Thus at the k^{th} level, $\alpha_i^{(k)}$ can be computed if $n \leqslant R - (k+1)\, r$. This gives the optimum value of k to be

$$k_{opt} \leqslant \text{int} \left\{ \frac{1}{r}(R - n) - 1 \right\}, \quad r \neq 0.$$

when $r = 0$ all previous storage locations can be used so that

$$1 \leqslant k_{opt} < \infty$$

Computed Values of Series for $r=3$, $s=5$

k \ n	1	2	3	4	5	6
1	0.16667	0.30952	0.43452	0.54563	0.54563	0.73654
2	0.09094	0.18317	0.27524	0.36634	0.45602	0.54404
3	0.06106	0.12620	0.19421	0.26424	0.33571	0.40819
4	0.04404	0.09200	0.14302	0.19651	0.25200	0.30917
5	0.03275	0.06875	0.10740	0.14826	0.19101	0.23540
6	0.02471	0.05200	0.08142	0.11267	0.14550	0.17973
7	0.01878	0.03956	0.06203	0.08584	0.11113	0.13744
8	0.01432	0.03020	0.04738	0.06569	0.08499	0.10519
9	0.01095	0.02309	0.03624	0.05026	0.06506	0.08055
10	0.00838	0.01767	0.02774	0.03848	0.04982	0.06169
11	0.00641	0.01353	0.02124	0.02947	0.03816	0.04726
12	0.00491	0.01036	0.01627	0.02257	0.02923	0.03620
13	0.00376	0.00749	0.01246	0.01729	0.02239	0.02773
14	0.00288	0.00608	0.00955	0.01325	0.01715	0.02125
15	0.00221	0.00466	0.00731	0.01015	0.01314	0.01628

With these estimates a large-size multiple series of the type (1.1) can be computed.

The table above gives computed values of a k^{th}-dimensional series, using equation (2.4) with $r = 3$, $s = 5$ and $1 \leqslant k \leqslant 15$, of the type

$$\sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{i_k+r} \cdots \sum_{i_2=1}^{i_3+r} \sum_{i_1=1}^{i_2+r} \Phi(\xi_1, \xi_2, \ldots, \xi_k), \quad 1 \leqslant n \leqslant 6, \text{ where } \xi_i = 1/(i+s).$$

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Department of Computer Science
University of Ife,
Ile - Ife. NIGERIA

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SCHAUDER DECOMPOSITIONS AND ITERATIVE METHODS FOR SOLUTIONS OF LINEAR OPERATOR EQUATIONS

BY

C. ILIOI

In the paper [1] we have studied the class of conjugate directions methods for solution of linear equations in a Hilbert space H, by using polynomial bases of H. That idea can be further exploited, in order to extend the domain of convergence of these iterative methods.

In the present paper, a broad class of iterations is introduced by using Schauder decompositions of the space of definition. The special case of finite dimensional decomposition methods is considered in greater detail, because of its practical interest.

1. Let us start with a brief survey of results concerning Schauder decompositions of a Banach space, which we need later.

Definition. *A sequence $\{M_n\}$ of linear subspaces of the Banach space X is called a decomposition of X, if and only if, for each x in X, there exists a unique sequence $\{x_n\}$ such that*

$$x_n \in M_n \text{ for all } n \text{ and } x = \lim_{n \to \infty} \sum_{i<n} x_i = \sum_{n=1}^{\infty} x_n \text{ in } X.$$

The uniqueness implies the existence of associated projections E_n of X on M_n such that $E_i E_j = \delta_{ij} E_j$, where δ_{ij} is the usual Kronecker symbol. If all subspaces M_n are closed, then $\{M_n\}$ is called a Schauder decomposition of X.

We shall denote by $\{M_n, E_n\}$ the decomposition of X, whenever it will be necessary to emphasize the projection involved, E_n. According to J. T. Marti's monograph [4] we have:

Proposition 1. *In a Banach space X the following conditions are equivalent:*

(i) *There is a sequence $\{M_n\}$ of closed linear subspaces of X, which is a Schauder decomposition of X.*

(ii) *There is a sequence $\{E_n\}$ of continuous projections of X such that $E_i E_j = \delta_{ij} E_j$ and $\lim_{n \to \infty} \sum_{i<n} E_i x = x$, for each x in X.*

