$$\sup_k \theta\,(H_{(k)}, AH_{(k)}) < 1. \tag{24}$$

Proof. We shall prove that the method (23) belongs to the general class (8). To show that, let us define the endomorphisms E_k by :

$$E_k(x) = \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(x, p_l)\, p_j \in M_k .$$

It is standard to verify the condition (7) of Theorem 1.
Next, since A is self-adjoint,

$$\begin{aligned} E_k A x &= \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(Ax, p_l)\, p_j = \sum_{j \in J_k} \sum_{l \in J_k} \beta_{jl}(x, Ap_l)\, P_j) = \\ &= \sum_{j \in J_k} \alpha_j(x) p_j = v^{(k)}(x) - v^{(k-1)}(x),\ x \in H, \end{aligned} \tag{25}$$

here $v^{(k)}(x)$ are the partial sums (21).

Then, according to Theorem 1, the iterations (23) converge to u, if and only if, $\{M_k, E_k A\}$ is a Schauder decomposition of H. But, by virtue of Theorem 3 and of (25), for that it is necessary and sufficient to have (18), which is equivalent to (24) because $\varepsilon_k^2 + |\theta(H_{(k)}, AH_{(k)})|^2 = 1$ as it is proved in [3], p. 196.

The iterative method (23) is obviously an extension of the conjugate directions method, where a certain number of steps of the ordinary method (namely the dimension of M_k) are performed simultaneously at one iteration. Certainly for practical purposes, we have to dispose of sequences of A-conjugate subspaces and also, to make sure that the conditions (24) or (18) are satisfied. In [2] some procedures to get simple sequences of A-conjugate subspaces are described and a dissertation about the class of operators satisfying the convergence conditions is done.

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Faculty of Mathematics
University of Iași
România

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CONSTRAINED CONVEX DISCRETE-TIME CONTROL PROBLEMS IN HILBERT SPACES

BY

V. POPESCU

Introduction. In [1], the following continuous control problem was investigated :

Minimize

$$\int_0^T L(x(t), u(t))\,dt + l(x_0, x_N), \tag{0.1}$$

subject to

$$x(t) = S(t, 0)\,x_0 + \int_0^t S(t, s)\,B(s)\,u(s)\,ds,\quad 0 \leq t \leq T, \tag{0.2}$$

$$x(t) \in K,\ t \in [0, T]. \tag{0.3}$$

where K is a closed convex subset of H and L, l are lower semicontinuous convex functions, nonidentically $+\infty$, and $S(t, s) \in L(H, H)$ is an evolution operator, $B : [0, T] \to L(U, H)$ is strong measurable such as $B^* : [0, T] \to L(H, U)$. Moreover $B \in L^\infty(0, T; L(U, H))$.

The purpose of our paper is to study discrete-time control problem, i.e.

Minimize

$$\sum_{i=0}^{N-1} L_i(x_i, u_i) + l(x_0, x_N), \tag{0.4}$$

subject to

$$x_{i+1} = S_i x_i + D_i u_i,\ i \in \sigma_{N-1}, \tag{0.5}$$

$$x_i \in K_i,\ i \in \sigma_{N-1} \tag{0.6}$$

where σ_{N-1} ; L_i, l, S_i, D_i *and* K_i *will be defined in* §1 *and* §2, which is an analogue of the above one.

Indeed starting with (0.1), (0.2), (0.3) in an only formal manner, we can set (see e.g. [8], [10]) $N\delta = T$, $t_i = i\delta$, $i \in \sigma_N = \{0, 1, \ldots, N\}$ and $u(t) \equiv u(t_i) = u_i$, $t \in [t_i, t_{i+1})$ and denoting $x(t_i)$ by x_i, $i \in \sigma_N$, we obtain $x_{i+1} = S_i x_i + D_i u_i$, $i \in \sigma_{N-1}$, where $S_i = S(t_{i+1}, t_i)$, $i \in \sigma_{N-1}$ and $D(t, t_i) u_i = \int_{t_i}^{t} S(t, s) B(s) u_i \, ds$, $t \in [t_i, t_{i+1})$ and $D_i = D(t_{i+1}, t_i)$, $i \in \sigma_{N-1}$. On the other hand, the cost function is $F(x, u) = \int_0^T L(t, x(t), u(t))\, dt + l(x_0, x_N)$, which becomes $\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} L(t, x(t), u_i)\, dt + l(x_0, x_N)$. If we substitute $\int_{t_i}^{t_{i+1}} L(t, x(t), u_i)\, dt$ by $L_i(x_i, u_i)$, we obtain our discrete-time problem.

§ 1. Notations, preliminaries and plan of the paper. We denote by

1. H a real Hilbert space with $(.,.)$ and $|\,.\,|$ the inner product and the norm, respectively; H will be the states space;
2. U another real Hilbert space with $<.,.>$ and $\|\,.\,\|$ the inner product and the norm, respectively; U will be the space of controls;
3. $l^2(\sigma_N; H)$ and $l^2(\sigma_{N-1}; U)$ the two real Hilbert spaces considered in [7], [8];
4. $S_i \in L(H, H)$ for $i \in \sigma_{N-1}$ and $D_i \in L(U, H)$ for $i \in \sigma_{N-1}$;
5. $x_i \in H$, $i \in \sigma_N$, $u_i \in U$, $i \in \sigma_{N-1}$,
6. K_i, $i \in \sigma_{N-1}$ and K some closed convex subsets of H.

The problem

$$x_{i+1} = S_i x_i + D_i u_i, \; i \in \sigma_{N-1}; \; x_0 \in H \tag{1.1}$$

has a unique solution for every $x_0 \in H$ and $u \in l^2(\sigma_{N-1}; U)$ and the functions $u \to x$ and $x_0 \to x$ of $l^2(\sigma_{N-1}; U)$ into $l^2(\sigma_N; H)$ and of H into $l^2(\sigma_N; H)$, respectively, defined by the difference equation (1.1), are continuous (see [7]).

