with

(26) $$u(0) = h(0).$$

Applying Corollary 1 one obtains a solution $u_1(t)$ of (25), (26) such that

(27) $$u_1 \in C([0, t_0] : H) \cap L^2(0, t_0 : V),$$

(28) $$u_1', Au_1 \in L^2(0, t_0 : H),$$

(29) $t \to \varphi(u_1(t))$ is absolutely continuous on $[0, t_0]$.

To extend $u_1(t)$ on $(t_0, 2t_0]$ consider the problem

(30) $$\begin{cases} v'(t) + m Av(t) + \int_0^t a(t-s) Av(s)\,ds \ni f(t+t_0) - Au_1(t) - \\ \qquad - \int_0^{t_0} a(t+t_0-s) Au_1(s)\,ds, \text{ a.e. on } (0, t_0), \\ v(0) = u_1(t_0). \end{cases}$$

From Corollary 1 one derives the existence of a function $v : [0, t_0] \to H$ satisfying (30). Furthermore (27) — (29) remain valid with v instead of u_1. It is now obvious that

$$u(t) = \begin{cases} u_1(t), & 0 \leq t \leq t_0, \\ v(t - t_0), & t_0 < t \leq 2t_0 \end{cases}$$

is a solution of (23), (24) on $[0, 2t_0]$.

In a similar manner one continues $u(t)$ on $(2t_0, \infty)$. Clearly, by (27)—(29) assertions (20)—(22) are satisfied.

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WEAKLY ALMOST PERIODIC SYSTEMS

BY

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1. Introduction. In this note we announce some results concerning discrete actions (X, Φ), where Φ is a continuous self map (not necessarily a homeomorphism) of a compact metric space X. The proofs of these and related results will appear elsewhere [9]. Let T denote the operator on $C(X)$ (complex valued continuous maps on X) induced by Φ, thus $T(f) = f \circ \Phi$.

Definition. *We say T and Φ are weakly almost periodic, or that (X, Φ) is a w.a.p. system, provided the semigroup $\{T^n \mid n \in Z\}$ is precompact in the weak operator topology.*

This definition may be restated in various ways. For example it is nontrivially equivalent to the statement that in $C(X)$ with the weak topology (bounded pointwise convergence) each orbit $\{T^n(f)\}$ is precompact. If the condition in the definition holds in the strong operator topology, we say T and Φ are strongly almost periodic (s.a.p.). Weak almost periodicity arises and has been studied in various contexts: topology [3], [7], symbolic dynamics [11], and operator theory [10], [11].

2. General Theory. It can be shown [4], [6], that T induces a decomposition $C(X) = C_0 \oplus C_1$ where for each $f \in C_0$ the weak closure of $\{T^n f\}$ contains 0 and each f in C_1 is the uniform limit of finite linear combinations of unimodular eigenfunctions. This decomposition, together with results on Markov operators from [5], provide the key tools for developing the general theory.

We begin with the minimal sets of Φ, that is, sets which are nonempty, closed, Φ-invariant and minimal with these properties. Since X is compact, minimal sets exist.

Theorem 2.1. *If (X, Φ) is a w.a.p. system and E is minimal, then $\Phi | E$ is a s.a.p. homeomorphism. In fact, E is a monothetic group, [2], and $\Phi | E$ is translation by a transitive group element.*

It can also be shown in the above context that E supports exactly one invariant probability measure. The regularity described in Theorem 2.1 carries over to the union of all the minimal sets, which we call *center* of the system and which we denote hereafter by M. The results about M

set forth in the next theorem depend to a surprising degree on the „de-Leeuw-Glicksberg" decomposition of $C(X)$ mentioned earlier and does not seem to follow from purely topological considerations.

Theorem 2.2. *Let (X, Φ) be a w.a.p. system with center M. Then M is a retract of X, Φ is s.a.p. on M, and the orbit closure decomposition of M is both upper and lower-semicontinuous. There is a sequence N of integers, such that, for all $x \in X$, $\{\Phi^n(x) \mid n \in N\}$ clusters in M and $\{\Phi^n \mid n \in N\}$ converges uniformly to the identity on M.*

Thus, on M, Φ exhibits strong recurrence properties reminiscent of almost periodicity as the term is used in classical dynamics. In connection with the decomposition of M mentioned in Theorem 2.2, there is a related result which shows that the structure of certain minimal sets cannot change abruptly. This contrasts sharply with the orbit structure of even such nice maps as a twist map of R^2 given by $(r, \theta) \to (r, \theta + 2\pi r)$. The result is

Theorem 2.3. *If (X, Φ) is a w.a.p. system and E is a minimal set of dimension 1, then there is a neighborhood of E containing no 0-dimensional minimal set. On the other hand, if E is totally disconnected, there is a neighborhood of E containing no connected minimal set.*

3. Actions on Manifolds. We summarize briefly some of the more detailed information available when it is assumed that the state space X is a compact, closed, connected manifold. If X is topologically equivalent to a circle then by Theorem 2.2, M is a point, a proper arc, or all of M. In the first case, various examples show that not much can be said. In the last case, Φ must be conjugate to a rotation or reflection. In the remaining case, Φ must be inessential, in fact $\Phi(X) \subset X$, and is also s.a.p. on X.

When X is a 2-manifold the situation is more complicated but still tractable. Beginning with an annulus A in R^2, let Φ be an „irrational" rotation of A, followed possibly by reflection through a central circle. By making various identifications on the boundary of A or by embedding A in a larger 2-manifold (in both cases, A becomes M) one obtains a variety of actions on 2-manifolds with M homeomorphic to an annulus, a disk, a sphere, a torus, projective space, a Klein bottle or a Möbius band. We can now state the main result.

Theorem 3.1. *Let (X, Φ) be a w.a.p. system, where X is a compact, closed, connected, 2-manifold. Then exactly one of the following holds:*

(a) *Every minimal set is 0-dimensional and Φ is periodic on M.*

(b) *$M = X$ is a torus and is minimal under Φ.*

(c) *The only minimal set is a single simple closed curve.*

(d) *$\Phi \mid M$ is one of the annular actions described above.*

The proof of this theorem is a strenuous exercise in classical cut-and-paste topology, using results from [8] and [12]. Finally, for higher dimensional manifolds some positive results have been obtained. For example, as is the case with flows [1], we can say that if E is a minimal set of a w.a.p. action (X, Φ) where X is a compact n-manifold, and if E has dimension n or $n-1$, then E is a finite union of tori (the point being that E is not some generalized solenoid with messy topological structure as described in [1]). Finally, there is an interesting result about w.a.p. homeomorphisms of manifolds, the proof of which uses the retract part of Theorem 2.2 and some algebraic topology.

Theorem 3.2. *Let Φ be a w.a.p. homeomorphism of a compact, closed manifold X. If the interior of M is not empty then $M = X$, and thus Φ is s.a.p.*

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