(iii) *There is a sequence* $\{S_n\}$ *of continuous projections of* X *such that* $S_m S_n = S_{\min(m,n)}$ *and* $\lim_{n\to\infty} S_n x = x$ *for each* x *in* X. *The subspaces* M_n *and the projections* E_n *and* S_n *are related by*: $E_n(X) = M_n$ *and* $S_n = \sum_{i\le n} E_i$ *(partial sum projections)*, $n = 1, 2, \ldots$

For any set $M \subset X$ we shall denote by $\mathrm{sp}\, M$ the linear subspace spaned by M and by $\overline{\mathrm{sp}}\, M$ the closure of this subspace. The sequence $\{M_n\}$ of closed subspaces of X is complete in X, if $\overline{\mathrm{sp}} \bigcup_{n=1}^{\infty} M_n = X$; we denote for any natural n:

$$X_{(n)} = \mathrm{sp} \bigcup_{i=1}^{n} M_i, \quad X^{(n)} = \overline{\mathrm{sp}} \bigcup_{i=n+1}^{\infty} M_i.$$

Let us introduce also the operators $U_k : \bigcup_{n=1}^{\infty} X_{(n)} \to X$ by:

$$p = \sum_{i=1}^{n} x_i \to U_k(p) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k} x_i & \text{for } 1 \le k \le n, \\ \sum_{i=1}^{n} x_i & \text{for } k > n, \end{cases}$$

where $x_i \in M_i$.

For any pair of linear subspaces $F, G \subset X$ the real number

$$(\widehat{F:G}) = \inf\{\|x - y\|,\ x \in F,\ \|x\| = 1,\ y \in G\} \tag{1}$$

is called [5] the *inclination* of F to G.

Using these concepts, we get some intrinsec characterizations of Schauder decompositions.

Proposition 2. *Let* $\{M_n\}$ *be a complete sequence of closed linear subspaces of* X. *Then the following statements are equivalent*:

1°. $\{M_n\}$ *is a Schauder decomposition of* X.

2°. *There exists a sequence of endomorphisms* $\{S_n\} \subset L(X, X)$, *satisfying*

$$S_n x = x, \text{ for any } x \in X_{(n)},$$

and

$$S_n x = 0, \text{ for any } x \in X^{(n)},$$

and such that

$$1 \le C_1 = \sup \|S_n\| < \infty;$$

in this case the sequence is uniquely determined and coincides with the sequence of the partial sum projections associated to the decomposition $\{M_n\}$.

3°. *There exists a constant* $1 \le C_2 < \infty$, *such that we have*

$$\left\|\sum_{i=1}^{n} x_i\right\| \le C_2 \left\|\sum_{i=1}^{n+m} x_i\right\|,$$

for all positive integers n, m *and any* $x_i \in M_i$.

4°. *We have*

$$C_3 = \inf_n (\widehat{X_{(n)}, X^{(n)}}) > 0.$$

5°. *There exists a constant* C_4, $1 \le C_4 < \infty$ *such that, for every* n *and every* $p_0 \in X_{(n)}$, *there exists an* $f \in X^*$ *(the dual space of* X*) satisfying* $1 \le \|f\| \le C_4$ *and*

$$\langle f, p_0 \rangle = \|p_0\|, \quad \langle f, y \rangle = 0, \text{ for any } y \in X^{(n)}.$$

6°. *We have*

$$C_5 = \sup_{n \ge 1} \ \sup_{p \in \bigcup_{n=1}^{\infty} X_{(n)},\ \|p\| = 1} \|U_n(p)\| < \infty.$$

Furthermore, for the above constants we have

$$1 \le C_1 = \inf C_2 = 1/C_3 = \inf C_4 = C_5 < \infty. \tag{2}$$

if and only if $\{M_n\}$ *is a Schauder decomposition of* X.

We omit the proof, since it follows the ideas of that given in [5], p. 57, for the special case of the Schauder bases.

The constant defined by (2) is called the norm of the sequence $\{M_n\}$. In analogy with the similar concept of the theory of bases, *the inverse of the norm*

$$\gamma = \inf_n (\widehat{X_{(n)}, X^{(n)}})$$

is called the index of $\{M_n\}$. For Schauder decomposition we have $0 < \gamma \le 1$ and the greatest value $\gamma = 1$ holds, if we get

$$\left\|\sum_{i=1}^{n} x_i\right\| \le \left\|\sum_{i=1}^{n+m} x_i\right\|$$

for any positive integers n, m and any $x_i \in M_i$. In this case we shall say that the decomposition $\{M_n\}$ is monotone.

