where

$$P(t) = |c_0| + (|c_1| + |c_2|)\left(1 + 2t + \frac{t^2}{2}\right), \ t \in I.$$

and

$$Q(t) = \int_0^t k(s)\, P(s) \exp\left(\int_s^t [(1+\tau) + k(\tau) + (1+\tau)\, k(\tau)]\, d\tau\right) ds, \ t \in I.$$

Thus (43) gives a bound on $|y'(t)|$ in terms of the known functions, and hence integrating both sides of (43) from 0 to t we can obtain a bound on $|y(t)|$ for $t \in I$, where $y(t)$ is a solution of (40).

We next consider the Volterra integral equation of the form

$$z(t) = f(t) + z(t)\, H\left(t, \int_0^t k(t, s, z(s))\, ds\right), \tag{44}$$

where $z, f \in C[I, R]$, $K \in C[I \times I \times R, R]$, and $H \in C[I \times R, R]$. Suppose that the functions k and H in (44) satisfy

$$|k(t, s, z(s))| \leqslant g(s)\, |z(s)|, \ t, s \in I, \tag{45}$$

$$|H(t, u)| \leqslant M\, |u|, \ t \in I, \tag{46}$$

where $M > 0$ is a constant and $g(t)$ is a real-valued nonnegative continuous function defined on I. Now, in view of the assumptions (45) and (46), we can establish very easily the bound on the solution $z(t)$ of (44) by using Theorem 4 with $n(t) = 1$ on a suitable subinterval of I.

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NOTE ON AN ABSTRACT VOLTERRA EQUATION

BY

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Following Barbu [1], [2] we use monotonicity methods to approach a class of abstract Volterra integral equations. Specifically we are concerned with establishing the existence of solutions of the integral equation

$$u(t) + \int_0^t a(t-s)\, Au(s)\, ds \ni f(t), \quad t \geqslant 0, \tag{1}$$

under weak assumptions on the convolution kernel $a : [0, \infty) \to R$. Here $u(t)$ is to be an element of a Hilbert space H, A — a nonlinear monotone operator on H and $f : [0, \infty) \to H$, a prescribed function. For background material on monotone operators, relevant to the present note, we refer the reader to [3], [4].

Let H be a real Hilbert space of norm $|\cdot|$ and inner product $(,)$. Let V be a real Banach space such that $V \subset H$, algebraically and topologically. Consider a convex, lower semicontinuous, proper function $\varphi : H \to (-\infty, \infty]$ and denote by A its subdifferential ($A = \partial\varphi$). It is known that A is maximal monotone in H.

We treat Eq. (1) in case $A = \partial\varphi$ making the following hypotheses:

(i) $a : [0, \infty) \to R$ *is locally absolutely continuous,*

(ii) $a(0) > 0$,

(iii) *the injection* $V \subset H$ *is compact,*

(iv) $D(\varphi) \subset V$ *and for each* $k > 0$ *the set* $\{x ; |\varphi(x)| + |x| \leqslant k\}$ *is bounded in* V,

(v) $f, f' \in L^2_{loc}(0, \infty ; H)$,

(vi) $f(0) \in D(\varphi)$.

Definition. *By a solution of (1) we mean a function* $u : [0, \infty) \to H$ *satisfying*

$$u \in C([0, \infty) ; H), \ u' \in L^2_{loc}(0, \infty ; H), \quad u(t) \in D(A), \text{ a.e. } t > 0.$$

There exists $w \in L^2_{loc}(0, \infty ; H)$ such that $w(t) \in Au(t)$, a.e. on $(0, \infty)$,

$$u(t) + \int_0^t a(t-s)\, w(s)\, ds = f(t), \quad t \geq 0.$$

For simplicity we will write Au instead of w.

Theorem. *Suppose* (i)—(vi) *hold. Then Eq.* (1) *has a solution satisfying:*

$u \in L^\infty_{loc}(0, \infty ; V)$, $t \to \varphi(u(t))$ *is locally absolutely continuous.*

Remark 1. This result is related to that of Londen [6, Theorem 1]. Note that we remove Londen's restrictive requirement that a' be of bounded variation on an interval $[0, T]$. Our condition (iv) is not used in [6]. It should however be emphasized that there are important classes of φ's of physical interest which satisfy (iv) (see e.g. [3], Chapter IV]).

