$$n, y) \prod_{r=n}^{n+\nu} \alpha(r)^{-1} \Big| = \sup_{\nu \geq 0} \Big[\omega\left(x(n+\nu : n, x) - y(n+\nu : n, \bar{y})\right) \prod_{r=n}^{n+\nu} \alpha(r)^{-1} \Big| \leq$$
$$\leq K(n) \mid \omega(x - y) - \omega(\bar{x} - \bar{y}) \mid .$$

Theorem 4.1 is thus proved.

Remark 4.1. In the case where $K(n) = K = \text{const.}$, $\alpha(n) = \alpha = \text{const.}$, the property (P) means the ω-exponential asymptotic stability in the sense of Definition 4.1. In this case we can prove a converse of Theorem 4.1. To prove this, assume that exists a real scalar function such that conditions a), b) and c) hold. Let us substitute in b) x and y by $x(n : n_0, x_0)$ and $y(n : n_0, y_0)$ respectively. It follows then $\Delta V(n, x(n), y(n)) = V(n+1, x(n+1), y(n+1)) - V(n, x(n), y(n)) \leq -(1-\alpha) V(n, x(n), y(n))$ from which $V(n+1, x(n+1), y(n+1)) \leq \alpha V(n, x(n), y(n))$ or $V(n, x(n : n_0, x_0), y(n : n_0, y_0)) \leq \alpha^{n-n_0} V(n_0, x_0, y_0)$.

From this relation and a) we obtain

$$\omega\left(x(n : n_0, x_0) - y(n : n_0, y_0)\right) \leq K\omega(x_0 - y_0)\,\alpha^{n-n_0}.$$

This completes the proof of our assertion.

REFERENCES

1. Mc Clamroch N. H. and Aggarwall J. K. — *On the convergence of an improper integral evaluated along the solution of a differential equation*. J. Math. Anal. Appl. 24 (1968), pp. 608–612.
2. Fodor J. — *Remark on the paper of Hahn „A new type of stability*. An. Univ. Bucureşti, 21 (1972), pp. 19–31.
3. Gordon S. P. — *A stability theory for perturbed difference equations*. SIAM J. Control, 10 (1972), pp. 671–678.
4. — — *On converses to the stability theorems for difference equations*. SIAM J. Control, 106 (1972), pp. 76–81.
5. — — *Stability and summability of solution of difference equations*. Math. System Theory, 5 (1971), pp. 56–65.
6. Hahn W. — *Über die Anwendung der Methode von Ljapunov auf Differentialgleichungen* Math. Ann., 136 (1958), 430–441.
7. — — *On a new type of stability*, J. Diff. Eqs., 3 (1967), pp. 440–448.
8. Halanay A. — *Quelques questions de la théorie de la stabilité pour les systèmes aux différences finies*, Arch. Rat. Mech. Anal., 12 (1963), pp. 150–154.
9. Kalman R. E. and Bertram J. E. — *Control system analysis and design via second method of Liapunov. II. Discrete-time systems*. Trans. ASME, J. Basic Engrg., Ser. D, 82 (1960), pp. 394–400.
10. Ramakrishna Rao D. — *L^p — stability of non-linear differential difference equations*. Funkcialaj Ekvacioj, 10 (1967), pp. 163–166.
11. Sinha A. S. C. — *On stability of a class of discrete systems*. J. Franklin Institute, 297 (1974), pp. 287–292.
12. Strauss A. — *Liapunov functions and L^p-stability of differential equations*. Trans. Am. Math. Soc., 119 (1965), pp. 37–50.

Received 30.X. 1976

Department of Mathematics
Polytechnic Institute of Iaşi
Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXIII, s. I a, 1977, f. 1

ON SOME FUNDAMENTAL INTEGRODIFFERENTIAL AND INTEGRAL INEQUALITIES

BY

B. G. PACHPATTE

1. Introduction. In the theory of differential and integral equations one has often to deal with certain integral inequalities. In particular, Gronwall's inequality [3], also known as Bellman's Lemma [1] and its generalization due to Bihari [2] have been used extensively in the study of differential and integral equations. In this paper we wish to establish some fundamental integrodifferential and integral inequalities of the Gronwall - Bellman - Bihari type which can be used in some applications as handy tools. The author believes that the inequalities established in this paper are new to the literature.