If K is a closed convex subset of H, we denote by I_K the indicator function of K, i.e.

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{otherwise.} \end{cases}$$

Given a proper lower semicontinuous convex function φ on H and a point $x \in H$, as usual, we denote by $\partial\varphi(x)$ the subdifferential of φ at x. Also, by φ^* we denote the conjugate of φ. We state below some properties of the function

$$\varphi_\lambda(x) = \inf\left\{\frac{|x-y|^2}{2\lambda} + \varphi(y) : y \in H\right\}, \quad x \in H, \; \lambda > 0. \tag{1.2}$$

The function φ_λ is convex and everywhere finite on H. Furthermore φ_λ is Fréchet differentiable on H and its Fréchet differential $\partial\varphi_\lambda$ is Lipschitz on H and coincides with the Yosida approximation $(\partial\varphi)_\lambda$ of $\partial\varphi$. The following relations hold:

$$\varphi_\lambda(x) = \frac{\lambda}{2} |\partial\varphi_\lambda(x)|^2 + \varphi((I + \lambda\partial\varphi)^{-1} x), \tag{1.3}$$

$$\lim_{\lambda\to 0} \varphi_\lambda(x) = \varphi(x), \tag{1.4}$$

(1.5) if $x_\lambda \in H$ and $\lim_{\lambda\to 0} x_\lambda = x$ weakly in H then $\liminf_{\lambda\to 0} \varphi_\lambda(x_\lambda) \geq \varphi(x)$.

Remark. If $\varphi = I_K$, then we have:

$$\partial I_K(x) = \{z \in H; (x - y, z) \geq 0 \text{ for every } y \in K\} \tag{1.6}$$

and

$$I_K^*(x) = \sup_{y\in K} (x, y), \quad x \in H. \tag{1.7}$$

Hence I_K^* is the support function of K.

The above results can be found in [4, Chap. 2].

The plan of the paper is as follows. In §2 and §3 we state and prove an optimality theorem. A dual control problem is introduced in §4 and studied in relation with the optimality conditions. Some remarks on synthesis of optimal controls are proved in §5.

§ 2. **Optimality theorem.** We consider the problem (0.4) – (0.6) denoted by (P).

Definition. *An attainable pair of* $\{L_i\}_{i=0}^{N-1}$ *is a couple* $(x_0, x_N) \in H \times H$ *for which there exist* $x_i \in K_i$, $i \in \sigma_{N-1}$ *and* $u_i \in U$, $i \in \sigma_{N-1}$ *such that*

$$x_{i+1} = S_i x_i + D_i u_i, \; i \in \sigma_{N-1} \text{ and } \sum_{i=0}^{N-1} L_i(x_i, u_i) < +\infty.$$

We denote by C_L the set of all attainable pairs of $\{L_i\}_{i=0}^{N-1}$.

We now list the hypotheses that will be in effect throughout this paper:

(A) For every $i \in \sigma_{N-1}$, L_i is a lower semicontinuous convex function from $H \times U$ to $]-\infty, +\infty]$ and nonidentically $+\infty$. The function l is a lower semicontinuous convex function on $H \times H$ with values in $]-\infty, +\infty]$ and nonidentically $+\infty$, $S_i \in L(H)$, $D_i \in L(U, H)$ and K_i is a closed convex subset of H, for every $i \in \sigma_{N-1}$.

(B) For every $i \in \sigma_{N-1}$ the Hamiltonian function $H_i : H \times U \to]-\infty, +\infty]$ defined by

$$H_i(x, u) = \sup\{<u, v> - L_i(x, v);\; v \in U\} \tag{2.1}$$

is such that

$$D_1(H_i) = H \text{ and } 0 \in \operatorname{int} D_2(H_i) \tag{2.2}$$

where

$$D_1(H_i) = \{x \in H;\ H_i(x, u) > -\infty \text{ for every } u \in U\} \tag{2.3}$$

and

$$D_2(H_i) = \{u \in U;\ H_i(x, u) < +\infty \text{ for every } x \in H\}. \tag{2.4}$$

(C) There exist $x \in l^2(\sigma_N; H)$ and $u \in l^2(\sigma_{N-1}; U)$ such that:

$$(x_0, x_N) \in C_L \cap D(l) \tag{2.5}$$

and

$$x_i \in \operatorname{int} K_i,\ i \in \sigma_{N-1} \tag{2.6.}$$

where $D(l)$ is the effective domain of l, i.e.

$$D(l) = \{(\alpha, \beta) \in H \times H;\ l(\alpha, \beta) < +\infty\}.$$

(D) There exists a pair $(\alpha, \beta) \in C_L \cap D(l)$ such that one of the following two conditions holds:

$$\beta \in \operatorname{int}\{h \in H;\ (\alpha, h) \in C_L\} \tag{2.7}$$

$$\beta \in \operatorname{int}\{h \in H;\ (\alpha, h) \in D(l)\}. \tag{2.8}$$

Theorem 2.1. *Let hypotheses (A), (B), (C) and (D) hold. Then the pair $(x, u) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; U)$ is a solution of (P) if and only if there exist $\mu_i \in \partial I_{K_i}(x_i)$, $i \in \sigma_{N-1}$; $p_i \in H$, $i \in \sigma_N$ and $q_i \in H$, $i \in \sigma_{N-1}$ such that:*

$$x_{i+1} = S_i x_i + D_i u_i,\ i \in \sigma_{N-1}. \tag{2.9}$$

$$x_i \in K_i,\ i \in \sigma_{N-1}, \tag{2.10}$$

$$(q_i, D_i^* p_{i+1}) \in \partial L_i(x_i, u_i),\ i \in \sigma_{N-1}, \tag{2.11}$$

$$p_i = S_i^* p_{i+1} - q_i - \mu_i,\ i \in \sigma_{N-1}, \tag{2.12}$$

$$(p_0, -p_N) \in \partial l(x_0, x_N) \text{ (transversality).} \tag{2.13}$$

Remarks: 1) As usual, we denoted above by D_i^* (and S_i^*) the adjoint of D_i (and S_i, respectively).