Finally let us generalize the notion of strictly polynomial bases, which played a leading part in the theory of the conjugate directions methods [1].

Definition. *Let* $\{x_n\}$ *be a linearly independent sequence, complete in* H *and* $H_n = \mathrm{sp}\{x_1, x_2, \ldots, x_n\}$. *Then any decomposition of* X, *consisting in finite dimensional subspaces of the form*

$$\mathrm{sp}\{y_{j+1}, \ldots, y_{j+k}\},$$

where $y_i \in H_i$, *for every* i, *but* $y_i \notin H_{i-1}$, *for any* $i > 1$, *is called polynomial decomposition of* X *with respect to* $\{x_n\}$. It is quite obvious that polynomial one-dimensional decompositions are strictly polynomial Schauder bases of X. The theorem of existence from [1] can be extended adequately as follows:

Proposition 3. *In the Banach space X, let $\{x_n\}$ be linearly independent and complete in X. Then there exists a polynomial decomposition with respect to $\{x_n\}$, if and only if a sequence of endomorphisms $\{v_n\}$ can be defined and a subsequence $\{v_{n_k}\}$ can be chosen, such that, for every natural integers n, k:*

(a) $v_n(x) \in H_n$, *for any* $x \in X$;

(b) $v_{n_k}(h) = h$, *for any* $h \in H_{n_k}$;

(c) $v_{n_k}[v_n(x)] = v_{n_k}(x)$, *for every* $n_k \leq n$ and $x \in X$;

(d) *there exists* $M > 0$, *so that* $\|v_{n_k}\| \leq M$, $k = 1, 2, \ldots$

Proof. According to (b) we have $v_{n_k}(h) = h$ for any $h \in H_p$ and $n_k \geq p$. Since $\{x_n\}$ is complete in X and because of (d), we get $\lim\limits_{k \to \infty} v_{n_k}(h) = h$, for each $h \in X$.

Let us introduce the norm

$$|x| = \sup_k \|v_{n_k}(x)\|,$$

which is equivalent to the current norm of X because

$$\|x\| = \lim_{k \to \infty} \|v_{n_k}(x)\| \leq |x| \leq M\|x\|, \text{ for any } x \in X.$$

By using (c) it follows:

$$|v_{n_p}(x)| = \sup_k \|v_{n_k}(v_{n_p}(x))\| = \sup_{k \leq p} \|v_{n_k}(x)\| \leq \tag{3}$$

$$\leq \sup_{k \leq p+1} \|v_{n_k}(x)\| = |v_{n_{p+1}}(x)|.$$

Define $\{y_n\}$ by:

$$y_1 = x_1, \quad y_{n+1} = x_{n+1} - v_n(x_{n+1}), \quad n = 1, 2, \ldots$$

Because for $j \geq n_k$, $v_{n_k}(y_{j+1}) = v_{n_k}(x_{j+1}) - v_{n_k}(v_j(x_{j+1})) = 0$, and by using (3), we can write, for any sequence $\{\alpha_i\}$, $i = 1, 2, \ldots, n_{k+1}$, of real numbers:

$$\left|\sum_{i=1}^{n_k} \alpha_i y_i\right| = \left|v_{n_k}\left(\sum_{i=1}^{n_k} \alpha_i y_i\right)\right| = \left|v_{n_k}\left(\sum_{i=1}^{n_{k+1}} \alpha_i y_i\right)\right| \leq \tag{4}$$

$$\leq \left|v_{n_{k+1}}\left(\sum_{i=1}^{n_{k+1}} \alpha_i y_i\right)\right| = \left|\sum_{i=1}^{n_{k+1}} \alpha_i y_i\right|.$$

Denote by

$$M_k = \mathrm{sp}\,\{y_{n_{k-1}+1}, \ldots, y_{n_k}\}, \quad (n_0 = 0). \tag{5}$$

Because of the norm equivalence and according to Proposition 2 (3°), the relation (4) is necessary and sufficient for $\{M_n\}$ to be a Schauder decomposition.

Conversely, if there exists a polynomial decomposition $\{M_n\}$ of the form (5), it is quite easy to verify that the partial sum operators satisfy (a), (b), (c). The last condition is implied by Proposition 2 (2°).

For $n_k = k$, $k = 1, 2, \ldots$, we obtain the existence theorem for strictly polynomial bases ([5], p. 185).