2. Integrodifferential Inequalities. In this section we state and prove some new integrodifferential inequalities which can be used in the analysis of a class of differential equations. A useful integrodifferential inequality is embodied in the following theorem.

Theorem 1. *Let $x(t)$, $x'(t)$, $a(t)$, $b(t)$, and $c(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$x'(t) \leq a(t) + b(t)\left(\int_0^t c(s)\left(x(s) + x'(s)\right) ds\right), \quad t \in I, \tag{1}$$

holds. Then

$$x'(t) \leq a(t) + b(t)\left(\int_0^t c(s)\left[A(s) + B(s)\right] \exp\left(\int_s^t c(\tau)\,|b(\tau) - 1|\, d\tau\right) ds\right), \tag{2}$$

for all $t \in I$, where

$$A(t) = x(0) + a(t) + \int_0^t a(s)\, ds, \quad t \in I, \tag{3}$$

and

(4) $$B(t)=\int_0^t c(s)\,A(s)\exp\left(\int_s^t [\,b(\tau)+c(\tau)+b(\tau)\,c(\tau)]\,d\tau\right)ds,\quad t\in I.$$

Proof. Define

(5) $$m(t)=\int_0^t c(s)\,(x(s)+x'(s))\,ds \qquad m(0)=0,$$

then (1) can be restated as

(6) $$x'(t)\leqslant a(t)+b(t)\,m(t)\,.$$

Differentiating (5) and using (6) we have

(7) $$m'(t)\leqslant c(t)\,[x(t)+a(t)+b(t)\,m(t)].$$

Integrating both sides of (6) from 0 to t we have

(8) $$x(t)\leqslant x(0)+\int_0^t a(s)\,ds+\int_0^t b(s)\,m(s)\,ds.$$

Now, using (8) in (7) we obtain

$$m'(t)\leqslant c(t)\left[x(0)+a(t)+\int_0^t a(s)\,ds+b(t)\,m(t)+\int_0^t b(s)\,m(s)\,ds\right].$$

Adding $c(t)\,m(t)$ to both sides of the above inequality and denoting

$$A(t)=x(0)+a(t)+\int_0^t a(s)\,ds,\ \text{ we have}$$

(9) $$m'(t)+c(t)\,m(t)\leqslant c(t)\,A(t)+b(t)\,c(t)\,m(t)+ + c(t)\left[m(t)+\int_0^t b(s)m(s)\,ds\right].$$

If we put

(10) $$v(t)=m(t)+\int_0^t b(s)\,m(s)\,ds,\quad v(0)=m(0)=0,$$

then it follows by differentiating (10) and using the facts that $m'(t)\leqslant$ $\leqslant c(t)\,A(t)+b(t)\,c(t)\,m(t)+c(t)\,v(t)$ and $m(t)\leqslant v(t)$ from (9) and (10), we see that the inequality

$$v'(t)\leqslant [b(t)+c(t)+b(t)\,c(t)]\,v(t)+c(t)\,A(t),$$

is satisfied, which implies the estimate for $v(t)$ such that

$$v(t)\leqslant\int_0^t c(s)\,A(s)\exp\left(\int_s^t [b(\tau)+c(\tau)+b(\tau)\,c(\tau)]\,d\tau\right)ds=B(t).$$

Substituting this value of $v(t)$ in (9) we have

$$m'(t)\leqslant c(t)\,[\,b(t)-1]\,m(t)+c(t)\,[\,A(t)+B(t)],$$

which implies the estimate for $m(t)$ such that

$$m(t)\leqslant\int_0^t c(s)\,[A(s)+B(s)]\exp\left(\int_s^t c(\tau)\,[\,b(\tau)-1]\,d\tau\right)ds.$$

Now, substituting this value of $m(t)$ in (6) we obtain the desired bound in (2).

