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ON THE ANALYTIC SOLUTION OF CERTAIN LINEAR DIFFERENCE EQUATIONS

BY

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Introduction. Consider a linear difference equation

$$\sum_{i=0}^{m} \mu_i(n) Z_{n-i} = \lambda\beta_n Z_{n+r} \tag{1.1}$$

$$Z_0(\lambda) = f(\lambda), \; Z_n(0) = C^1, \text{ a constant,}$$

where $\mu_i(n)$ are functional coefficients, λ is a parameter and β_n a function of the discrete variable n. Of the few ways of obtaining analytic solutions of (1.1) already studied, the formal series generation is of fundamental importance.

The existence of analytic solutions of linear as well as non-linear difference equations have been discussed by Stephens [7] and Trijitzinsky [8] who showed that analytic formal solutions of such equations exist and also by Milne-Thomson [5] who applied Nörlund's approach to obtain an integral form of the solution. Milne-Thomson attempts to solve a given difference equation of a very low order via a much-higher order differential equation thereby leading to several limitations.

However the questions arise:

If there exists a formal solution $y_n(\lambda)$ of (1.1), *which converges to an analytic solution, to what analytic solution does $y_n(\lambda)$ converge?*

What is the radius of convergence of this formal solution?

Can an analytic solution be so constructed in a well defined region while every singular solution is sieved off?

These questions are fully answered in this paper. It will be shown here that if $Z_n(\lambda)$ is an analytic non-formal solution of (1.1) then $y_n(\lambda)$ converges to $Z_n(\lambda)/Z_0(\lambda)$ and that the radius of convergence of $y_n(\lambda)$ is given by the first zero of $Z_0(\lambda)$, the contour integral form of which is obtained in the discourse that follows.

We shall construct the solution $y_n(\lambda)$ by taking arbitrary initial values and transforming (1.1) into the two-layer equation

$$\sum_{i=0}^{m} \mu_i(n) y_{n-i}^{(k+1)} = \lambda\beta_n y_{n+r}^{(k)}, \tag{1.1 A}$$

$$y_0^{(k)} = y_n^{(0)} = C^1, \; k \geqq 0, \; n \geqq 1,$$

which uses previously computed values at the k^{th} layer to obtain values at the $(k+1)^{th}$ layer. The main advantages of this method over the traditional method [8] are that the initial values may be completely arbitrary, the method is easily adaptible on a digital machine and the region of convergence of the solution can be characteristically predicted.

The method here is an improvement on Milne-Thomson's as it will become evident that every arbitrarily high order linear difference equation can be replaced by a simple first order ordinary differential equation.

We will here assume the existence of a uniquely defined solution of the difference equation (1.1A) and that such solution converges to an analytic function, $y_n(\lambda)$ satisfying

$$\sum_{i=0}^{m} \mu_i(n) Z_{n-i} = \lambda \beta_n Z_{n+r}$$

We will discuss the cases in which

(i) $\mu_i(n)$ is a constant independent of n, $\beta_n \equiv 1$,

(ii) $\mu_i(n)$ is linear in n, of the form $\mu_i n + \omega_i$, and $\beta_n \equiv 1$ where μ_i, ω_i are constants and,

(iii) $\mu_i(n)$ is a factorial coefficient or admits of a factorial series.

Definition. We shall say that a solution $Z_n(\lambda)$ of (1.1) is of type G if it can be written as

$$Z_n(\lambda) = \int_{(c)} e^{nt} \gamma_\lambda(t) dt, \tag{1.2}$$

where $\gamma_\lambda(t)$ is an analytic function of λ and t specified throughout a contour c.

We shall regard the $\mu_i(n)$'s as of arbitrary signs unless otherwise stated. We note that the case $\mu_0(n) = 1$, $\mu_1(n) = n - 1$, $\mu_i(n) \equiv 0 \ \forall i \geq 2$ has a solution of type G. In fact this is the equation

$$Z_n - (n-1) Z_{n-1} = 0$$

the solution of which is the gamma function.

