$\forall t \in R_+$; iv) g is monotone increasing and continuous on R_+; v) $h(t, s, u) = \lambda(s) u$, where $\lambda : R_+ \to R_+$ is continuous on R_+ : vi) $f(t) = \alpha t$, $0 \leq \alpha < 1$, then, the above theorem reduces to theorem 1 of [4].

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COINCIDENCE POINTS AND APPLICATIONS TO VOLTERRA FUNCTIONAL EQUATIONS

BY

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1. Let $R_+ = [0, +\infty[$ be the positive half-axis, $N = \{0, 1, \dots\}$, the positive integers, $F(R_+, R_+) = \{f; f : R_+ \to R_+\}$. Denote :

i) $M = \{f \in F(R_+, R_+); f(0) = 0, f(t) < t, \ \forall t > 0\}$,

ii) $P = \{f \in M; \ \lim\limits_{n \to \infty} f^{(n)}(t) = 0, \ \forall t > 0\}$ (where, $f^{(n)}$ stands for the n-th iterate of f, $\forall n \in N$).

iii) $Q = \{f \in M; \ \limsup\limits_{s \to t+} f(s) < t, \ \forall t > 0$, (i.e., $\forall t > 0, \exists a(t) \in]0, t[$, such that $s \in]t, t + a(t)[\Rightarrow f(s) \leq t - a(t)$). Let us denote by P^* (respectively Q^*) the set all (φ, Ψ), $\varphi, \Psi \in F(R_+, R_+)$, for which there exists $f \in P$ (resp. $f \in Q$), such that, the following implication holds

$$u > 0, \ v > 0, \ \varphi(u) \leq \Psi(v) \Rightarrow u \leq f(v). \tag{1.1}$$

Note [3] that

$$Q \subset P \ (\Rightarrow Q^* \subset P^*), \tag{1.2}$$

In what follows, a partial characterization of the class Q^* is given.

Theorem 1.1. *Suppose that*

$$\forall t > 0, \ \exists s \in [0, t[\ \text{such that } \varphi(s) \leq \Psi(t), \tag{1.3}$$

$$u > 0, \ v > 0, \ \varphi(u) \leq \Psi(v) \Rightarrow u < v, \tag{1.4}$$

$$\liminf_{s \to t} \varphi(s) > \limsup_{s \to t+} \Psi(s), \ \forall t > 0, \ (i.\ e., \ \forall t > 0, \ \exists a(t) \in]0, t[, \text{ such that } \varphi(u) > \Psi(v), \ \forall u \in]t - a(t), t + a(t)[, \ \forall v \in]t, t + a(t)[\,). \tag{1.5}$$

Then, $(\varphi, \Psi) \in Q^*$.

Proof. Define $f \in F(R_+, R_+)$ by $f(0) = 0$, and

$$f(t) = \sup\{s \in R_+; \ \varphi(s) \leq \Psi(t)\}, \ \forall t > 0. \tag{1.6}$$

From (1.3) and (1.4), $f(t)$ is well defined and $f(t) \leq t$, $\forall t \in R_+$. Now, let

$t > 0$ be given. Suppose there exists $v \in]t, t + a(t)[$ such that $f(v) > t - a(t)$. By (1.6) we can find $u \in]t - a(t), f(v)[\subseteq]t - a(t), v[\subseteq]t - a(t), t + a(t)[$, such that $\varphi(u) \leqslant \Psi(v)$, which contradicts (1.5). Therefore, $f(v) \leqslant t - a(t)$, $\forall v \in]t, t + a(t)[$; since $t > 0$ was arbitrary, it follows that $f \in Q$. On the other hand, obviously, (1.6) $\Rightarrow$ (1.1). Q.E.D.

2. Let (X, d) be a complete metric space, $F(X, X) = \{T : T : X \to X\}$.

Definition 2.1. *A point $z \in X$ will be called a coincidence point (abbreviated c.p.) of $U, T \in F(X, X)$, if $Uz = Tz$.* Denote $\Lambda(U, T) = \{z \in X : Uz = Tz\}$. The main result concerning the existence and uniqueness of c.p. can be stated as follows.