2) p_i is called the adjoint state at time i.

§ 3. Proof of Optimality theorem. *Sufficiency.* Let x, u, p, q and μ be the elements for which (2.9) ~ (2.13) hold. Then the following relations obviously hold:

$$L_i(x_i, u_i) - L_i(y_i, v_i) \leqslant (q_i, x_i - y_i) + \langle D_i^* p_{i+1}, u_i - v_i\rangle,\ i \in \sigma_{N-1} \tag{3.1}$$

where $y_i \in H$ and $v_i \in U$;

$$l(x_0, x_N) - l(y_0, y_N) \leqslant (p_0, x_0 - y_0) - (p_N, x_N - y_N), \tag{3.2}$$

for every $y_0, y_N \in H$;

$$(\mu_i, x_i - y_i) \geqslant 0,\ i \in \sigma_{N-1} \tag{3.3}$$

for each $y_i \in K_i$.

It follows from (3.1) ~ (3.3) and (2.12) that

$$\sum_{i=0}^{N-1} L_i(x_i, u_i) + l(x_0, x_N) \leqslant \sum_{i=0}^{N-1} L_i(y_i, v_i) + l(y_0, y_N).$$

for every $(y, v) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; U)$ such that (2.9) and (2.10) hold and this completes the proof of the sufficiency.

Necessity. Let $(x, u) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; U)$ be an optimal pair of (P). For every $\lambda > 0$, we consider:

$$\begin{aligned} F_\lambda(h, v) = {} & \sum_{i=0}^{N-1} (L_i)_\lambda(y_i, v_i) + \frac{1}{2}\sum_{i=0}^{N-1} \|u_i - v_i\|^2 + \sum_{i=0}^{N-1} (I_{K_i})_\lambda(y_i) + \\ & + l_\lambda(y_0, y_N) + \frac{1}{2}\|y_0 - x_0\|^2, \end{aligned} \tag{3.4}$$

where

$$y_0 = h,\ y_{i+1} = S_i y_i + D_i v_i,\ i \in \sigma_{N-1}, \tag{3.5}$$

and $(L_i)_\lambda$, $(I_{K_i})_\lambda$, l_λ are defined by (1.7), respectively. $h \in H$ and $v \in l^2(\sigma_{N-1}; U)$ are arbitrary elements.

In our hypotheses, $F_\lambda : H \times l^2(\sigma_{N-1}; U) \to]-\infty, +\infty[$ is a strict convex coercive everywhere finite and (strong) lower semicontinuous function. Hence there exists a unique pair $(h_\lambda, u_\lambda) \in H \times l^2(\sigma_{N-1}; U)$ such that

$$F_\lambda(h_\lambda, u_\lambda) \leqslant F_\lambda(h, v),\ \lambda > 0 \tag{3.6.}$$

for every $(h, v) \in H \times l^2(\sigma_{N-1}; U)$ (see e.g. [9]).

We denote by $x_\lambda \in l^2(\sigma_N; H)$ the solution of the problem

$$x_\lambda^0 = h_\lambda,\ x_\lambda^{i+1} = S_i x_\lambda^i + D_i u_\lambda^i,\ i \in \sigma_{N-1}, \tag{3.7}$$

and we set

$$q_\lambda^i = \partial_x (L_i)_\lambda (x_\lambda^i, u_\lambda^i),\ i \in \sigma_{N-1}, \tag{3.8}$$

$$(\alpha_\lambda, \beta_\lambda) = \partial l_\lambda(x_\lambda^0, x_\lambda^N). \tag{3.9}$$

Now we define $p_\lambda \in l^2(\sigma_N; H)$ in

$$p_\lambda^N = -\beta_\lambda,\ p_\lambda^i = S_i^* p_\lambda^{i+1} - q_\lambda^i - \partial(I_{K_i})_\lambda(x_\lambda^i),\ i \in \sigma_{N-1}. \tag{3.10}$$

We observe that these elements verify the followings:

Lemma 3.1. *For every $\lambda > 0$ there exist $x_\lambda \in l^2(\sigma_N; H)$, $u_\lambda \in l^2(\sigma_{N-1}; U)$ and $p_\lambda \in l^2(\sigma_N; H)$ such that:*

$$x_\lambda^{i+1} = S_i x_\lambda^i + D_i u_\lambda^i,\ i \in \sigma_{N-1}. \tag{3.11}$$

$$p_\lambda^i = S_i^* p_\lambda^{i+1} - q_\lambda^i - \partial (I_{K_i})_\lambda (x_\lambda^i),\ i \in \sigma_{N-1}, \tag{3.12}$$

$$q_\lambda^i = \partial_x (L_i)_\lambda (x_\lambda^i, u_\lambda^i),\ i \in \sigma_{N-1}, \tag{3.13}$$

$$D_i^* p_\lambda^{i+1} = u_\lambda^i - u_i + \partial_u (L_i)_\lambda (x_\lambda^i, u_\lambda^i),\ i \in \sigma_{N-1}, \tag{3.14}$$

$$(p_\lambda^0 - x_\lambda^0 + x_0, -p_\lambda^N) = \partial l_\lambda (x_\lambda^0, x_\lambda^N). \tag{3.15}$$

Furthermore

$$\lim_{\lambda \to 0} x_\lambda^i = x_i \qquad \textit{strongly in } H, \tag{3.16}$$

$$\lim_{\lambda \to 0} u_\lambda^i = u_i \qquad \textit{strongly in } U, \tag{3.17}$$

for every $i \in \sigma_N$ and $i \in \sigma_{N-1}$, respectively.