2. Now consider a Hilbert space H. In fact, all results of this section could be easily adapted to Banach spaces, but for the particular methods of practical interest, a Hilbert space setting is quite adequate.

Let us consider in H the equation:

$$Au = f, \tag{6}$$

where A is a linear bounded inversable operator defined on H and $f \in R(A)$, the range of A.

Theorem 1. *If $\{M_n\}$ is a sequence of closed linear subspaces complete in H, and $\{E_n\}$ a sequence of endomorphisms from H to M_n, satisfying:*

$$E_i A E_j = \delta_{ij} E_j, \text{ for any natural } i, j, \tag{7}$$

then the iterations $\{u_n\}$ given by

$$u_{n+1} = u_n + E_{n+1} r_n, \quad r_n = f - Au_n, \quad n = 1, 2, \ldots \tag{8}$$

converge to solution u of the equation (6), for every $u_0 \in H$, if and only if $\{M_n, E_n A\}$ is a Schauder decomposition of H.

Proof. We have

$$u_n - u_0 = (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \ldots + (u_1 - u_0) =$$

$$= E_n r_{n-1} + E_{n-1} r_{n-2} + \ldots + E_1 r_0$$

and

$$r_n = r_0 - AE_n r_{n-1} - AE_{n-1} r_{n-2} - \ldots - AE_1 r_0,$$

where from, by using (7), we get

$$E_{n+1} r_n = E_{n+1} r_0, \quad n = 0, 1, \ldots$$

So

$$u_n = u_0 + \sum_{j=1}^{n} E_j r_0 = u_0 + \sum_{j=1}^{n} E_j A(u - u_0). \tag{9}$$

If $\{M_n, E_n A\}$ is a Schauder decomposition, then $\lim\limits_{n \to \infty} \sum_{j=1}^{n} E_j A(u - u_0) = u - u_0$ and (9) implies $\lim\limits_{n \to \infty} u_n = u$, for any $u_0 \in H$.

Conversely, if $\{u_n\}$ is convergent to u when $n \to \infty$, for any $u_0 \in H$, we get

$$u_n - u_0 = \sum_{j=1}^{n} E_j A(u - u_0) \to u - u_0, \text{ for } n \to \infty. \tag{10}$$

Then, Proposition 1 (ii) shows that the sequence $\{M_n\}$ is a Schauder decomposition with $E_n A$ as associated projections. The theorem is now completely proved.

By using Proposition 2, necessary and sufficient conditions of convergence for the iteration (8) can be obtained. Here it is one of these criterions.

Theorem 2. *Under the conditions of Theorem 1, let us denote*

$$H_{(n)} = \overline{\mathrm{sp}} \bigcup_{j=1}^{n} M_j, \quad H^{(n)} = \overline{\mathrm{sp}} \bigcup_{j=n+1}^{\infty} M_j .$$

and consider the factor-space $H/H^{(n)}$ related to the subspace $H^{(n)}$ with the usual norm. Consider also the natural surjection χ_n of H on $H/H^{(n)}$ ($h \in \chi_n(h)$, for any $h \in H$). Then for the convergence of the iterations u_n given by (8) to the solution u of (6) it is necessary and sufficient that, for any $u_0 \in H$ and any natural n,

$$\tau_n = \inf_{x \in H_{(n)}, \|x\|=1} \|\chi_n(x)\| \geqslant \tau > 0. \tag{11}$$

In this case, the following error estimation holds

$$\|u - u_n\| \leqslant \left(1 + \frac{1}{\tau}\right) \|s_0 - P_n s_0\|, \tag{12}$$

where $s_0 = u - u_0$ and P_n is the orthogonal projection on $H_{(n)}$.

Proof. According to Proposition 2 (4°), $\{M_n\}$ is a Schauder decomposition and therefore, $\lim\limits_{n \to \infty} u_n = u$, if and only if we have:

$$\tau = \inf_{n \geqslant 1} (\widehat{H_{(n)}, H^{(n)}}) > 0. \tag{13}$$

But

$$\tau_n = (\widehat{H_{(n)}, H^{(n)}}) = \inf_{x \in H_{(n)}, \|x\|=1} \; \inf_{y \in H^{(n)}} \|x - y\| = \inf_{x \in H_{(n)}, \|x\|=1} \|\chi_n(x)\| .$$

So (13) is equivalent to (11) and the first part of the theorem is proved.