It is intresting to note that, the advantage of inequality (2) over inequality (1) becomes apparent if we consider $a(t)$, $b(t)$ as known functions and $x(t)$ and $x'(t)$ as unknown. that is inequality (2) gives us a completely known function which majorizes $x'(t)$ and consequently $x(t)$ after integration.

Another interesting and useful integrodifferential inequality of the Bihari type may be stated as follows.

Theorem 2. *Let $x(t)$, $x'(t)$ be real-valued nonnegative continuous functions defined on I, and $b(t)\geqslant 1$, be a real-valued continuous function defined on I, $W(u)$ be a positive, continuous, strictly increasing function for $u>0$, $W(0)=0$, and suppose further that the inequality*

(11) $$x'(t)\leqslant a+\int_0^t b(s)\,W(x(s)+x'(s))\,ds,$$

is satisfied for all $t\in I$, where a is a nonnegative constant. Then for $t\in I_1$

(12) $$x'(t)\leqslant a+\int_0^t b(s)\,W\left(G^{-1}\left[G(x(0)+a)+\int_0^s b(\tau)\,d\tau\right]\right)ds.$$

where

(13) $$G(r)=\int_{r_0}^{r}\frac{ds}{r_0 s+W(s)},\qquad r\geqslant r_0>0,$$

and G^{-1} is the inverse function of G, and

$$I_1 = \left\{ t \in I : G(\infty) \geq G(x(0) + a) + \int_0^t b(s)\, ds \right\}.$$

Proof. Define

$$(14) \qquad m(t) = a + \int_0^t b(s)\, W(x(s) + x'(s))\, ds, \quad m(0) = a,$$

then (11) can be restated as

$$(15) \qquad x'(t) \leq m(t).$$

Differentiating (14) and using (15) we have

$$(16) \qquad m'(t) \leq b(t)\, W(x(t) + m(t)).$$

Integrating both sides of (15) from 0 to t, we have

$$(17) \qquad x(t) \leq x(0) + \int_0^t m(s)\, ds.$$

Now, using (17) in (16), we obtain

$$(18) \qquad m'(t) \leq b(t)\, W\left(x(0) + m(t) + \int_0^t m(s)\, ds\right).$$

If we put

$$(19) \qquad v(t) = x(0) + m(t) + \int_0^t m(s)\, ds, \quad v(0) = x(0) + m(0) = x(0) + a,$$

then it follows by differentiating (19) and using the facts that $m'(t) \leq b(t)\, W(v(t))$ and $m(t) \leq v(t)$ from (18) and (19) and $b(t) \geq 1$ we see that the inequality

$$(20) \qquad v'(t) \leq b(t)\, [v(t) + W(v(t))],$$

is satisfied. Dividing both sides of (20) by $[v(t) + W(v(t))]$, using (13) and then integrating both sides from 0 to t we have

$$(21) \qquad G(v(t)) - G(x(0) + a) \leq \int_0^t b(s)\, ds.$$

Now, substituting the value of $v(t)$ from (21) in (18) and then integrating both sides from 0 to t we obtain

$$m(t) \leq a + \int_0^t b(s)\, W\left(G^{-1}\left[G(x(0) + a) + \int_0^s b(\tau)\, d\tau\right]\right) ds.$$

Now, substituting this value of $m(t)$ in (15), we obtain the desired bound in (12).

We note that (12) gives the estimate on $x'(t)$ and consequently on $x(t)$ after integration, which is different from the given by Bihari in [2]. For similar estimates on integral inequalities we refer the interested reader to the author's recent papers [4] and [5].

We next establish the following integrodifferential inequality which can be used in some applications.