Proof of Lemma 3.1. From (3.6) $\sim$ (3.10), it is easy to verify (3.14) and (3.15). We note that

$$F_\lambda (h_\lambda, u_\lambda) \leqslant F_\lambda (x_0, u) \leqslant F(x_0, u) < +\infty, \tag{3.18}$$

(see (1.2)), where $F(x_0, u) = \sum_{i=0}^{N-1} L_i (x_i, u_i) + l(x_0, x_N)$.

On the other hand there exist $a \in l^2(\sigma_{N-1} : H)$, $b \in l^2(\sigma_{N-1} : U)$ and $\alpha, \beta \in H$ such that :

$$(L_i)_\lambda (x_\lambda^i, u_\lambda^i) \geqslant (a_i, x_\lambda^i) + \langle b_i, u_\lambda^i \rangle,\ i \in \sigma_{N-1}, \text{ for every } \lambda > 0, \tag{3.19}$$

and

$$l_\lambda (x_\lambda^0, x_\lambda^N) \geqslant (\alpha, x_\lambda^0) + (\beta, x_\lambda^N) \quad \text{for every} \quad \lambda > 0. \tag{3.20}$$

By (3.19), (3.20) and the properties of (1.1) mentioned in § 1, we deduce that :

$$F_\lambda (h_\lambda, u_\lambda) \geqslant C_1 + C_2 |x_\lambda^0 - x_0|^2 + C_3 \| u_\lambda - u \|^2_{l^2(\sigma_{N-1}; U)} \tag{3.21}$$

where C_1, C_2, C_3 are some positive constants.

From (3.18) and (3.21) we may infer that :

$$\lim_{\lambda \to 0} u_\lambda^i = \tilde{u}_i \ \textit{weakly in } U, \textit{ for every } i \in \sigma_{N-1}, \tag{3.22}$$

$$\lim_{\lambda \to 0} x_\lambda^0 = \tilde{x}_0 \ \textit{weakly in } H. \tag{3.23}$$

By (3.22), (3.23) and (3.7) it follows

$$\lim_{\lambda \to 0} x_\lambda^i = \tilde{x}_i \text{ weakly in } H \text{ for each } i \in \sigma_N, \tag{3.24}$$

where

$$\tilde{x}_{i+1} = S_i \tilde{x}_i + D_i \tilde{u}_i,\ i \in \sigma_{N-1}. \tag{3.25}$$

From (3.18), (3.22) $\sim$ (3.24) and (1.7), (1.10) we deduce that (3.16) and (3.17) hold.

We continue the proof of necessity by :

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$| p_\lambda^N | \leqslant C \textit{ for every } \lambda > 0. \tag{3.26}$$

Proof of Lemma 3.2. For every $v \in l^2(\sigma_{N-1}; U)$, we define

$$G(v) = \sum_{i=0}^{N-1} L_i (y_i, v_i) + \frac{1}{2} \| v - u \|^2_{l^2(\sigma_{N-1}; U)} + \frac{1}{2} \sum_{i=0}^{N-1} I_{K_i}(y_i), \tag{3.27}$$

where

$$y_{i+1} = S_i y_i + D_i v_i,\ i \in \sigma_{N-1} \tag{3.28}$$

and $y_0 \in H$ is fixed.

We denote

$$\Phi (h_1, h_2) = \inf \{ G(v) ;\ v \in l^2 (\sigma_{N-1} ; U),\ y_0 = h_1,\ y_N = h_2 \} \tag{3.29}$$

for every $(h_1, h_2) \in H \times H$.

Obviously, the set $\{ v \in l^2 (\sigma_{N-1} ; U) ;\ y_0 = h_1,\ y_{i+1} = S_i y_i + D_i v_i,\ y_i \in K_i,\ y_N = h_2 \}$ is closed and convex in $l^2 (\sigma_{N-1} ; U)$.

It should be observed that $C_L \subset D(\Phi)$ and that the infimum defining $\Phi (h_1, h_2)$ is attained for every $(h_1, h_2) \in C_L$. The function Φ is convex lower semicontinuous and nowhere $-\infty$ on $H \times H$.

We suppose that condition (2.7) holds. It follows that there are $y \in l^2(\sigma_N ; H)$ and $\rho > 0$ such that

$$l(y_0, y_N) < +\infty \tag{3.30}$$

and

$$\Phi (y_0, y_N + \rho h) \leqslant C \text{ for every } h \in H,\ |h| = 1. \tag{3.31}$$

(since a proper convex and lower semicontinuous function is continuous at every interior point of its effective domain).

If

$$\begin{cases} z_0 = y_0, \\ z_{i+1} = S_i z_i + D_i v_i,\ i \in \sigma_{N-1},\ z_N = y_N + \rho h, \end{cases} \tag{3.32}$$

then

$$\Phi (z_0, z_N) \leqslant C. \tag{3.31'}$$

On the other hand, from (3.8), (3.9), (3.14) and (3.15) we find that

$$\begin{aligned} F_\lambda (x_\lambda^0, u_\lambda) \leqslant & \sum_{i=0}^{N-1} L_i (z_i, v_i) + \frac{1}{2} \sum_{i=0}^{N-1} \| u_i - v_i \|^2 + l(y_0, y_N) - \\ & - (p_\lambda^N, \rho h) - (x_\lambda^0 - x_0, x_\lambda^0 - y_0). \end{aligned} \tag{3.33}$$

From (3.30), (3.31)' and (3.33) we deduce

(3.34) $$F_\lambda (x^0_\lambda, u_\lambda) \leqslant C - \rho (p^N_\lambda, h) \text{ for every } h \in H, \ |h| = 1.$$

Lemma 3.1, (3.34) and (3.21) give (3.26).