2. Constant coefficients. Let

$$\sum_{i=0}^{m} \mu_i Z_{n-i} = \lambda Z_{n+r}, \tag{2.1}$$

where μ_i are constants, $\mu_0 \neq 0$. Substituting (1.2) into (2.1) we have

$$\frac{e^{nt}}{n} \gamma_\lambda(t) V(t) \Big|_c - \int_c [\gamma'_\lambda(t) V(t) + \gamma_\lambda(t) V'(t) + n\lambda e^{rt} \gamma_\lambda(t)] \frac{e^{nt}}{n} dt = 0, \tag{2.2}$$

where

$$V(t) = \sum_{i=0}^{m} \mu_i e^{-it}, \tag{2.2 a}$$

and $V'(t) \equiv \frac{d}{dt} V(t)$.

We seek c such that the primitive in (2.2) remains single-valued throughout c and vanishes at both ends if c is an open contour. This implies that

$$\frac{e^{nt}}{n} \gamma_\lambda(t) V(t) \Big|_c = 0. \tag{2.3}$$

Thus we have that

$$\gamma'_\lambda(t) V(t) + \gamma_\lambda(t) V'(t) + n\lambda e^{rt} \gamma_\lambda(t) = 0, \tag{2.4}$$

so that

$$\gamma_\lambda(t) = \frac{A(\lambda)}{V(t)} \exp\left\{ - \int^{t} \frac{n\lambda e^{r\tau}}{V(\tau)} d\tau \right\}, \tag{2.5}$$

where $A(\lambda)$ is a constant (with respect to t) of integration. (2.3) then becomes

$$A(\lambda) \frac{e^{nt}}{n} \exp\left\{ - \int^{t} \frac{n\lambda e^{r\tau}}{V(\tau)} d\tau \right\} \Big|_c = 0. \tag{2.6}$$

Now consider the line

$$L(t) = l \tag{2.7}$$

where l is a constant. We seek an estimate of $\gamma_\lambda(t)$ along L. As $t \to -\infty$,

$$\exp\left\{ - \int^{t} \frac{n\lambda e^{r\tau}}{V(\tau)} d\tau \right\} \approx o(1)$$

and in fact $1/V(t) \to 0$. This means $\gamma_\lambda(t) \to 0$. Let $t \to +\infty$.

$$\frac{n\lambda e^{rt}}{V(t)} = \frac{n\lambda e^{r\tau}}{\mu_0 + o(1)} \tag{2.8}$$

and definitely, $\exp\left\{ - \int^{t} \frac{n\lambda e^{r\tau}}{\mu_0 + o(1)} d\tau \right\} \to 0.$

Thus, (2.6) holds. We note that as $t \to +\infty + il$,

$$n\lambda \int^{t} \frac{e^{r\tau}}{\mu_0 + o(1)} d\tau \approx \frac{n\lambda e^{rt}}{r(\mu_0 + o(1))},$$

and

$$\arg\left\{ \int^{t} \frac{n\lambda e^{r\tau}}{\mu_0 + o(1)} d\tau \right\} \approx \arg \bar{A} + rL(t) = \arg \bar{A} + rl = \pi, \tag{2.9}$$

where $\bar{A} = n\lambda / r(\mu_0 + o(1))$.

Hence, we have

$$Z_n(\lambda) = \int_{-\infty + il}^{\infty + il} e^{nt} \gamma_\lambda(t) dt, \tag{2.10}$$

where $\gamma_\lambda(t)$ is given by (2.5), and $i = \sqrt{-1}$.

Suppose $V(t)$ has at least one simple zero, u_1, on line L and $\gamma_\lambda(t)$ is not single-valued in a neighbourhood of the first zero, u_1, of $V(t)$. Then the contour c comprises the loop from $-\infty$, round u_1 and back to $-\infty$. In this case we write

$$Z_n(\lambda) = \int_{-\infty}^{(u_1+)} e^{nt} \gamma_\lambda(t) dt. \tag{2.11}$$

We see that (2.10) and (2.11) are solutions of type G.

