

where x, y are the asymptotic parameters. Boruvka has obtained these surfaces by integrating the system of differential equations established by E. Cartan in his mémoire „Sur la déformation projective des surfaces“ ([19] p. 311—314). O. Boruvka showed that in the research of the surfaces with ∞^2 projective transformations into themselves with a projective subgroup, there is a case which leads to a second order differential equation with two regular singular points, and with an irregular point. In all other cases one finds the precedent classes of surfaces. We must observe that in Boruvka's paper, [18] the second fun-

damental mémoire „Fondamenti di geometria proiettivo-differenziale“ of G. Fubini, [5], 1919, are not quoted. The curvatures K of Fubini's first differential form [13] of the last three classes of surfaces are respectively:

$$K = -\frac{2a^2}{(a+2)^2}; \quad K = -8; \quad K = -8. \tag{7.7}$$

Because these curvatures satisfy $K(K+2) \neq 0$, all the surfaces belong to the surfaces (1.2). Therefore they enjoy the properties given in our paper [4], among others, to be projective rotation surfaces, projectively non-deformable.

Consider now the surfaces (7.3). We shall show that they are precisely the surfaces $(\mathcal{J})$. Indeed, if we multiply the coordinates given by (M), with

$$\sqrt{\alpha}(u-v)^{-1/2}, \tag{7.8}$$

and putting

$$u = x; \quad v = y, \text{ and } a^2\alpha = 1, \tag{7.9}$$

we obtain:

$$\overline{x}^1 = \sqrt{\alpha}\left(\frac{u}{\alpha} - v\right) = ax - \frac{y}{a} = x_3, \quad \overline{x}^2 = \sqrt{\alpha}\left(\frac{u^2}{\alpha} - v^2\right) = au^2 - \frac{y}{a} = x_2,$$

$$\overline{x}^3 = \sqrt{\alpha}\,(1+\alpha)\left(\frac{u^3}{\alpha} - v\right) - (u-v) = a^2x^3 + 3x^2y - 3xy^2 - \frac{1}{a^2}y^3 = x_1 \tag{7.10}$$

where x_1, x_2, x_3 are precisely the coordinates (7.3) found by O. Boruvka. This proves once more, that the sixth order surfaces with elliptical sections which admit ∞^2 (and no more) projective transformations into themselves are given in finite form by (M).

Concerning the surfaces (7.2) we can prove that they are projectively applicable on the surfaces (7.3). Indeed putting in (7.2)

$$x = v, \quad y = u, \quad a^2\alpha = 1, \tag{7.11}$$

and

$$x = u; \quad y = v; \quad a^2\alpha = 1, \tag{7.12}$$

in (7.3), we obtain for these surfaces $\beta = 1/[\alpha(u-v)]$; $\gamma = -\alpha/(u-v)$, and this proves the assertion, *which did not seem self-evident*. Therefore we can state:

The surfaces (7.2) are projectively applicable on the surfaces (7.3).

8. Some other remarks. Finally we want to put in evidence another class of algebraic surfaces of the sixth order, which admit ∞^2 p. t. into themselves.

In their paper quoted above, Lane and MacQueen [17] have mentioned some results in relation to the problem of determining the analytic non-ruled surfaces whose asymptotic curves are twisted cubics. They have reduced this problem to the problem of integration of an ordinary differential equation. If S is a surface whose asymptotic curves $u =$ const., and $v =$ const. belong to linear complexes, then one can by a suitable choice of the parameters u, and v, obtain for β and γ the equation

$$\frac{\partial^2 \log\beta}{\partial u\,\partial v} = \beta^2, \quad (\beta = \gamma) \tag{8.1}$$

whose general solution is

$$\beta = \frac{\sqrt{\mathcal{U}'\mathcal{V}'}}{\mathcal{U} + \mathcal{V}} \quad (\mathcal{U}'\mathcal{V}' \neq 0), \tag{8.2}$$

where $\mathcal{U}$ and $\mathcal{V}$ are two arbitrary functions of u alone and v alone, respectively.