Proof of the Theorem. For each $\lambda > 0$, consider the regularized equation

(2) $$u_\lambda(t) + \int_0^t a(t-s)\, A_\lambda u_\lambda(s)\, ds = f(t), \ t \geq 0,$$

where

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad J_\lambda = (I + \lambda A)^{-1}.$$

Inasmuch as A_λ is lipschitzian it follows that (2) has a unique locally absolutely continuous solution $u_\lambda : [0, \infty) \to H$. Differentiating (2) yields

(3) $$u'_\lambda(t) + a(0)\, A_\lambda u_\lambda(t) + \int_0^t a'(t-s)\, A_\lambda u_\lambda(s)\, ds = f'(t), \text{ a.e. on } (0, \infty).$$

Remember that $A_\lambda = \partial \varphi_\lambda$, where

(4) $$\begin{cases} \varphi_\lambda(x) = \min\limits_{y \in H} \left\{ \dfrac{1}{2\lambda} |y - x|^2 + \varphi(y) \right\}, & x \in H,\ \lambda > 0, \\ \varphi(J_\lambda x) \leq \varphi_\lambda(x) \leq \varphi(x), & x \in H. \end{cases}$$

Multiply (3) by $A_\lambda u_\lambda(t)$ and integrate over $(0, t)$. This gives

(5) $$\varphi_\lambda(u_\lambda(t)) + a(0) \int_0^t |A_\lambda u_\lambda(s)|^2 ds \leq \varphi_\lambda(f(0)) + \int_0^t |a'(s)|\, ds \int_0^t |A_\lambda u_\lambda(s)|^2 ds + \int_0^t |f'(s)|\, |A_\lambda u_\lambda(s)|\, ds, \quad t \geq 0.$$

We fix $T > 0$ such that

(6) $$2 \int_0^T |a'(s)|\, ds \leq a(0),$$

and restrict ourselves to $t \in [0, T]$. Then using (4), (5), (6) one has

(7) $$\varphi(J_\lambda u_\lambda(t)) + \frac{a(0)}{2} \int_0^t |A_\lambda u_\lambda(s)|^2 ds \leq \varphi(f(0)) + \int_0^t |f'(s)|\, |A_\lambda u_\lambda(s)|\, ds, \quad t \in [0, T].$$

Since we may assume (without loss of generality) that $\varphi \geq 0$, we infer from (7) that

(8) $\{A_\lambda u_\lambda, \lambda > 0\}$ is bounded in $L^2(0, T ; H)$.

Hence, by (2), (3) one concludes

(9) $\{u_\lambda, \lambda > 0\}$ is bounded in $L^\infty(0, T ; H)$,

(10) $\{u'_\lambda, \lambda > 0\}$ is bounded in $L^2(0, T ; H)$.

Set $v_\lambda(t) = J_\lambda u_\lambda(t)$ and remark that for sufficiently small λ, (9) and (10) remain satisfied if u_λ is replaced by v_λ. Consequently, by (7),

(11) $\varphi(v_\lambda(t)) + |v_\lambda(t)| \leq C, \quad t \in [0, T]$, for λ sufficiently small

Here and in the sequel C denotes various finite positive constants independent of λ. Now use (iii), (iv) and a well-known compacity criterion [5, Theorem 5.1, Chapter 1] to deduce that

(12) $v_{\lambda_n} \to u$ strongly in $C(0, T ; H)$,

for a suitable sequence $\lambda_n \to 0$. Since $A_\lambda u_\lambda(t) \in A v_\lambda(t)$ one gets by (8), (12) and the maximal monotonicity of A,