Now we assume that condition (2.8) holds.

By this it follows that there is $\rho > 0$ such that

(3.35) $$l (y_0, y_N + \rho h) \leqslant C \text{ for every } h \in H, \ |h| = 1.$$

and

(3.36) $$\Phi (y_0, y_N) < +\infty.$$

By (3.9), (3.15) and Lemma 3.1 we find that

(3.37) $$\rho (p^N_\lambda, h) \leqslant C - l_\lambda (x^0_\lambda, x^N_\lambda) + (p^0_\lambda, x^0_\lambda - y_0) - (p^N_\lambda, x^N_\lambda - y_N).$$

From (3.8), (3.14) and (3.37) we deduce

(3.38) $$F_\lambda (x^0_\lambda, u_\lambda) + \rho (p^N_\lambda, h) \leqslant C.$$

(3.38) and (3.21) give again (3.26).

We show now that the following Lemma holds :

Lemma 3.3. *There exists $C > 0$ such that*

(3.39) $$|p^i_\lambda| \leqslant C, \ i \in \sigma_N, \text{ for every } \lambda > 0 \text{ and sufficiently small}$$

(3.40) $$|\partial (I_{K_i})_\lambda (x^i_\lambda)| \leqslant C, \text{ for every } \lambda > 0 \text{ and } i \in \sigma_{N-1}.$$

Furthermore,

(3.41) $$|q^i_\lambda| \leqslant C, \ i \in \sigma_{N-1}, \text{ for every } \lambda > 0 \text{ and sufficiently small.}$$

Proof of Lemma 3.3. By hypothesis (C) there are $(y, v) \in l^2 (\sigma_N ; H) \times l^2(\sigma_{N-1} ; U)$ such that

(3.42) $$\begin{cases} y_{i+1} = S_i y_i + D_i v_i, \ i \in \sigma_{N-1}, \ y_i \in \operatorname{int} K_i, \ i \in \sigma_{N-1}, \\ (y_0, y_N) \in C_L, \ l (y_0, y_N) < +\infty \end{cases}$$

Therefore, there exists $\rho > 0$ such that

(3.42)' $$y_{i+1} = S_i y_i + D_i v_i, \ i \in \sigma_{N-1}, \quad y_i + \rho h \in K_i, \ i \in \sigma_{N-1},$$
$$\text{for every } h \in H, \ |h| = 1, \quad (y_0, y_N) \in C_L, \ l (y_0, y_N) < +\infty.$$

Since

$$(I_{K_i})_\lambda (x^i_\lambda) - (I_{K_i})_\lambda (y_i + \rho h) \leqslant (\partial (I_{K_i})_\lambda (x^i_\lambda), x^i_\lambda - y_i - \rho h),$$

it follows

(3.43) $$|\partial (I_{K_i})_\lambda (x^i_\lambda)| \leqslant \frac{1}{\rho} (\partial (I_{K_i})_\lambda (x^i_\lambda), x^i_\lambda - y_i), \ i \in \sigma_{N-1}.$$

On the other hand, from (2.12), (3.11), (3.13), (3.15) and (3.42) we obtain that :

(3.44) $$\sum_{i=0}^{N-1} (\partial (I_{K_i})_\lambda (x^i_\lambda), x^i_\lambda - y_i) + \sum_{i=0}^{N-1} (L_i)_\lambda (x^i_\lambda, u^i_\lambda) + l_\lambda (x^0_\lambda, x^N_\lambda)$$
$$- \sum_{i=0}^{N-1} (L_i)_\lambda (y_i, v_i) - l_\lambda (y_0, y_N) \leqslant - (x^0_\lambda - x_0, x^0_\lambda - y_0) - \sum_{i=0}^{N-1} <$$
$$< u^i_\lambda - u_i, u^i_\lambda - v_i > .$$

From (3.43) and (3.44) we find that

(3.45) $$\rho \sum_{i=0}^{N-1} |\partial (I_{K_i})_\lambda (x^i_\lambda)| + F_\lambda (x^0_\lambda, u_\lambda) \leqslant C + \sum_{i=0}^{N-1} L_i (y_i, v_i) + l (y_0, y_N).$$

By (3.45) and (3.18), (3.21) we may infer that (3.40) holds. Now we use the hypothesis (B). Since $H_i (x, p)$ and $\partial H_i (x, p)$ are locally bounded at every point $(x, p) \in \operatorname{int} \operatorname{Dom} H_i$, there exists $\rho > 0$ such that

(3.46) $$|H_i (x_i + \rho h, 0)| \leqslant C, \ i \in \sigma_{N-1}, \text{ for every } h \in H, \ |h| = 1$$

(3.47) $$\sup \{ \|z\|, z \in \partial_p H_i (x_i + \rho h, 0) \} \leqslant C, \ i \in \sigma_{N-1}, \text{ for every } h \in H, \ |h| = 1.$$

Let $v_i \in \partial_p H_i (x_i + \rho h, 0)$. It follows then

$$L_i (x_i + \rho h, v_i) = -H_i (x_i + \rho h, 0),$$

hence

(3.48) $$\begin{cases} L_i (x_i + \rho h, v_i) \leqslant C, \text{ for every } h \in H, \ |h| = 1 \text{ and } i \in \sigma_{N-1}, \\ \|v_i\| \leqslant C, \ i \in \sigma_{N-1}. \end{cases}$$

From (3.13) and (3.14) we find that

(3.49) $$|q^i_\lambda| \leqslant C_1 + C_2 |p^{i+1}_\lambda|, \ i \in \sigma_{N-1}.$$

From (3.12), (3.40) and (3.48) we deduce that $|p^i_\lambda| \leqslant C$, $i \in \sigma_N$, for $\lambda > 0$ and sufficiently small.