Theorem 3. *Let $x(t)$, $x'(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

$$(22) \qquad x'(t) \leq a + \int_0^t b(s)\, x'(s)\, (x(s) + x'(s))\, ds, \quad t \in I,$$

holds, where a is a positive constant. Then

$$(23) \qquad x'(t) \leq a \exp\left(\int_0^t b(s) \left[\frac{[x(0) + a]\, e^s}{1 - [x(0) + a] \int_0^s e^\tau b(\tau)\, d\tau} \right] ds \right),$$

for all $t \in I$, where $\int_0^t e^\tau\, b(\tau)\, d\tau < [x(0) + a]^{-1}$, for all $t \in I$.

Proof. Define a function $m(t)$ by the right hand member of (22). Then

$$m'(t) = b(t)\, x'(t)\, (x(t) + x'(t)), \quad m(0) = a,$$

which in view of (22) implies

$$(24) \qquad m'(t) \leq b(t)\, m(t)\, (x(t) + m(t)).$$

Since $x'(t) \leq m(t)$ from (22), we have $x(t) \leq x(0) + \int_0^t m(s)\, ds$. Now, using this value of $x(t)$ in (24) we have

$$(25) \qquad m'(t) \leq b(t)\, m(t) \left(x(0) + m(t) + \int_0^t m(s)\, ds\right).$$

If we put

$$(26) \qquad v(t) = x(0) + m(t) + \int_0^t m(s)\, ds, \quad v(0) = x(0) + m(0) = x(0) + a,$$

then if follows by differentiating (26) and using the facts that $m'(t) \leq b(t)\, m(t)\, v(t)$ and $m(t) \leq v(t)$ from (25) and (26) we see that the inequality

$$v'(t) \leq b(t)\, v^2(t) + v(t),$$

is satisfied, which implies the estimate for $v(t)$ such that (see [7])

$$v(t) \leqslant \frac{[x(0)+a]\,e^t}{1-[x(0)+a]\int_0^t e^s\, b(s)ds}, \quad t \in I.$$

substituting this value of $v(t)$ in (25) and then integrating both sides from to t we have

$$m(t) \leqslant a \exp\left(\int_0^t b(s)\left[\frac{[x(0)+a]\,e^s}{1-[x(0)+a]\int_0^s e^\tau\, b(\tau)\, d\tau}\right] ds\right) t \in I.$$

Now, substituting this value of $m(t)$ in (22) we obtain the desired bound in (23).

3. Integral Inequalities. In this section we establish some interesting integral inequalities which can be used in the study of many problems concerning the behavior of solutions of some differential and integral equations. Evidently, some of our results in this section include as special cases the well-known results of Gronwall [3], Bellman [1] and Bihari [2].

Theorem 4. *Let $x(t)$, $a(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on I, and $b(t)$ is a positive, monotonic, nondecreasing continuous function defined on I, for which the inequality*

$$x(t) \leqslant a(t)\, n(t) + \frac{x(t)}{n(t)}\left(\int_0^t b(s)\, x(s)\, ds\right), \quad t \in I, \tag{27}$$

holds. Then

$$x(t) \leqslant \frac{a(t)\, n(t)}{\left[1-2\int_0^t a(s)\, b(s)\, ds\right]^{1/2}}, \quad 0 \leqslant t < t_1, \tag{28}$$

where $t_1 = \sup\left\{t \in I : 2\int_0^t a(s)\, b(s)\, ds < 1\right\}.$

Proof. Since $n(t)$ is positive, monotonic, nondecreasing, we observe from (27) that

$$\frac{x(t)}{n(t)} \leqslant a(t) + \frac{x(t)}{n(t)}\left(\int_0^t b(s)\frac{x(s)}{n(t)}\, ds\right) \leqslant a(t) + \frac{x(t)}{n(t)}\left(\int_0^t b(s)\frac{x(s)}{n(s)}\, ds\right). \tag{29}$$