3. $\mu_i(n)$ linear. Let $\mu_i(n) = \mu_i n + \omega_i$ where μ_i, ω_i are constants. Substituting (1.2) into (1.1) we have that

$$V_2(t) \gamma_\lambda(t) e^{nt} \Big|_c - \int_c \{ V_2(t) \gamma^1_\lambda(t) + [V^1_2(t) - V_1(t) + \lambda e^{rt}] \gamma_\lambda(t) \} e^{nt} dt = 0, \tag{3.1}$$

where

$$V_1(t) = \sum_{i=0}^{m} \omega_i e^{-it}, \quad V_2(t) = \sum_{i=0}^{m} \mu_i e^{-it}. \tag{3.1 a}$$

By our earlier choice of c we will have

$$V_2(t) \gamma_\lambda(t) e^{nt} \Big| = 0 \tag{3.2}$$

and $\gamma_\lambda(t)$ satisfies the differential equation

$$V_2\gamma_\lambda' + [V_2' - V_1 + \lambda e^{rt}]\gamma_\lambda = 0,$$

from which

$$\gamma_\lambda(t) = \frac{B(\lambda)}{V_2(t)}\exp\left\{\int^t \frac{V_1(\tau)-\lambda e^{r\tau}}{V_2(\tau)}\,d\tau\right\}, \tag{3.3}$$

where $B(\lambda)$ is a constant (with respect to t) of integration. (3.2) becomes

$$\frac{B(\lambda)}{V_2(t)}\exp\left\{nt+\int^t \frac{V_1(\tau)-\lambda e^{r\tau}}{V_2(\tau)}\,d\tau\right\}\Bigg|_c = 0. \tag{3.4}$$

Now suppose $\mu_0 \neq 0$. As $t \to -\infty$, the integrand in (3.4) tends to $\alpha_0/\mu_0 = \delta$, say. Also $1/V_2(t) \to 0$. Thus (3.4) holds. As $t \to +\infty$, the integrand in (3.4) again tends asymptotically to

$$\delta - (\lambda/\mu_0)e^{rt}, \quad r > 0$$

so that

$$\left\{nt + \int^t \frac{V_1(\tau)-\lambda e^{r\tau}}{V_2(\tau)}\,d\tau\right\} \approx nt + \delta - \frac{\lambda}{r\mu_0}e^{rt} \to -\infty \text{ as } t\to\infty.$$

Therefore

$$\exp\left\{nt + \int^t \frac{V_1-\lambda e^{r\tau}}{V_2}\,d\tau\right\} \to 0.$$

Considering again the line L defined by (2.7) and using the arguments that follow, the latter to estimate $\gamma_\lambda(t)$ along L, we have

$$Z_n(\lambda) = \int_{-\infty+il}^{\infty+il} e^{nt}\gamma_\lambda(t)dt, \tag{3.5}$$

where $\gamma_\lambda(t)$ is given by (3.3). This again is a solutions of type G.

If however we have u_1 as a singularity of $\gamma_\lambda(t)$ in the t-plane, namely the first simple zero of $V_2(t)$, then we have a solution of the form

$$Z_n(\lambda) = \int_{-\infty}^{(u_1+)} e^{nt}\gamma_\lambda(t)dt.$$

4. Factorial polynomial and factorial series coefficients. These are cases in which $\mu_i(n)$ admits of a factorial polynomial (respectively factorial series) representation. Let

$$\mu_i(n) = \mu_i n^{[i]} \tag{4.1}$$

where μ_i is a constant and

$$n^{[i]} = n(n+1)(n+2)\dots(n+i). \tag{4.2}$$

In this case we can obtain a solution of type G by simply repeating the arguments of section 2 above, while the function $V(t)$ is replaced by

$$V(t) = \sum_{i=0}^{m}\mu_i n^{[i]} e^{-it}. \tag{4.3}$$

Suppose we have factorial series coefficients viz

$$\text{(a)} \quad \mu_i(n) = \sum_{j=0}^{\infty}\mu_{ij}n^{[j]} \text{ or (b) } \mu_i(n) = \sum_{j=0}^{\infty}\mu_{ij}n^{[-j]}. \tag{4.4}$$

Here the situation is again similar to that in section 2 but with

$$V(t) = \begin{cases} \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{\infty}\mu_{ij}n^{[j]}e^{-it}, \text{ for (a)} \\ \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{\infty}\mu_{ij}n^{[-j]}e^{-it} \text{ for (b)} \end{cases} \tag{4.5}$$

so that in each case $V(t)\gamma_\lambda(t)e^{nt}|_c = 0$, and $V(t)\gamma_\lambda'(t) + \gamma(t)V'(t) + n\lambda e^{rt}\gamma_\lambda(t) = 0$ which gives

$$\gamma_\lambda(t) = \frac{C(\lambda)}{V(t)}\exp\left\{-\int^t \frac{n\lambda e^{r\tau}}{V(\tau)}\,d\tau\right\}, \tag{4.6}$$

$C(\lambda)$ a constant of integration. This again gives a solution, similar to (2.11), of type G.