Next, let us remark that the partial sum projections associated with the decomposition $\{M_n, E_n A\}$ have the form $S_n A$, where $S_n = \sum\limits_{j=1}^{n} E_j$. It is easy to verify that $S_n A x = x$, for any $x \in H_{(n)}$ and $S_n A x = 0$ for $x \in H^{(n)}$. To obtain (12) write first:

$$u - u_n = (u - u_0) - (u_n - u_0) = s_0 - S_n A s_0 .$$

Since the sequence $\{M_n\}$ is complete in H, for any $u_0 \in H$, there exists $y_n \in H_{(n)}$ so that $\lim\limits_{n \to \infty} \|s_0 - y_n\| = 0$, and, so much the more, $\lim\limits_{n \to \infty} \|s_0 - P_n s_0\| = 0$. Then

$$\|u - u_n\| = \|s_0 - S_n A s_0\| \leqslant \|s_0 - P_n s_0\| + \|S_n A s_0 - S_n A P_n s_0\| \leqslant (1 + \|S_n A\|) \|s_0 - P_n s_0\| .$$

But, by virtue of Proposition 2, the partial sum projections $S_n A$ satisfy $\|S_n A\| \leqslant 1/\tau$ and (12) follows.

Remark. The estimation (12) shows that the rate of convergence for the method (8) is related to the speed of convergence of $P_n s_0$ to s_0, so it depends on the particular choice for the sequence $\{M_n\}$ complete in H. Also (12) suggests that the convergence is the better as the index of $\{M_n\}$ is greater. Thus the best estimation holds for the monotone decomposition.

3. Certainly, from the practical point of view, the most interesting case is that of the finite dimensional decomposition methods.

Let us consider briefly the one-dimensional case: $M_n = \mathrm{sp}\{p_n\}$. In this situation we get a Schauder basis and the iterative method (8) is a direct extension of the conjugate directions method [1], to linear, not necessarily symmetrical, operators. In fact, if we denote by $\alpha_n(x)$ the coefficient-functionals associated to this basis, we have $E_n(x) = \alpha_n(x) p_n$ and (7) becomes:

$$\alpha_j(x) \alpha_i(A p_j) p_i = \delta_{ij} \alpha_j(x) p_j,$$

for any $x \in H$, which implies

$$\alpha_i(A p_j) = \delta_{ij} . \tag{14}$$

Then the iterative method (8) becomes

$$u_{n+1} = u_n + \alpha_{n+1}(r_n) p_n, \quad r_n = f - A u_n , \tag{15}$$

with α_n satisfying (14). Practically, as we work in Hilbert space the condition (14) involves the determination of a sequence $\{q_n\} \subset H$ such that

$$(q_i, A p_j) = \delta_{ij}. \tag{14'}$$

The simplest method of this class is the classical conjugate directions method, for which A is self-adjoint and $q_n = p_n$, $n = 1, 2, \ldots$

We shall consider now a general class of finite dimensional decomposition methods for linear self-adjoint operators A, by using polynomial decompositions.

Let $\{p_n\}$ be any linearly independent sequence of H and $N = \bigcup\limits_{k=1}^{\infty} J_k$ a partition of the set N of natural numbers, where J_k are finite and such that J_{k+1} contains only natural succesors of the components of J_k, $k = 1, 2, \ldots$ Introduce the notations: $M_k = \mathrm{sp}\{p_j, j \in J_k\}$ and $H_{(k)} = \mathrm{sp}' \bigcup\limits_{j=1}^{k} M_j$.

Definition. *If $A : H \to H$ is a linear self-adjoint operator, the subspaces M_k defined as before, are called A-conjugate subspaces, if $\{p_n\}$ and $\{J_k\}$ satisfy:*

(16) $\quad (p_i, Ap_j) = 0$, *for any* i, j *belonging to distinct* $J_k - s$:

and

(17) $\quad \det G_k \neq 0$, *where* G_k *is the matrix with elements* $g_{l,m} = (p_l, Ap_m)$ *for* l, m *belonging to same* J_k, $k = 1, 2, \ldots$

This is surely a natural extension of the notion of conjugate directions [1].