Theorem 2.1. *Let $U, T \in F(X, X)$, $\varphi, \Psi \in F(R_+, R_+)$ be such that*

(2.1) $$d(Ux, Uy) \leqslant \varphi(s) \Rightarrow d(x, y) \leqslant s,$$

(2.2) $$d(x, y) \leqslant t \Rightarrow d(Tx, Ty) \leqslant \Psi(t),$$

(2.3) $$U(X) = X,$$

(2.4) $$(\varphi, \Psi) \in P^*.$$

Then, the following conclusions hold

(2.5) *$\Lambda(U, T)$ consists of a single element, $z \in X$,*

(2.6) *there exists $f \in P$, which has the property*

$\forall x \in X$, $\exists \rho(x) \geqslant 0$, such that $d(x_n, z) \leqslant f^{(n)}(t)$, $\forall t \geqslant \rho(x)$, $\forall n \in N$, where, $\{x_n ; n \in N\} \subset X$ is defined by $x_0 = x$, $U(x_{n+1}) = T(x_n)$, $\forall n \in N$.

Proof. Let $x \in X$ be given. From (2.3), the set $U^{-1}(Tx) = \{y \in X : Uy = Tx\}$ is not empty. Besides, let $y_1, y_2 \in U^{-1}(Tx)$, $y_1 \neq y_2$. If we choose $s \in]0, d(y_1, y_2)[$ then, $d(Uy_1, Uy_2) = 0 \leqslant \varphi(s)$, while, $d(y_1, y_2) > s$. Therefore, $y_1 = y_2$, and so, we can define $V \in F(X, X)$ by

(2.7) $$Vx = U^{-1}(Tx), \ \forall x \in X.$$

It is evident that

(2.8) $$UVx = Tx, \ \forall x \in X \text{ (i.e., } UV = T).$$

Consequently, $z \in X$ is a c.p. of U and T iff it is a fixed point of V. Let $x, y \in X$, $t, u > 0$, be such that $d(x, y) \leqslant t$, $d(Vx, Vy) > u$. From (2.1) and (2.8) we get $d(Tx, Ty) > \varphi(u)$. On the other hand, from (2.2), $d(Tx, Ty) \leqslant \Psi(t)$. Therefore, $\varphi(u) < \Psi(t)$, and this implies $u \leqslant f(t)$, where, $f \in P$ is given by (2.4). Since $u < d(Vx, Vy)$ was arbitrary, we get $d(Vx, Vy) \leqslant f(t)$. Thus, we have proved that

(2.9) $$d(x, y) \leqslant t \Rightarrow d(Vx, Vy) \leqslant f(t).$$

Now, the result follows from theorem 2.2 of [3]. Q.E.D.

3. The main result of the preceding paragraphe may be extended if, instead of (ordinary) complete metric spaces we consider generalized complete metric spaces (g.c.m.s.), [1]. Let (X, d) be such a g.c.m.s. ; denote

i) $\Omega(x) = \{y \in X : d(x, y) < +\infty\}$, $\forall x \in X$,

ii) $\Omega(T) = \{x \in X : d(x, Tx) < +\infty\}$, $\forall T \in F(X, X)$

iii) $\Omega(U, T) = \{x \in X : \exists y \in \Omega(x)$, such that $Uy = Tx\}$.

The extension of the main result is the following

Theorem 3.1. *Let $U, T \in F(X, X)$, $\varphi, \Psi \in F(R_+, R_+)$ be such that*

(3.1) *all the conditions (2.1) — (2.4) hold.*

(3.2) *$\Omega(U, T)$ is not empty,*

Then, the following conclusions hold :

(3.3) *$\Lambda(U, T)$ is not empty ;*

(3.4) *there exists $f \in P$, which has the property :*

if $z \in \Lambda(U, T)$ is given, then, $\forall x \in \Omega(U, T) \cap \Omega(z) \exists \rho(x) \geqslant 0$, such that, $d(x_n, z) \leqslant f^{(n)}(t)$, $\forall t \geqslant \rho(x)$, $\forall n \in N$, where, $\{x_n ; n \in N\} \subset X$ is defined by $x_0 = x$, $U(x_{n+1}) = T(x_n)$, $\forall n \in N$.