Fundamental results. It has become evident, from the foregoing, that the path of integration depends on the distribution of the zeros, in the t-plane, of $V(t)$ (or $V_2(t)$ in the linear coefficient case) but the existence of a convergent solution, as a function of λ, depends on the distribution of the singularities of $\gamma_\lambda(t)$ in the λ-plane. From this last statement if we apply the iterative scheme as in (1.0) then the solution of (1.0) exists uniquely and converges in the λ-plane up to the first zero of $Z_0(\lambda)$. To these effects we have the following fundamental results. Here we put $x = e^{-t}$.

Lemma 1. *Let*

$$V(x) = \sum_{i=0}^{m}\mu_i x^i. \tag{4.7}$$

Let $\mu_0 > 0$ and μ_i real, $0 \leqslant i \leqslant m$. $V(x)$ has only real zeros within the bound of the series in (34.7) if and only if the conditions

$$4\mu_i\mu_{i+2} \leqslant \mu_{i+1}^2 \text{ for } 0 \leqslant i \leqslant m-2$$

holds.

Proof

$$V(x) = \sum_{i=0}^{m}\mu_i x^i,$$

which, on grouping the terms

$$= \sum_{i=0}^{m-2} x^i(\mu_i + \mu_{i+1}x + \mu_{i+2}x^2), \quad i = 3(j-1),\ 1 \leqslant j \leqslant \frac{m+1}{3},\ 3\,|\,(m+1).$$

Since $x = e^{-t}$, then $x \neq 0$. For $V(x)$ to have a zero, therefore, we will have

$$\mu_{i+2}x^2 + \mu_{i+1}x + \mu_i = 0 \text{ for } 0 \leqslant i \leqslant m-2.$$

There will exist at least one complex zero if for some i

$$4\mu_i\mu_{i+2} > (\mu_{i+1})^2. \tag{4.8}$$

Whence, all zeros lie on the real line if and only if for $0 \leqslant i \leqslant m-2$

$$4\mu_i\mu_{i+2} \leqslant (\mu_{i+1})^2. \tag{4.9}$$

From the above Lemma we see that if the conditions of the lemma hold for every α_i, $\gamma_\lambda(t)$ can have only branch points on the real line. But if for some $i \geqslant 0$, (4.8) holds then the upper limits of the integral solutions obtained will be defined by

$$\log_e u_1 = \log_e\frac{1}{r} - i\theta,\ |u_1| = \frac{1}{r} \neq 0,$$

where $i = \sqrt{-1}$. This consideration holds especially for the factorial polynomials and series coefficient cases where, in particular, $\mu_0 > 0$.

The solutions are therefore obtained by cutting the plane through the first singularity u_1 of $\gamma_\lambda(t)$ (respectively zero of $F(t)$) and taking two infinite loops round u_1 respectively for (4.8) and (4.9)

Theorem 1. Let the difference equation (1.1) have a formal solution which converges to an analytic function

$$y_n(\lambda) = \sum_{\nu=0}^{\infty} \alpha_n^{(\nu)} \lambda^{(\nu)}$$

which satisfies (1.1). Let the initial values be correspondingly defined such that $y_0(\lambda) \equiv C'$ a constant, $k \geqslant 1$ and $Z_0(\lambda) = f(\lambda)$. Then $y_n(\lambda) = O\,[Z_n(\lambda)/f(\lambda)]$.

The radius of convergence of the solution of (1.0), if given by a formal series, is determined by the first zero in the λ-plane of the initial solution $Z_0(\lambda)$ of (1.1)

Proof. By our earlier discussion, if the solution sequence $\{y_n^{(k)}(\lambda)\}$ of (1.0) converges to $y_n(\lambda)$ then

$$y_n(\lambda) = \lim_{k\to\infty} [\alpha_n^{(0)} + \alpha_n^{(1)}\lambda + \alpha_n^{(2)}\lambda^2 + \dots + \alpha_n^{(k)}\lambda^k],$$

where $\alpha_0^{(k)} = 0$, $k \geqslant 1$, $\alpha_n^{(0)} = C'$, $n \geqslant 0$.