Define

$$z(t) = \int_0^t b(s)\frac{x(s)}{n(s)}\, ds, \; z(0) = 0.$$

Then

$$z'(t) = b(t)\frac{x(t)}{n(t)};$$

which in view of (29) implies

$$z'(t) \leqslant b(t)\left[a(t) + \frac{x(t)}{n(t)} z(t)\right] = a(t)\, b(t) + z(t)\, z'(t).$$

Integrating both sides of the above inequality from 0 to t we have

$$z^2(t) - 2z(t) + 2\int_0^t a(s)\, b(s)\, ds \geqslant 0, \tag{30}$$

The left side in (30) is a quadratic expression and both the factors of this quadratic expression are either $\geqslant 0 \leqslant$ or 0 at the same time so as to satisfy (30). Considering these cases we have

$$z(t) = 1 \pm \left[1 - 2\int_0^t a(s)\, b(s)\, ds\right]^{1/2}. \tag{31}$$

The first value of $z(t)$ in (31) is not admissible here, because this value of $z(t)$ will give us the lower bound for $x(t)$ after substituting it in (29). Hence, substituting the second value of $z(t)$ given in (31) in (29), we obtain the desired bound in (28).

We note that by setting $n(t) = 1$ in the above inequality we arrive at the another interesting inequality which can be used in some applications.

We now apply Theorem 4 to establish the following new integral inequalities.

Theorem 5. *Let $x(t)$, $a(t)$, $b(t)$, and $c(t)$ be real-valued nonnegative continuous functions defined on I, let $W(u)$ be a positive, continuous, monotonic, nondecreasing and submultiplicative function for $u > 0$, $W(0) = 0$, and suppose further that the inequality*

$$x(t) \leqslant a(t)\left(x_0 + \int_0^t c(s)\, W(x(s))\, ds\right) + \frac{x(t)\left(\int_0^t b(s) x(s)\, ds\right)}{\left(x_0 + \int_0^t c(s)\, w(x(s)) ds\right)}, \tag{32}$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leqslant k(t)\, \Omega^{-1}\left[\Omega(x_0) + \int_0^t c(s)\, W(k(s))\, ds\right], \quad 0 \leqslant t < t_1, \tag{33}$$

in which

$$k(t) = \frac{a(t)}{\left[1-2\int_0^t a(s)\, b(s)\, ds\right]^{\frac{1}{2}}}, \quad 0 \leqslant t < t_1, \tag{34}$$

where t_1 is as defined in Theorem 4, and

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{W(s)}, \quad r \geq r_0 > 0, \tag{35}$$

and Ω^{-1} is the inverse function of Ω, and t is in the subinterval $[0, t_1)$ of I so that

$$\Omega(x_0) + \int_0^t c(s)\, W(k(s))\, ds \in \operatorname{Dom}(\Omega^{-1}).$$

Proof. Define

$$n(t) = x_0 + \int_0^t c(s)\, W(x(s))\, ds, \quad n(0) = x_0. \tag{36}$$

Then (32) can be restated as

$$x(t) \leq a(t)\, n(t) + \frac{x(t)}{n(t)}\left(\int_0^t b(s)\, x(s)\, ds\right).$$

The rest of the proof proceeds much as that of Theorem 2, given by this author in [6, p. 143], by making use of Theorem 4, and we leave the details to the reader.

An interesting observation about the integral inequality established in Theorem 5 is that, when $b(t) = 0$ and $a(t) = 1$, it reduces to the Bihari's inequality given in [2], which in itself contains as a special case the well-known Gronwall-Bellman inequality given in [3], [1]. In the special case, when $c(t) = 0$ and $x_0 = 1$, then Theorem 5 reduces to Theorem 4 with $n(t) = 1$.

We next establish a still more general form of Theorem 5 which basically involves the comparison principle.