Proof. Let $V \in F(X, X)$ be defined by (2.7). Condition (3.2) implies that $\Omega(V)$ is not empty. On the other hand, (3.1) $\Rightarrow$ (2.9). Therefore, theorem 3.2 of [3] is applicable Q.E.D.

4. In this paragraphe, as an application, some existence and uniqueness results about a general class of Volterra functional equations [2] are presented.

Let $(R^n, \|.\|)$ be the euclidean n-dimensional space, $C(R_+, R^n) = \{x ; x : R_+ \to R^n, x \text{ is continuous}\}$, $C([0, t], R^n) = \{x ; x : [0, t] \to R^n, x \text{ is continuous}\}$, $\forall t \in R_+$, $D = \{(t, s) ; 0 \leqslant s \leqslant t < +\infty\}$, $C(D \times R_+, R_+) = \{x ; x : D \times R_+ \to R_+, x \text{ is continuous}\}$. For a given $x \in C(R_+, R^n)$, $t \in R_+$, x_t stands for the restriction $x_t(s) = x(s)$, $\forall s \in [0, t]$; clearly, $x_t \in C([0, t], R^n)$, $\forall t \in R_+$. Consider two families of mappings

(4.1) $$t \to k(t) : C([0, t], R^n) \to R^n, \ \forall t \in R_+,$$

(4.2) $$(t, s) \to K(t, s) : C([0, s], R^n) \to R^n, \ \forall (t, s) \in D$$

such that

(4.3) $\forall x \in C(R_+, R^n)$, the mapping $t \to k(t)(x_t)$ of R_+ into R^n is continuous on R_+,

(4.4) $\forall x \in C(R_+, R^n)$, the mapping $(t, s) \to K(t, s)(x_s)$ of D into R^n is continuous on D.

Now, let us consider the following Volterra functional equation

(4.5) $$k(t)(x_t) = x^0(t) + \int_0^t K(t, s)(x_s)ds, \ \forall t \in R_+,$$

(where, $x^0 \in C(R_+, R^n)$ is fixed) or, in an abstract form

(4.6) $$Ux(t) = Tx(t), \ \forall t \in R_+ \text{ (i.e., } Ux = Tx)$$

where, $U, T : C(R_+, R^n) \to C(R_+, R^n)$ are defined by

(4.7) $\quad Ux(t) = k(t)(x_t), \ \forall t \in R_+, \ \forall x \in C(R_+, R^n)$

(4.8) $\quad Tx(t) = x^0(t) + \int_0^t K(t, s)(x_s)\, ds, \ \forall t \in R_+, \ \forall x \in C(R_+, R^n).$

Consequently, $x \in C(R_+, R^n)$ is a solution of (4.5) iff it is a c.p. of U and T and so, all the preceding results are applicable, provided that a (generalized) metric structure on $C(R_+, R^n)$ might be defined. To this end, let $g \in F(R_+, R_+)$ be such that g = continuous, $g(t) > 0$, $\forall t \in R_+$. Define a generalized norm $\| . \|_g$ on $C(R_+, R^n)$ by

(4.9) $\quad \| x \|_g = \sup \{ \| x(t) \| / g(t) : t \in R_+ \}, \ \forall x \in C(R_+, R^n)$

and denote $C_g(R_+, R^n) = \{ x \in C(R_+, R^n) ; \| x \|_g < +\infty \}$.
On the other hand, suppose that a family of mappings

(4.10) $\quad t \to | . |_t$ = seminorm on $C([0, t], R^n), \forall t \in R_+$.