Theorem 3. *Let* $\{p_n\}$ *be linearly independent and complete in* H *and* $\{M_k\}$ *a sequence of* A*-conjugate subspaces related to* $\{p_n\}$. *Denote by* Q_k *the orthogonal projection on* $AH_{(k)}$. *Then* $\{M_k\}$ *is a polynomial decomposition of* H, *with respect to* $\{p_n\}$, *if and only if, there exists* τ *such that* :

$$\inf_{h_k \in H_{(k)}, \|h_k\| = 1} \|Q_k h_k\| = \tau_k \geqslant \tau > 0. \tag{18}$$

Proof. We shall use Proposition 3 by actually constructing endomorphisms $\{\tau_n\}$ which satisfy $(a) - (d)$.

Because of (17), there exists $G_k^{-1} = [\beta_{j,l}, j, l \in J_k]$. For each j natural j, let us consider the functional α_j given by :

$$\alpha_j(x) = \sum_{l \in J_k} \beta_{jl}(x, Ap_l), \quad \text{if } j \in J_k. \tag{19}$$

Define endomorphisms τ_n simply by :

$$\tau_n(x) = \sum_{j=1}^{n} \alpha_j(x) p_j. \tag{20}$$

(a) is obviously satisfied since $\tau_n(x) \in \mathrm{sp}\{p_1, \ldots, p_n\}$.

Consider the subsequence of (20) defined by

$$\tau^{(k)}(x) = \sum_{j \in \bigcup_{i=1}^{k} J_i} \alpha_j(x) p_j. \tag{21}$$

In order to verify (b), consider $h_k = \sum_{j \in \bigcup_{i=1}^{k} J_i} \gamma_j p_j \in H_{(k)}$ with arbitrary γ_j; then we have, for any natural j :

$$\alpha_j(h_k) = \sum_{l \in J_p} \sum_{i \in J_p} \beta_{jl} \gamma_i (p_i, Ap_l) = \sum_{i \in J_p} \delta_{ji} \gamma_i = \gamma_j,$$

if $j \in J_p$. From here

$$\tau^{(k)}(h_k) = \sum_{j \in \bigcup_{i=1}^{k} J_i} \alpha_j(h_k) p_j = \sum_{j \in \bigcup_{i=1}^{k} J_i} \gamma_j p_j = h_k,$$

that is (b). By similar calculation which we omit here by sake of brevity, the condition (c) is also verified. Finally for the last one, let $x \in H$, k arbitrary and $j \in J_p$, for $p = 1, 2, \ldots, k$. We have :

$$(x - \tau^{(k)}(x), Ap_j) = (x, Ap_j) - \sum_{i \in \bigcup_{s=1}^{k} J_s} \alpha_i(x)(p_i, Ap_j) = (x, Ap_j) -$$

$$- \sum_{i \in J_p} \sum_{l \in J_p} \beta_{il}(x, Ap_l)(p_i, Ap_j) = (x, Ap_j) - \sum_{l \in J_p} \delta_{jl}(x, Ap_l) = 0.$$

Therefore

$$Q_k(x - \tau^{(k)}(x)) = 0. \tag{22}$$

If we denote by $\tilde{Q}_k$ the restriction of Q_k to $H_{(k)}$, the assumption (18) implies the existence of $\tilde{Q}_k^{-1}$ and then, by virtue of (22) we have: $\tau^{(k)}(x) = \tilde{Q}_k^{-1} Q_k(x)$, $x \in H$. From here $\|\tau^{(k)}\| \leqslant \|\tilde{Q}_k^{-1}\| \leqslant 1/\tau_k \leqslant 1/\tau$ and (d) is satisfied.

Conversely, if $\{M_k\}$ is a polynomial decomposition of H with respect to $\{p_n\}$, then, for each $x \in H$, there exists $x_i \in M_i$, uniquely determined, such that $x = \sum_{i=1}^{\infty} x_i$. The partial sum of this series is uniquely expressed by $\sigma_k(x) = \sum_{i=1}^{k} x_i = \sum_{j \in \bigcup_{s=1}^{k} J_s} \alpha'_j(x) p_j$.

But it is quite simple to verify the identity $\alpha'_j(x) = \alpha_j(x)$ and so, by the same argument we obtain : $\sigma_k(x) = \tilde{Q}_k^{-1} Q_k(x)$.