The proof of Lemma 3.3 is complete.

These imply by a standard argument that :

(3.50) $$\lim_{\lambda \to 0} p^i_\lambda = p_i, \ (i \in \sigma_N), \text{ weakly in } H,$$

(3.51) $$\lim_{\lambda \to 0} q^i_\lambda = q^i, \ (i \in \sigma_{N-1}), \text{ weakly in } H,$$

(3.52) $$\lim_{\lambda \to 0} \partial (I_{K_i})_\lambda (x^i_\lambda) = \mu_i, \ (i \in \sigma_{N-1}), \text{ weakly in } H,$$

where

(3.53) $$(p_0, -p_N) \in \partial l (x_0, x_N),$$

(3.54) $$(q_i, D_i^* p_{i+1}) \in \partial L_i (x_i, u_i), \ i \in \sigma_{N-1},$$

(3.55) $$\mu_i \in \partial I_{K_i} (x_i), \ i \in \sigma_{N-1}.$$

This completes the proof of the necessity.

Remarks. 1) Some details of the proof of Theorem 2.1 are lefted for the reader (see and [1], [2], or [5]).

2) The optimality theorem can be formulated in an equivalent form as follows:

Let the hypotheses (A), (B), (C) and (D) hold. Then the pair $(x, u) \in l^2(\sigma_N : H) \times l^2(\sigma_{N-1} : U)$ is optimal for (P) iff there exist $\mu_i \in \partial I_{K_i}(x_i)$ $i \in \sigma_{N-1}$ and $p \in l^2(\sigma_N : H)$ such that

$$(3.56)\quad \begin{cases} x_{i+1} = S_i x_i + D_i u_i, \ i \in \sigma_{N-1}, \ x_i \in K_i, \ i \in \sigma_{N-1}, \\ (p_i - S_i^* p_{i+1} + \mu_i, u_i) \in \partial H_i(x_i, D_i^* p_{i+1}), \ i \in \sigma_{N-1}, \ (p_0, -p_N) \in \partial l(x_0, x_N) \end{cases}$$

(We called (3.56) the Hamiltonian form of the optimality theorem).

§ 4. Duality theorem. For every $(h_1, h_2) \in H \times H$ we define

$$(4.1)\quad m(h_1, h_2) = l^*(h_1, -h_2) = \sup\{(h_1, x) - (h_2, y) - l(x, y);\ (x, y) \in H \times H\}.$$

$$(4.2)\quad \mathcal{A}_{K_i}(h_1) = I_{K_i}^*(h_1) = \sup_{x \in K_i}(x, h_1), \ i \in \sigma_{N-1}.$$

For each $(u, p) \in U \times H$ we define

$$(4.3)\quad M_i(u, p) = L_i^*(p, u) = \sup\{(p, x) + \langle u, v \rangle - L_i(x, v);\ (x, v) \in H \times U\}, \ i \in \sigma_{N-1}.$$

We consider the problem:

Minimize

$$(4.4)\quad \sum_{i=0}^{N-1} M_i(D_i^* p_{i+1}, q_i) + \sum_{i=0}^{N-1} \mathcal{A}_{K_i}(\mu_i) + m(p_0, p_N)$$

subject to

$$(4.5)\quad p_i = S_i^* p_{i+1} - q_i - \mu_i, \ i \in \sigma_{N-1}.$$

This problem is called the dual of problem (P) and it will be denoted by (P^*).

Theorem 4.1. *Let Hypotheses (A), (B), (C) and (D) hold. Then the pair* $(x, u) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; U)$ *is optimal for* (P) *if and only if the problem* (P^*) *has a solution* $(p, q, \mu) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; H) \times l^2(\sigma_{N-1}; H)$ *and*

$$(4.6)\quad \min P + \min P^* = 0.$$

Furthermore, the optimal pair (x, u) for (P) and the optimal triplet (p, q, μ) for (P^*) verify „the conditions of extremality" (2.9) ~ (2.13).

Proof. From (4.1) we deduce that

$$(4.7)\quad l(y_0, y_N) + m(\tilde{p}_0, \tilde{p}_N) \geq (y_0, \tilde{p}_0) - (y_N, \tilde{p}_N),$$

for every $(y_0, y_N) \in H \times H$ and $(\tilde{p}_0, \tilde{p}_N) \in H \times H$.

The following two conditions are equivalent:

$$(4.8)\quad (p_0, -p_N) \in \partial l(x_0, x_N).$$

$$(4.9)\quad l(x_0, x_N) + m(p_0, p_N) = (x_0, p_0) - (x_N, p_N).$$

Similarly, from (4.2) we may infer that

$$(4.10)\quad I_{K_i}(y_i) + \mathcal{A}_{K_i}(\tilde{\mu}_i) \geq (y_i, \tilde{\mu}_i), \ i \in \sigma_{N-1},$$

for every $y \in l^2(\sigma_{N-1} : H)$, $\tilde{\mu} \in l^2(\sigma_{N-1}, H)$, and the following two conditions are equivalent:

$$(4.11)\quad \mu_i \in \partial I_{K_i}(x_i), \ i \in \sigma_{N-1},$$

$$(4.12)\quad I_{K_i}(x_i) + \mathcal{A}_{K_i}(\mu_i) = (x_i, \mu_i), \ i \in \sigma_{N-1}.$$

From (4.3) we find that

$$(4.13)\quad L_i(y_i, v_i) + M_i(D_i^* \tilde{p}_{i+1}, \tilde{q}_i) \geq (y_i, \tilde{q}_i) + \langle v_i, D_i^* \tilde{p}_{i+1} \rangle, \ i \in \sigma_{N-1}$$

for every $y \in l^2(\sigma_N : H)$, $v \in l^2(\sigma_{N-1} : U)$, $\tilde{p} \in l^2(1, N; H)$ and $\tilde{q} \in l^2(\sigma_{N-1} : H)$.