Theorem 6. *Let $x(t)$, $a(t)$, $b(t)$, and $c(t)$ be real-valued nonnegative continuous functions defined on I, let $W(t, u)$ be a positive, continuous, monotonic, nondecreasing in u, $u > 0$, for each fixed $t \in I$; let the function $\Psi(u) \geq 0$ be nondecreasing and continuous for $u \geq 0$, $\Psi(0) = 0$; and suppose further that the inequality*

$$x(t) \leq a(t)\left(x_0 + \Psi\left(\int_0^t c(s)\, W(s, x(s))\, ds\right)\right) + \frac{x(t)\left(\int_0^t b(s)\, x(s)\, ds\right)}{\left(x_0 + \Psi\left(\int_0^t c(s)\, W(s, x(s))\, ds\right)\right)} \tag{37}$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leq k(t)\,[x_0 + \Psi(r(t))], \quad 0 \leq t < t_1. \tag{38}$$

where $k(t)$ and t_1 are as defined in Theorem 4 and $r(t)$ is the maximal solution of

$$r'(t) = c(t)\, W(t, k(t)\,[x_0 + \Psi(r(t))]), \quad r(0) = 0, \tag{39}$$

existing on the subinterval $0 \leq t < t_1$ of I.

The proof of this theorem follows by an argument similar to that in the proof of Theorem 4 given by this author in [6, p. 145], by making use of Theorem 4. We omit the detalils.

4. Some Applications. In this section, we indicate some applications of the inequalities established in Theorem 1 and 4 to obtain the bounds for the solutions of some differential and integral equations. There are many possible applications of the inequalities established in this paper, but those indicated here are sufficient to convey the importance of our results.

We first consider the third order differential equation of the form

$$y'''(t) = h(t, y(t), y'(t)), \tag{40}$$

where $h \in C[I \times R \times R, R]$, and $y(0) = c_0$, $y'(0) = c_1$, and $y''(0) = c_2$ are given constants. Assume that the function h satisfies

$$|h(t, y(t), y'(t))| \leq k(t)\,(|y(t)| + |y'(t)|), \quad t \in I, \tag{41}$$

where $k(t)$ is a real-valued nonnegative continuous function defined on I. Integrating (40) twice from 0 to t, $t \in I$, we obtain

$$y'(t) = c_1 + c_2 t + \int_0^t (t - s)\, h(s, y(s), y'(s))\, ds. \tag{42}$$

Taking the absolute values on both sides and using the condition (41) in (42) we have

$$|y'(t)| \leq |c_1| + |c_2|\, t + t\int_0^t k(s)\,(|y(s)| + |y'(s)|)\, ds \leq$$

$$\leq (1 + t)(|c_1| + |c_2|) + (1 + t)\left(\int_0^t k(s)\,(|y(s)| + |y'(s)|)\, ds\right).$$

Now, an application of Theorem 1 yields

$$|y'(t)| \leq (1+t)(|c_1| + |c_2|) + (1+t)\left(\int_0^t k(s)\,[P(s) + Q(s)]\exp\left(\int_s^t k(\tau)\,\tau\, d\tau\right) ds\right). \tag{43}$$

where

$$P(t) = |c_0| + (|c_1| + |c_2|)\left(1 + 2t + \frac{t^2}{2}\right), \quad t \in I,$$

and

$$Q(t) = \int_0^t k(s)\, P(s) \exp\left(\int_s^t [(1+\tau) + k(\tau) + (1+\tau)\, k(\tau)]\, d\tau\right) ds, \quad t \in I.$$

Thus (43) gives a bound on $|y'(t)|$ in terms of the known functions, and hence integrating both sides of (43) from 0 to t we can obtain a bound on $|y(t)|$ for $t \in I$, where $y(t)$ is a solution of (40).