is given (of course, $| . |_t$ is also a seminorm on $C(R_+, R^n)$ if we take $| x |_t = | x_t |_t, \ \forall t \in R_+, \ \forall x \in C(R_+, R^n)$). Suppose in addition that

(4.11) the family $\{ | . |_t : t \in R_+ \}$ is a suficient family of seminorms on $C(R_+, R^n)$

(4.12) $\quad \| x \|_g = \sup \{ | x |_t / g(t) ; t \in R_+ \}, \ \forall x \in C(R_+, R^n)$

The main result about (4.5) is the following

Theorem 4.1. *Suppose that there exist* $\varphi, \Psi \in F(R_+, R_+)$, $h \in C(D \times R_+, R_+)$, *such that*

(4.13) $\quad t, \sigma \in R_+, \ x, y \in C(R_+, R^n), \ \| k(t)(x_t) - k(t)(y_t) \| \leqslant \varphi(\sigma) g(t) \Rightarrow | x - y |_t \leqslant \sigma . g(t)$

(4.14) $\quad (t, s) \in D, \ x, y \in C(R_+, R^n), \ \sigma \in R_+, \ | x - y |_s \leqslant \sigma \Rightarrow \| K(t, s)(x_s) - K(t, s)(y_s) \| \leqslant h(t, s, \sigma)$

(4.15) $\quad \int_0^t h(t, s, \tau g(s))\, ds \leqslant g(t) \Psi(\tau), \ \forall t, \tau \in R_+$

(4.16) $\quad \forall y \in C(R_+, R^n), \ \exists x \in C(R_+, R^n)$, *such that* $k(t)(x_t) = y(t), \ \forall t \in R_+$

(4.17) $\quad (\varphi, \Psi) \in P^*$

(4.18) *there exists* $x \in x^0 + C_g(R_+, R^n)$, *such that*

$$k(t)(x_t) = x^0(t) + \int_0^t K(t, s)(x_s^0)\, ds, \ \forall t \in R_+ .$$

Then, the following conclusions hold

(4.19) *the Volterra functional equation (4.5) has at least one solution in* $C(R_+, R^n)$ *and exactly one solution in* $x^0 + C_g(R_+, R^n)$

(4.20) *there exists* $f \in P$ *which has the property: the sequence* $\{ x^{]n[} ; n \in N \} \subset C(R_+, R^n)$, *defined by* $x^{]0[}(t) = x^0(t), \ \forall t \in R_+, \ k(t)(x_t^{]n+1[}) = x^0(t) + \int_0^t K(t, s)(x_s^{]n[})\, ds, \ \forall t \in R_+, \ \forall n \in N$, *converges to the unique solution* $z \in x^0 + C_g(R_+, R^n)$, *and the error of approximation is given by* $\| x^{]n[} - z \|_g \leqslant f^{(n)}(\tau), \ \forall \tau \geqslant \rho = \sup \{ \| x^{]p[} - x^{]q[} \|_g ; \ p, q \in N \}$.

Proof. It is easy to see that (4.13) $\Rightarrow$ (2.1). Furthermore, by the same way as in theorem 4.1 of [3], one shows that (4.14) and (4.15) imply (2.2). Finally, it is evident that (4.16) $\Rightarrow$ (2.3), (4.17) $\Rightarrow$ (2.4) and (4.18) $\Rightarrow$ (3.2). Therefore, theorem 3.1 is applicable, and this completes the proof. Q.E.D.

5. As an application of the above result, a certain interesting class of Volterra functional equations will be presented. Suppose that a family of mappings

(5.1) $\quad (t, s) \to H(t, s) : C([0, s], R^n) \to R^n, \ \forall (t, s) \in D$

is given, such that

(5.2) $\forall x \in C(R_+, R^n)$, the mapping $(t, s) \to H(t, s)(x_s)$ of D into R^n is continuous on D. Consider the following Volterra functional equation

(5.3) $$\int_0^t H(t, s)(x_s)\, ds = y^0(t), \ \forall t \in R_+,$$

where, $y^0 \in C(R_+, R^n)$ is fixed. This class of functional equations represents a „functional" extension of the so-called Volterra integral equations of the first type. On the other hand, (5.3) is — from a formally point of view — a particular case of (4.5) but an effective reduction to (4.5) is possible only under some restrictive conditions. More precisely, suppose that there exists a family of mappings.