The following two conditions are equivalent

$$(4.14)\quad (q_i, D_i^* p_{i+1}) \in \partial L_i(x_i, u_i), \ i \in \sigma_{N-1};$$

$$(4.15)\quad L_i(x_i, u_i) + M_i(D_i^* p_{i+1}, q_i) = (x_i, q_i) + \langle u_i, D_i^* p_{i+1} \rangle,$$

$i \in \sigma_{N-1}$.

From (4.7) ~ (4.15) it is very easy to verify the theorem.

§ 5. Some remarks on synthesis of optimal controls. In this paragraph, we consider the problem

Minimize

$$(5.1)\quad \sum_{i=0}^{N-1} L_i(x_i, u_i),$$

subject to

$$(5.2)\quad \begin{cases} x_{i+1} = S_i x_i + D_i u_i, \ i \in \sigma_{N-1}, \\ x_0 = \hat{x}_0 \in H, \end{cases}$$

where $\hat{x}_0$ *is fixed in* H.

We note that this is a problem of type (P) with $K_i = H$, $i \in \sigma_{N-1}$ and

$$(5.3)\quad l(x_0, x_N) = \begin{cases} 0 \ \text{if} \ x_0 = \hat{x}_0, \\ +\infty, \ \text{otherwise}. \end{cases}$$

We assume that (A), (B) and (α) hold, where (α) means $C_h \neq \emptyset$. Then the hypotheses, (A), (B) (C) and (D) hold.

Therefore the optimality theorem and the duality theorem hold also.

For each $i \in \sigma_{N-1}$ and $h \in H$ we define :

$$J_i(h) = \inf\left\{ \sum_{j=i}^{N-1} L_j(x_j, u_j);\ x_i = h,\ x_{j+1} = S_j x_j + D_j u_j,\ j = i, \right. \tag{5.4}$$
$$\left. i+1, \ldots, N-1 \right\}.$$

We set

$$l_i(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha = h, \\ +\infty, & \text{otherwise} \end{cases} \tag{5.5}$$

where $(\alpha, \beta) \in H \times H$.

Hence (5.4) is a problem (P) for which (A), (B) and (α) hold. We denote this problem by $(P)_i$.

From the optimality theorem we have the following result :

Let hypotheses (A), (B) and (α) hold. Then (x^i, u^i) is an optimal pair for $(P)_i$ iff there exist (p^i, q^i) such that.

$$x^i_{j+1} = S_j x^i_j + D_j u^i_j, \quad j = i, i+1, \ldots, N-1, \tag{5.6}$$

$$x^i_i = h, \tag{5.7}$$

$$(q^i_j, D^*_j p^i_{j+1}) \in \partial L_j(x^i_j, u^i_j), \quad j = i, i+1, \ldots, N-1, \tag{5.8}$$

$$p^i_j = S^*_j p^i_{j+1} - q^i_j, \quad j = i, i+1, \ldots, N-1, \tag{5.9}$$

$$p^i_N = 0. \tag{5.10}$$

It is easy to verify that the dual problem for $(P)_i$ is :

Minimize

$$\sum_{j=1}^{N-1} M_j(D^*_j p_{j+1}, q_j) + (p_i, h) \tag{5.11}$$

subject to

$$p_j = S^*_j p_{j+1} - q_j,\ j = i, i+1, \ldots, N-1,\ p_N = 0. \tag{5.12}$$

We denote this problem by $(P^*)_i$.

The duality theorem becomes :

Suppose that (A), (B) and (α) hold. Then the pair (x^i, u^i) is optimal for $(P)_i$ iff the problem $(P^*)_i$ has a solution (p^i, q^i) and

$$\min (P)_i + \min (P^*)_i = 0. \tag{5.13}$$

Furthermore, the optimal pair (x^i, u^i) for $(P)_i$ and the optimal pair (p^i, q^i) for $(P^*)_i$ verify the conditions of extremality (5.6) ~ (5.10).

Proposition 5.1. *Let hypotheses (A), (B) and (α) hold. Then for every $i \in \sigma_{N-1}$, $J_i : H \to]-\infty, +\infty]$ are convex lower semicontinuous functions and nonidentically $+\infty$.*

Proof. By using (α) we obtain that $D(J_0) \neq \emptyset$ such that for each $i \in \sigma_{N-1}$, $D(J_i) \neq \emptyset$. Here, $D(J_i)$ is, as usual,

$$D(J_i) = \{x \in H;\ J_i(x) < +\infty\}.$$

But, from duality theorem we have :

$$J_i(h) = \sup\left\{ (h, -p_i) - \sum_{j=1}^{N-1} M_j(D^*_j p_{j+1}, q_j);\ p_N = 0,\ p_j = S^*_j p_{j+1} - q_j \right\}.$$

Hence we may infer that $\{J_i\}_{i \in \sigma_{N-1}}$ are convex lower semicontinuous functions and nowhere $-\infty$.

Proposition 5.2. *Let $h \in D(J_i)$ be such that the infimum defining $J_i(h)$ is attained. Then $h \in D(\partial J_i)$.*

Proof. Let us suppose that $h \in D(J_i)$ is such that infimum defining $J_i(h)$ is attained. Then by Optimality theorem there exist (x^i, u^i) and (p^i, q^i) such that (5.6) ~ (5.10) hold.

Let be $\bar{h} \in H$ and $\{v_j\}_{j=i}^{N-1} \subset U$.