$$A_{\lambda_n} u_{\lambda_n} \to Au, \ \textit{weakly in } L^2(0, T ; H).$$

We also have

$$u_{\lambda_n} \to u, \ \textit{strongly in } L^2(0, T ; H),$$

$$u'_{\lambda_n} \to u', \ \textit{weakly in } L^2(0, T ; H).$$

Take $\lambda = \lambda_n$ in (2) and let $\lambda_n \to 0$. One obtains

$$u(t) + \int_0^t a(t-s)\, Au(s)\, ds = f(t), \quad t \in [0, T].$$

In addition, from (11) it follows

$$|\varphi(u(t))| \leqslant C, \quad 0 \leqslant t \leqslant T. \tag{13}$$

This implies (cf. (iv))

$$u \in L^{\infty}(0, T; V).$$

Then, using the fact that $u', Au \in L^2(0, T: H)$ we infer (see [4, Chapter 3, Lemma 3.3]) that the function $t \to \varphi(u(t))$ is absolutely continuous on $[0, T]$.

To conclude the proof, it remains to show that $u(t)$ can be extended outside $[0, T]$. Let us consider the equation

$$z(t) + \int_0^t a(t-s) Az(s)\, ds \ni f(t+T) - \int_0^T a(t+T- \tag{14}$$
$$- s) Au(s)\, ds, \qquad 0 \leqslant t \leqslant T.$$

Remark that $f_1, f_1' \in L^2(0, T; H)$, where $f_1(t)$ denotes the right-hand side of (14). By (13) one also has $f_1(0) = u(T) \in D(\varphi)$. Therefore the preceding arguments lead to a solution $z: [0, T] \to H$ of (14). It follows that

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \leqslant t \leqslant T, \\ z(t-T), & T < t \leqslant 2T. \end{cases}$$

is the desired extension of $u(t)$ on $[0, 2T]$. The same method enables us to prolong $u(t)$ on the whole half-axis.

Remark 2. The above proof can be easily adapted to the case where $a(t)$ is replaced by an operator valued kernel $E: [0, \infty) \to \mathcal{L}(H, H)$, such that

For each $x \in H$, $t \to E(t)x$ *is locally absolutely continuous on* $[0, \infty)$,

$$E' \in L^1_{loc}(0, \infty; \mathcal{L}(H, H)),$$

$$(E(0)x, x) \geqslant k|x|^2, \ k > 0, \ x \in H.$$

By $\mathcal{L}(H, H)$ we have denoted the space of all linear continuous operators on H.

Corollary 1. *Let (iii), (iv) be satisfied. In addition suppose*

$$a \in L^1_{loc}(0, \infty: R) \tag{15}$$

$$f \in L^2_{loc}(0, \infty; H). \tag{16}$$

$$u_0 \in D(\varphi). \tag{17}$$

Then the equation

$$u'(t) + mAu(t) + \int_0^t a(t-s) Au(s)\, ds \ni f(t), \text{ a.e. } t > 0, \ m > 0, \ A = \partial\varphi, \tag{18}$$

with initial condition

$$u(0) = u_0. \tag{19}$$

has a solution $u(t)$ *such that*

$$u \in C([0, \infty); H) \cap L^{\infty}_{loc}(0, \infty: V). \tag{20}$$

$$u', Au \in L^2_{loc}(0, \infty: H). \tag{21}$$

$$t \to \varphi(u(t)) \textit{ is locally absolutely continuous.} \tag{22}$$

Proof. We need only observe that (18), (19) are equivalent to

$$u(t) + \int_0^t b(t-s) Au(s)\, ds \ni g(t),$$

where

$$b(t) = m + \int_0^t a(s)\, ds, \quad t \geqslant 0.$$

$$g(t) = u_0 + \int_0^t f(s)\, ds, \quad t \geqslant 0.$$

Consequently, by virtue of (15)—(17) our Theorem applies.