We next consider the Volterra integral equation of the form

$$z(t) = f(t) + z(t)\, H\left(t, \int_0^t k(t, s, z(s))\, ds\right), \tag{44}$$

where $z, f \in C[I, R]$, $K \in C[I \times I \times R, R]$, and $H \in C[I \times R, R]$. Suppose that the functions k and H in (44) satisfy

$$|k(t, s, z(s))| \leqslant g(s)\, |z(s)|, \quad t, s \in I, \tag{45}$$

$$|H(t, u)| \leqslant M\, |u|, \quad t \in I, \tag{46}$$

where $M > 0$ is a constant and $g(t)$ is a real-valued nonnegative continuous function defined on I. Now, in view of the assumptions (45) and (46), we can establish very easily the bound on the solution $z(t)$ of (44) by using Theorem 4 with $n(t) = 1$ on a suitable subinterval of I.

REFERENCES

1. R. Bellman — *The stability of solutions of linear differential equations*. Duke Math. Joun 10 (1943), 643—647.
2. I. Bihari — *A generalization of a lemma of Bellman and applications to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar 7 (1856), 81—94.
3. T. H. Gronwall — *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*. Ann. Math. 20 (1919), 292—296.
4. B. G. Pachpatte — *An integral inequality similar to Bellman — Bihari inequality*, Bull. Soc. Math. Grèce 15 (1974), 7—12.
5. B. G. Pachpatte — *On some integral inequalities similar to Bellman — Bihari inequalities*, J. Math. Anal. Appl. 49 (1975), 794—802.
6. B. G. Pachpatte — *On some generalizations of Bellman's Lemma*, J. Math. Anal. Appl. 51 (1975), 141—150.
7. B. G. Pachpatte — *On some new integral inequalities for differential and integral equations*, J. Math. Physical Sci. 10 (1976), 101—116.

Received 22. XI 1976

Department of Mathematic
Deogiri College
Aurangabad (Maharashtra)
India

Analele ştiinţifice ale Universităţii „Al. I. CUZA" din Iaşi
Tomul XXIII, s. I a, 1977, f. 1

NOTE ON AN ABSTRACT VOLTERRA EQUATION

BY

SERGIU AIZICOVICI

Following Barbu [1], [2] we use monotonicity methods to approach a class of abstract Volterra integral equations. Specifically we are concerned with establishing the existence of solutions of the integral equation

$$u(t) + \int_0^t a(t-s)\, Au(s)\, ds \ni f(t), \qquad t \geqslant 0, \tag{1}$$

under weak assumptions on the convolution kernel $a : [0, \infty) \to R$. Here $u(t)$ is to be an element of a Hilbert space H, A — a nonlinear monotone operator on H and $f : [0, \infty) \to H$, a prescribed function. For background material on monotone operators, relevant to the present note, we refer the reader to [3], [4].

Let H be a real Hilbert space of norm $|\cdot|$ and inner product $(,)$ Let V be a real Banach space such that $V \subset H$, algebraically and topologically. Consider a convex, lower semicontinuous, proper function $\varphi : H \to (-\infty, \infty]$ and denote by A its subdifferential ($A = \partial\varphi$). It is known that A is maximal monotone in H.

We treat Eq. (1) in case $A = \partial\varphi$ making the following hypotheses:

(i) *$a : [0, \infty) \to R$ is locally absolutely continuous,*
(ii) $a(0) > 0$,
(iii) *the injection $V \subset H$ is compact,*
(iv) *$D(\varphi) \subset V$ and for each $k > 0$ the set $\{x ; |\varphi(x)| + |x| \leqslant k\}$ is bounded in V,*
(v) $f, f' \in L^2_{\mathrm{loc}}(0, \infty ; H)$,
(vi) $f(0) \in D(\varphi)$.

Definition. *By a solution of (1) we mean a function $u : [0, \infty) \to H$ satisfying*

$$u \in C([0, \infty) ; H), \quad u' \in L^2_{\mathrm{loc}}(0, \infty ; H), \quad u(t) \in D(A), \text{ a.c. } t > 0.$$

There exists $w \in L^2_{\mathrm{loc}}(0, \infty ; H)$ such that $w(t) \in Au(t)$, a.e. on $(0, \infty)$,