(5.4) $\quad (t, s) \to K(t, s) : C([0, s], R^n) \to R^n, \ \forall (t, s) \in D.$

such that

(5.5) $\forall x \in C(R_+, R^n)$, the mapping $(t, s) \to K(t, s)(x_s)$ of D into R^n is continuous on D

(5.6) $$H(t + a, s)(x_s) - H(t, s)(x_s) = - \int_t^{t+a} K(u, s)(x_s)\, du, \forall x \in C(R_+, R^n),$$
$$\forall (t, s) \in D, \ \forall a \in R_+.$$

Define the family of mappings

(5.7) $\quad t \to k(t) : C([0, t], R^n) \to R^n, \ \forall t \in R_+,$

by the following relation

$$k(t)(x_t) = H(t, t)(x_t), \quad \forall x \in C(R_+, R^n), \quad \forall t \in R_+. \tag{5.8}$$

Finally, suppose that there exists $x^0 \in C(R_+, R^n)$, such that

$$y^0(t) = \int_0^t x^0(s)\, ds, \quad \forall t \in R_+. \tag{5.9}$$

The main result about (5.3) can be stated as follows

Theorem 5.1. *Suppose that (4.13) — (4.18) hold for* $\{x^0\}$, $\{k(t)$ $t \in R_+\}$, $\{K(t, s) : (t, s) \in D\}$ *defined above. Then*

(5.10) *the Volterra functional equation (5.3) has at least one solution in* $C(R_+, R^n)$ *and exactly one solution in* $x^0 + C_g(R_+, R^n)$

(5.11) *there exists* $f \in P$, *which has the property: the sequence* $\{x^{[n]f} : n \in N\} \subset C(R_+, R^n)$, *defined as in (4.20) converges to the unique solution* $z \in x^0 + C_g(R_+, R^n)$ *of (5.3) and the error of approximation is given by the same inequality as in (4.20).*

Proof. Suppose that $x \in C(R_+, R^n)$ is a solution of (5.3). Let us write (5.3) for t, $t + a \in R_+$, where $a > 0$. A simple substraction of these equalities yields

$$\int_0^t [H(t + a, s)(x_s) - H(t, s)(x_s)]\, ds + \int_0^{t+a} H(t + a, s)(x_s)\, ds = y^0(t + a) - y^0(t). \tag{5.12}$$

Taking into account (5.5), (5.6), (5.9), dividing by $a > 0$, and interchanging the limits in the first integral of the left side, we get

$$\frac{1}{a}\int_t^{t+a} H(t + a, s)(x_s)\, ds = \frac{1}{a}\int_t^{t+a} \left[x^0(s) + \int_0^t K(s, u)(x_u)\, du\right] ds, \ \forall a > 0. \tag{5.13}$$

Performing the limit as $a \to 0+$, we get (4.5). Conversely, suppose that $x \in C(R_+, R^n)$ is a solution of (4.5). Integrating (4.5) on the interval $[0, t]$, we get (from (5.8))

$$\int_0^t H(s, s)(x_s)\, ds = \int_0^t x^0(s)\, ds + \int_0^t \left[\int_0^s K(s, u)(x_u)\, du\right] ds \tag{5.14}$$

Now, an interchange of limits in the second integral of the right side of (5.14), and (5.9), yields

$$\int_0^t H(s,s)(x_s)\, ds = y^0(t) + \int_0^t \left[\int_0^t K(s,u)(x_u)\, ds\right] du. \tag{5.15}$$

Finally, from (5.6) we obtain (4.5). Therefore, we have proved that (4.5) and (5.3) are equivalent, and this completes the proof. Q.E.D.

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