Corollary 2. *Let the assumptions of Corollary 1 be fulfilled. Suppose moreover we are given a continuous function* $h: [-t_0, 0] \to H$, $t_0 > 0$ *satisfying*

$$h(t) \in D(A), \text{ a.e. on } (-t_0, 0); \ h(0) \in D(\varphi).$$

$$Ah \in L^2(-t_0, 0: H).$$

Then there exists at least one continuous solution $u: [-t_0, \infty) \to H$ *of th problem*

$$u'(t) + Au(t-t_0) + mAu(t) + \int_0^t a(t-s) Au(s)\, ds \ni \tag{23}$$
$$\ni f(t), \qquad \text{a.e. on } (0, \infty), \ m > 0, \ A = \partial\varphi$$

$$u(t) = h(t), \qquad -t_0 \leqslant t \leqslant 0. \tag{24}$$

such that (20) — (22) *hold.*

Proof. On $(0, t_0)$ Eq. (23) can be rewritten as

$$u'(t) + mAu(t) + \int_0^t a(t-s) Au(s)\, ds \ni f(t) - Ah(t-t_0), \quad \text{a.e. } t \in (0, t_0), \tag{25}$$

with

$$u(0) = h(0). \tag{26}$$

Applying Corollary 1 one obtains a solution $u_1(t)$ of (25), (26) such that

$$u_1 \in C([0, t_0) : H) \cap L^2(0, t_0 : V), \tag{27}$$

$$u_1', Au_1 \in L^2(0, t_0 : H), \tag{28}$$

$$t \to \varphi(u_1(t)) \text{ is absolutely continuous on } [0, t_0]. \tag{29}$$

To extend $u_1(t)$ on $(t_0, 2t_0]$ consider the problem

$$\begin{cases} v'(t) + mAv(t) + \int_0^t a(t-s)\,Av(s)\,ds \ni f(t+t_0) - Au_1(t) - \\ \qquad - \int_0^{t_0} a(t+t_0-s)\,Au_1(s)\,ds, \text{ a.e. on } (0, t_0), \\ v(0) = u_1(t_0). \end{cases} \tag{30}$$

From Corollary 1 one derives the existence of a function $v : [0, t_0] \to H$ satisfying (30). Furthermore (27) — (29) remain valid with v instead of u_1. It is now obvious that

$$u(t) = \begin{cases} u_1(t), & 0 \leq t \leq t_0, \\ v(t - t_0), & t_0 < t \leq 2t_0 \end{cases}$$

is a solution of (23), (24) on $[0, 2t_0]$.

In a similar manner one continues $u(t)$ on $(2t_0, \infty)$. Clearly, by (27)—(29) assertions (20)—(22) are satisfied.

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WEAKLY ALMOST PERIODIC SYSTEMS

BY

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1. Introduction. In this note we announce some results concerning discrete actions (X, Φ), where Φ is a continuous self map (not necessarily a homeomorphism) of a compact metric space X. The proofs of these and related results will appear elsewhere [9]. Let T denote the operator on $C(X)$ (complex valued continuous maps on X) induced by Φ, thus $T(f) = f \circ \Phi$.

Definition. *We say T and Φ are weakly almost periodic, or that (X, Φ) is a w.a.p. system, provided the semigroup $\{T^n \mid n \in Z^+\}$ is precompact in the weak operator topology.*

This definition may be restated in various ways. For example it is nontrivially equivalent to the statement that in $C(X)$ with the weak topology (bounded pointwise convergence) each orbit $\{T^n(f)\}$ is precompact. If the condition in the definition holds in the strong operator topology, we say T and Φ are strongly almost periodic (s.a.p.). Weak almost periodicity arises and has been studied in various contexts: topology [3], [7], symbolic dynamics [11], and operator theory [10], [11].

2. General Theory. It can be shown [4], [6], that T induces a decomposition $C(X) = C_0 \oplus C_1$ where for each $f \in C_0$ the weak closure of $\{T^n f\}$ contains 0 and each f in C_1 is the uniform limit of finite linear combinations of unimodular eigenfunctions. This decomposition, together with results on Markov operators from [5], provide the key tools for developing the general theory.

We begin with the minimal sets of Φ, that is, sets which are nonempty, closed, Φ-invariant and minimal with these properties. Since X is compact, minimal sets exist.

Theorem 2.1. *If (X, Φ) is a w.a.p. system and E is minimal, then $\Phi \mid E$ is a s.a.p. homeomorphism. In fact, E is a monothetic group, [2], and Φ is translation by a transitive group element.*

It can also be shown in the above context that E supports exactly one invariant probability measure. The regularity described in Theorem 2.1 carries over to the union of all the minimal sets, which we call *center* of the system and which we denote hereafter by M. The results about M