Since $\lim_{k \to \infty} \sigma_k(x) = x$ for each $x \in H$, the Banach-Steinhaus theorem implies the existence of a constant $M > 0$ so that $\|\sigma_k\| \leqslant M$, $k = 1, 2, \ldots$

Finally we have

$$\sup_{x \in AH_{(k)}, \|x\| = 1} \|\tilde{Q}_k^{-1} x\| = \sup_{x \in AH_{(k)}, \|x\| = 1} \|\sigma_k(x)\| \leqslant M,$$

that is $\|\tilde{Q}_k^{-1}\| \leqslant M$, $k = 1, 2, \ldots$ which is equivalent to (18). This complete the proof of the theorem.

The Schauder decomposition which we have obtained leads to a class of iterative methods for solving (6). As in [1], for any pair of closed subspaces H_1, H_2 of H, if P_1, P_2 stand for the corresponding orthogonal projections, we denote by $\theta(H_1, H_2) = \|P_1 - P_2\|$ the opening between H_1, H_2 (see [3], p. 195).

Theorem 4. *Under the conditions of Theorem 3, the sequence* u_k *given by*

$$u_{k+1} = u_k + \sum_{j \in J_{k+1}} \alpha_j p_j, \quad \alpha_j = \sum_{l \in J_{k+1}} \beta_{jl}(r_k, p_l), \tag{23}$$

where $r_k = f - Au_k$, *is convergent to the solution* u *of (6) for any* $u \in H$, *if and only if*

$$\sup_k \theta\,(H_{(k)}, AH_{(k)}) < 1. \tag{24}$$

Proof. We shall prove that the method (23) belongs to the general class (8). To show that, let us define the endomorphisms E_k by:

$$E_k(x) = \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(x, p_l)\, p_j \in M_k .$$

It is standard to verify the condition (7) of Theorem 1.
Next, since A is self-adjoint,

$$\begin{aligned} E_k A x &= \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(Ax, p_l)\, p_j = \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(x, Ap_l)\, P_j) = \\ &= \sum_{j \in J_k} \alpha_j(x) p_j = v^{(k)}(x) - v^{(k-1)}(x),\ x \in H, \end{aligned} \tag{25}$$

here $v^{(k)}(x)$ are the partial sums (21).

Then, according to Theorem 1, the iterations (23) converge to u, if and only if, $\{M_k, E_k A\}$ is a Schauder decomposition of H. But, by virtue of Theorem 3 and of (25), for that it is necessary and sufficient to have (18), which is equivalent to (24) because $\gamma_n^2 + [\theta(H_{(n)}, AH_{(n)})]^2 = 1$ as it is proved in [3], p. 196.

The iterative method (23) is obviously an extension of the conjugate directions method, where a certain number of steps of the ordinary method (namely the dimension of M_k) are performed simultaneously at one iteration. Certainly for practical purposes, we have to dispose of sequences of A-conjugate subspaces and also, to make sure that the conditions (24) or (18) are satisfied. In [2] some procedures to get simple sequences of A-conjugate subspaces are described and a dissection about the class of operators satisfying the convergence conditions is done.

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Faculty of Mathematics
University of Iași
România

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CONSTRAINED CONVEX DISCRETE-TIME CONTROL PROBLEMS IN HILBERT SPACES

BY

V. POPESCU

Introduction. In [1], the following continuous control problem was investigated:

Minimize

$$\int_0^T L(x(t), u(t))\, dt + l(x_0, x_N), \tag{0.1}$$

subject to

$$x(t) = S(t, 0)\, x_0 + \int_0^t S(t, s)\, B(s)\, u(s)\, ds,\quad 0 \leq t \leq T, \tag{0.2}$$

$$x(t) \in K,\ t \in [0, T], \tag{0.3}$$

where K is a closed convex subset of H and L, l are lower semicontinuous convex functions, nonidentically $+\infty$, and $S(t, s) \in L(H, H)$ is an evolution operator, $B:]0, T[\to L(U, H)$ is strong measurable such as $B^*:]0, T[\to L(H, U)$. Moreover $B \in L^\infty(0, T; L(U, H))$.

The purpose of our paper is to study discrete-time control problem, i.e.

Minimize

$$\sum_{i=0}^{N-1} L_i(x_i, u_i) + l(x_0, x_N), \tag{0.4}$$

subject to

$$x_{i+1} = S_i x_i + D_i u_i,\ i \in \sigma_{N-1}, \tag{0.5}$$

$$x_i \in K_i,\ i \in \sigma_{N-1} \tag{0.6}$$

where σ_{N-1}, L_i, l, S_i, D_i *and* K_i *will be defined in* §1 *and* §2, which is an analogue of the above one.