Let $y_n(\lambda) \to g(\lambda) Z_n(\lambda)$ as $k \to \infty$. When $n = 0$,

$$C' = \alpha_0^{(0)} = y_0 \sim g(\lambda)\, Z_0(\lambda) = g(\lambda) f(\lambda),$$

whence $g(\lambda) \sim C/f(\lambda)$, so that in the limit

$$y_n(\lambda) = O\left[\frac{Z_n(\lambda)}{f(\lambda)}\right]. \tag{4.10}$$

The analytic solution of (1.1) has been given in the form

$$Z_n(\lambda) = P(\lambda) \int_c e^{nt}\, \bar{\gamma}(n, \lambda, t)\,dt,$$

where $P(\lambda)$ corresponds to the constant of integration in each of the cases considered and

$$P(\lambda)\bar{\gamma}(n, \lambda, t) \equiv \gamma_\lambda(t) \equiv \gamma(n, \lambda, t).$$

In order that $Z_n(\lambda)$ may satisfy the initial condition $Z_0(\lambda) = f(\lambda)$, we must have

$$f(\lambda) = Z_0(\lambda) = P(\lambda) \int_c \bar{\gamma}(0, \lambda, t)\,dt.$$

Thus

$$P(\lambda) = f(\lambda) \Big/ \int_c \bar{\gamma}(0, \lambda, t)\,dt.$$

Hence,

$$Z_n(\lambda) = f(\lambda) \int_c e^{nt}\, \bar{\gamma}(n, \lambda, t)\,dt \Big/ \int_c \bar{\gamma}(0, \lambda, t)\,dt. \tag{4.11}$$

Therefore the limiting solution is given by

$$y_n(\lambda) = C' \int_c e^{nt}\, \bar{\gamma}(n, \lambda, t)\,dt \Big/ \int_c \bar{\gamma}(0, \lambda, t)\,dt.$$

Since the initial value is completely arbitrary we can, without loss of generality take $C' = 1$. Then we have

$$y_n(\lambda) = \int_c e^{nt}\, \bar{\gamma}(n, \lambda, t)\,dt \Big/ \int_c \bar{\gamma}(0, \lambda, t)\,dt. \tag{4.12}$$

We thus see that in the λ-plane, the radius of convergence of the formal solution of (1.1) will be determined by the first zero of

$$\int_c \bar{\gamma}(0, \lambda, t)\,dt.$$

Example. Consider the limiting equation

$$(n-2)Z_n - (2n-3)Z_{n-1} - (3n-1)Z_{n-2} = \lambda Z_{n-r} \tag{4.13}$$

$$n \geqslant 2,\ r \geqslant 0,\ Z_0 = f(\lambda).$$

Substituting (1.2) into (4.13) we have

$$V_2(t)\gamma_\lambda(t)e^{nt}\,|_c - \int_c \{V_2(t)\gamma'_\lambda(t) + [V'_2(t) - V_1(t)]\gamma_\lambda(t) + \lambda\gamma_\lambda(t)e^{rt}\}\, e^{nt}\,dt = 0.$$

c is chosen such that

$$V_2(t)\gamma_\lambda(t)e^{nt}\,|_c = 0 \tag{4.16}$$

and

$$V_2(t)\,\gamma'_\lambda(t) + [V'_2(t) - V_1(t) + \lambda e^{rt}]\gamma_\lambda(t) = 0, \tag{4.17}$$

where

$$V_1(t) = \sum_{i=0}^{2} \omega_i\, e^{-it}, \quad V_2(t) = \sum_{i=0}^{2} \mu_i\, e^{-it},$$

$$\mu_0 = 1,\ \mu_1 = -2,\ \mu_2 = -3,\ \omega_0 = -2,\ \omega_1 = 3,\ \omega_2 = 1.$$