Denoting by S and T the Schwarzian derivatives of $\mathcal{U}$ and $\mathcal{V}$ namely

$$S = \left(\frac{\mathcal{U}''}{\mathcal{U}'}\right)' - \frac{1}{2}\frac{\mathcal{U}''}{\mathcal{U}'}\,; \quad T = \left(\frac{\mathcal{V}''}{\mathcal{V}'}\right)' - \frac{1}{2}\left(\frac{\mathcal{V}''}{\mathcal{V}'}\right), \tag{8.3}$$

then the differential equation

$$S'' + \frac{8}{20}S^2 + \frac{3}{5}SU_1 - \frac{16}{15}U_1^2 - 3U_1'' = 0, \tag{8.4}$$

where

$$U_1 = \frac{D\mathcal{U}'' - E\mathcal{U}' + F}{\mathcal{U}'}, \tag{8.5}$$

(D, E, F constants not all essential) give the necessary and sufficient condition that the asymptotic u-curves be twisted cubics. A similar equation in T and V_1 where

$$V_1 = \frac{\mathcal{V}'' - E\mathcal{V}' + F}{\mathcal{V}'}, \tag{8.6}$$

give the same condition for an asymptotic v-curve. The authors succeed in integrating these equations for the particular case

$$S = T = U_1 = V_1 = 0, \tag{8.7}$$

obtaining the following result:

The asymptotic curves of the surfaces.

$$x^1 = 1\,; \quad x^2 = \left(\frac{1}{2}cu^2 + du\right) - \left(\frac{1}{2}c'v^2 + d'v\right);$$

$$x^3 = \left(\frac{1}{2}au^2 + bu\right) + \left(\frac{1}{2}a'v^2 + b'v\right); \tag{Q}$$

$$x^4 = \frac{1}{6}(u^3 + v^3) + \left(\frac{1}{2}au^2 + bu\right)\left(\frac{1}{2}c'v^2 + d'v\right) + \left(\frac{1}{2}cu^2 + du\right)\left(\frac{1}{2}a'v^2 + b'v\right),$$

with

$$ad - bc = 1\,; \quad a'd' - b'c' = 1, \tag{8.8}$$

are twisted cubics and the directrix curves are indeterminate. These surfaces are algebraic and of order six or of order eight.

Some observations on these surfaces are necessary. First one observes that the conditions $U_1 = 0$, $V_1 = 0$ are not independent. Secondly, if $U_1 = 0$ or $V_1 = 0$, then according to our results from [4], it follows that the surfaces (Q) are *minimal projective surfaces*. Therefore they belong to the surfaces (A) of Wilczynski or, what is the same, to the surfaces (A') of Terracini *without the functions $U(u)$, $V(v)$ being all third degree polynomials, and nevertheless their asymptotic curves are twisted cubics*. To know which one of the surfaces (Q) are of the sixth order, we shall impose the conditions that they admit ∞^2 projective deformations into themselves.

From $S = T = 0$ and (8.2) follows

$$\beta = \gamma = \frac{1}{B}, \tag{8.9}$$

where

$$(B) = (ad' + b'c)u + (c'b + a'd)v + (ac' + a'c)uv + db' + b'd. \tag{8.10}$$

Assuming that the surfaces (Q) admit ∞^2 p. d. into themselves, we obtain from

$$\beta_u + \beta_v = 0\,; \quad \beta + u\beta_u + v\beta_v = 0,$$

the following equalities:

$$ac' + ca' = 0\,; \quad bd' + db' = 0\,; \quad ad' + a'd + cb' + bc' = 0. \tag{8.11}$$

Consequently

$$\beta = \gamma = \frac{1}{(ad' + b'c)(u - v)}. \tag{8.12}$$

From (8.11) follows

$$a' = ma\,; \quad d' = nd\,; \quad b' = -nb\,; \quad c' = -mc. \tag{8.13}$$

Then from $ad - bc = a'd' - b'c' = 1$, results

$$mn = 0. \tag{8.14}$$

Finally, using the last condition of (8.11) we obtain

$$m + n = 1\,; \tag{8.15}$$

Therefore

$$m^2 = -1, \tag{8.16}$$

and the sixth order surfaces which belong to the surfaces (Q) are:

$$x^1 = 1\,; \quad x^2 = u + mv\,; \quad x^3 = u^2 + mv^2\,;$$

$$x^4 = \frac{u^3 + v^3}{6} - \frac{m}{2}(u^2v - v^2u) \tag{8.17}$$

and that we get from

$$x^{*1} = 1\,; \quad x^{*2} = \frac{c}{2}x^3 + dx^2\,; \quad x^{*3} = \frac{ax^3}{2} + bx^2, \quad x^{*4} = x^4, \tag{8.18}$$

with $ad - bc = 1$.

From (8.12) and (Q) it follows that:

$$\beta = \gamma = -\frac{1}{m(u - v)}\,; \quad \theta_u = \frac{1}{u - v}\,; \quad \theta_v = -\frac{1}{u - v}$$

$$L = M = \frac{3}{2m^2(u - v)^2}, \tag{8.19}$$

and it is easy to see that L, M satisfy the conditions (5.9) and (5.10).

Therefore the surfaces (8.17), (8.18) admit ∞^2 (and no more) projective transformations into themselves.

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SUR UNE VARIÉTÉ PRESQUE PARACOKÄHLÉRIENNE MUNIE D'UNE CONNEXION SELF-ORTHOGONALE INVOLUTIVE

PAR

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On considère une C_∞ -variété pseudo-riemannienne V_k^{2n+1} d'indice d'inertie $n+1$ qui est douée d'une structure cosymplectique **C** et structurée par une connexion self-orthogonale involutive ∇_I [7]. Une hypersurface $V_{\mathcal{H}}$, horizontale par rapport à **C**, est une variété para-kählérienne minimale dont les $2n$ courbures principales sont $\pm (a^i a^{i'})^{1/2}$ $(i = 1, 2, \ldots, n;$ $i' = i+n)$. Eu égard à la connexion ∇_I, à la forme symplectique canonique Ω de $V_{\mathcal{H}}$ est naturellement associée une 2-forme presque symplectique $\Omega' = \Sigma\, a^i a^{i'}\, \theta^i \wedge \theta^{i'}$ qui est telle que les coefficients sont des invariants par rapport au groupe symplectique $Sp(n, \mathbf{R})$ de Ω et l'agrégat de Pfaff d'ordre maximal de Ω' est homothétique à la courbure gaussienne K de $V_{\mathcal{H}}$. En faisant usage de l'a-opérateur et de la notion de a-concourance définie dans [9] on démontre que si $V_{\mathcal{H}}$ admet un champ normal **N** a-concourant alors $V_{\mathcal{H}}$ ne possède que deux courbures principales $\pm \dfrac{1}{\|\mathbf{N}\|}$ qui sont de plus constantes. On retrouve ainsi un type d'hypersurfaces qui a été étudié par plusieurs auteurs mais dans des contextes différents [2, 6, 12]. Dans ce cas Ω' est homothétique à Ω et dans un sous-cas particulier $V_{\mathcal{H}}$ est einsteinienne. L'a-opérateur permet de plus de formuler la propriété intéressante suivante : Soient **X** et ${}_a\mathbf{X}$ respectivement un champ isotrope (d'un des sous-espaces tangentiels self-orthogonaux de $T_x(V_{\mathcal{H}})$) et le a-champ correspondant. Si $\alpha_{\mathbf{X}}$ este la 1-forme duale de **X** par rapport à Ω, alors la condition nécessaire et suffisante pour que $\alpha_{\mathbf{X}}$ soit cofermée est que ${}_a\mathbf{X}$ soit un automorphisme de la $SL(2n;\, \mathbf{R})$-structure sur $V_{\mathcal{H}}$. Enfin, la qualité soit pour **X** soit pour ${}_a\mathbf{X}$ d'être parallèle dans le faisceau tangent de $V_{\mathcal{H}}$ entraîne une série de propriétés qui sont liées aux systèmes hamiltoniens sur $(V_{\mathcal{H}}, \Omega)$, à l'harmonicité et à la Ω-harmonicité (dans le sens de P. Libermann [5]) des covecteurs hamiltoniens et aussi à certaines transformations infinitésimales conformes sur $V_{\mathcal{H}}$.

1. Soit V^{2n+1} une C^∞ -variété pseudo-riemannienne *d'indice d'inertie* $n+1$ [11]. En chaque point $x \in V^{2n+1}$ on peut écrire

$$T_x(V^{2n+1}) = \mathcal{H}_x^{2n} \oplus \mathcal{D}_x \tag{1}$$

où $T_x(V^{2n+1})$, $\mathcal{H}_x^{2n}$ et $\mathcal{D}_x$ sont respectivement l'espace tangent à V^{2n+1}, un sous-espace *parahermitien* [5] vectoriel de dimension $2n$ et une droite temporelle. Soient $\mathcal{S}_x^{+n}$ et $\mathcal{S}_x^{-n}$ respectivement l'espace *temporel* et l'espace *spatial* qui définissent un automorphisme $\mathcal{U}$ sur $\mathcal{H}_x^{2n}$ tel que $\mathcal{U}^2 = +1$ [5]. Le champ $\mathcal{D}_x$ est orthogonal à $\mathcal{H}_x^{2n}$ et $\mathcal{U}(\mathbf{X}) = 0$ pour tout champ $\mathbf{X} \in \mathcal{D}_x$.

Soient $\mathbf{h}_i, \mathbf{h}_{i'}$ $(i, j, k = 1, 2, \ldots, n;\ i' = i+n)$ les vecteurs isotropes réels qui définissent la *base canonique symplectique* de $\mathcal{H}_x^{2n}$ et $\mathbf{h} = \mathbf{h}_{2n+1}$ le *vecteur unitaire anisotrope* (temporel)