Solving the differential equation (4.17) we obtain

$$\gamma_\lambda(t) = \frac{B(\lambda)}{V_2(t)} \exp\left\{\int^t \frac{V_1(\tau) - \lambda e^{r\tau}}{V_2(\tau)}\, d\tau\right\}.$$

(4.16) holds at $t = -\infty$. Examining $V_2(t)$ we have

$$V_2(t) = 1 - 2e^{-t} - 3e^{-2t} = 1 - 2x - 3x^2, \quad x = e^{-t}$$

The coefficients of $V_2(t)$ satisfy the condition (4.9) of Lemma 1. In fact there are two simple zeros of $V_2(x)$, -1 and $1/3$. Hence the contour c is the loop from $-\infty$ round -1 and back. Thus a solution of (4.13), assuming $y_0^{(k)} = 1$, is

$$Z_n(\lambda) = \frac{f(\lambda) \displaystyle\int_{-\infty}^{-1} \frac{e^{nt}}{V_2(t)} \exp\left\{\int^t \frac{V_1 - \lambda e^{r\tau}}{V_2}\, d\tau\right\} d\tau}{\displaystyle\int_{-\infty}^{-1} \frac{1}{V_2(t)} \exp\left\{\int^t \frac{V_1 - \lambda e^{r\tau}}{V_2}\, d\tau\right\} dt}$$

and the formal solution converges to

$$y_n(\lambda) = \frac{\int_{-\infty}^{-1} \frac{e^{nt}}{V_2(t)} \exp\left\{\int^{t} \frac{V_1 - \lambda e^{r\tau}}{V_2} d\tau\right\} dt}{\int_{-\infty}^{-1} \frac{1}{V_2(t)} \exp\left\{\int^{t} \frac{V_1 - \lambda e^{r\tau}}{V_2} d\tau\right\} dt},$$

which is analytic in the half-plane bounded by $\lambda = u_1$ where u_1 is the first zero of the integral $Z_0(\lambda)$, in the denominator. However if $Z_0(\lambda)$ has no zero, then $y_n(\lambda)$ is analytic in the whole λ-plane.

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RUNGE-KUTTA METHODS FOR INITIAL VALUE PROBLEMS

BY

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In this paper we deduce some Runge-Kutta type methods for approximation of the solution for initial-value problems.

Let us consider the problem

$$\text{(I)}\quad \begin{aligned} &(1) \quad \frac{DU}{D\tau} = F\left(\tau, X, U, \frac{DU}{DX}\right), \\ &(2) \quad U(0, X) = \Phi(X), \\ &(3) \quad U(\tau, X)|_{X\in\Gamma} = 0, \end{aligned}$$

where

$$U : [0, T]^m \times \prod_{i=1}^{q} [a_i, b_i] \to \mathrm{R}^p;$$

$$\tau \in [0, T]^m;\ X \in \prod_{i=1}^{q} [a_i, b_i];\ \Gamma = \partial\left(\prod_{i=1}^{q} [a_i, b_i]\right);$$

$$\Phi : \prod_{i=1}^{q} [a_i, b_i] \to \mathrm{R}^p;$$

$$F : [0, T]^m \times \prod_{i=1}^{q} [a_i, b_i] \times \mathrm{R}^p \times \mathrm{R}^{p\cdot q} \to \mathrm{R}^{p\cdot m}$$

and $DU/D\tau$ (resp. DU/DX) is the Fréchet derivative with respect to τ (resp X) for each X (resp τ).

Suppose that (I) has an unique analytical solution. This results by an argument analogous with Cauchy-Kowalevska theorem [2].

For simplicity we consider only two cases: i) $\tau \in [0, T]$ and $U \in R^2$, $X \in R^2$ and ii) $\tau \in [0, T]^2$.

1. Runge-Kutta methods for Cauchy-Kowalevska problem.

Consider the system of partial differential equations:

$$(4) \qquad \frac{\partial U}{\partial t} = F\left(t, X, U, \frac{DU}{DX}\right); \quad U \in \mathrm{R}^2,\ x \in \mathrm{R}^2$$

with initial condition

$$(5) \qquad U(0, x) = \Phi(x),$$

and boundary condition

$$(6) \qquad U(t, X)|_{X\in\Gamma} = 0, \quad \Gamma = \partial([0, 1]\times[0, 1]).$$