We define

$$y_i = \bar{h},\ y_{j+1} = S_j y_j + D_j v_j,\ j = i, i+1, \ldots, N-1.$$

From (5.6) ~ (5.10) we find that

$$J_i(h) - J_i(\bar{h}) \leq -(p^i_i, h - \bar{h})$$

Hence the proposition holds, and furthermore

$$-p^i_i \in \partial J_i(h). \tag{5.14}$$

A sufficient condition for which the hypothesis of Proposition 5.2. holds is the following one :

(A) holds

and

(β) $H_i(x, p)$ is finite for each $(x,p) \in H \times U$ and there exists $C > 0$ such that

$$H_i(0, p) \leq C(1 + \|p\|^2) \text{ for every } p \in U \text{ and for all } i \in \sigma_{N-1}. \tag{5.15}$$

For every $r > 0$ there exists $M_r > 0$ such that

$$\sup\{|y|;\ y \in \partial_x H_i(x, p)\} \leq M_r(1 + \|p\|) \tag{5.16}$$

for all $p \in U$ and $x \in H$, $|x| \leq r$ (see [3] or [5]).

Proposition 5.3. *We assume that (A) and (β) hold. Then for all $h \in D(J_i)$ infimum defining $J_i(h)$ is attained.*

Proof. It sufficies to show that the sets

$$M_\lambda = \Big\{ \{u_j\}_{j=i}^{N-1} : \sum_{j=i}^{N-1} L_j(x_j, u_j) \leqslant \lambda,\ x_{j+1} = S_j x_j + D_j u_j,\ j = i, \tag{5.17}$$
$$i+1, \ldots, N-1,\ x_i = h \Big\}$$

are bounded in $l^2(i, N-1; U)$. Here $h \in D(J_i)$ is fixed.

From (2.1) we have

$$L_i(x, u) \geqslant \langle v, u \rangle - H_i(x, v),\ i \in \sigma_{N-1},$$

for all $u, v \in U$ and $x \in H$.

Let $y_i \in \partial_x H_i(0, p)$. Then

$$H_i(x, p) - H_i(0, p) \leqslant (y_i, x),$$ for every $x \in H$ and $p \in U$. Therefore

$$L_i(x, u) \geqslant \langle p, u \rangle - H_i(0, p) - (y_i, x_i),\ i \in \sigma_{N-1},$$

for every $x \in H$ and $u, p \in U$.

Now we use (β) to obtain that

$$L_i(x, u) \geqslant \langle p, u \rangle - C(1 + \|p\|^2) - |x|(1 + \|p\|). \tag{5.18}$$

We put $p = \rho u$, $\rho > 0$: we deduce

$$L_i(x, u) \geqslant C_\rho \|u\|^2 - C|x|^2 - C_1,\ i \in \sigma_{N-1} \tag{5.19}$$

for every $x \in H$ and $u \in U$. Here $C_\rho > 0$ for ρ positive and sufficiently small.

On the other hand, from

$$x_{j+1} = S_j x_j + D_j u_j,\ j = i, i+1, \ldots, N-1,\ x_i = h$$

we obtain

$$|x_j| \leqslant C_2 + C_3 \sum_{k=i}^{j-1} \|u_k\|. \tag{5.20}$$

From (5.17) $\sim$ (5.19) we find that

$$C_\rho \|u_k\|^2 \leqslant C_4 + C_5 |x_k|^2 + C_6 \sum_{\substack{j=i\\ j \neq k}}^{N-1} |x_j|, \tag{5.21}$$

for every $k \geqslant i$. (We use here (5.18) with $p = 0$). From (5.20) and (5.21) we may infer that $\{u_j\}$ is bounded in U for $j \in \{i, i+1, \ldots, N-1\}$ as claimed.

Remarks. 1) For every $h \in D(J_i)$, $-\partial J_i(h)$ coincides with the set of all elements p_i which appear in (5.6) $\sim$ (5.10) (see [3]).

2) If (A) and (β) hold, then

$$D(J_i) = D(\partial J_i),\ i \in \sigma_{N-1}. \tag{5.22}$$

For every $i \in \sigma_{N-1}$ and $x \in H$ we consider $g_x^i : H \to]-\infty, +\infty]$ defined by $g_x^i(p) = H_i(x, D_i^* p)$. We observe that $D(g_x^i) = D(\partial g_x^i) = H$ and g_x^i is convex and lower semicontinuous.

We note also that $\partial g_x^i(p) = D_i \partial_p H_i(x, D_i^* p)$ for all $x \in H$ and $p \in H$ (see [5]).

Proposition 5.4. *We assume that (A), (β) and*

$$\lim_{|y| \to +\infty} H_i(x, D_i^* y)/|y| = +\infty,\ x \in H \tag{5.23}$$

hold. Then, the equation

$$y - \partial g_x^i(p_i) \ni S_i x,\ i \in \sigma_{N-1}, \tag{5.24}$$

where $p_i \in -\partial J_i(y)$, has a solution $y \in D(\partial J_i) = D(J_i)$, for every $x \in H$.

Proof. Equation (5.24) is equivalent with $\partial J_i(y) + \partial(g_x^i)^*(y - S_i x) \ni 0$ (we use the fact that $(\partial g_x^i)^{-1} = \partial(g_x^i)^*$ (see [4])). It suffices to show that:

$$D(\partial(g_x^i)^*) = H, \tag{5.25}$$

and

$$\lim_{|y| \to +\infty} (g_x^i)^*(y)/|y| = +\infty \quad \text{(see [4, Chap. II])}. \tag{5.26}$$

Since (5.25) is equivalent with $R(\partial g_x^i) = H$, we may infer by (5.23) that (5.25) holds. (see [4, Chap. II]).

On the other hand, (5.26) shows that $(g_x^i)^*$ is cofinite and this is equivalent with $D(g_x^i) = H$. Hence Proposition 5.4 is proved.

Remarks. 1) We denote by $y \in \Lambda_i x$ a solution of (5.24).

2) If ∂J_i or $\partial(g_x^i)^*$ is strictly monotone then (5.24) has a unique solution.

We return now to problem (P) given by (5.1) and (5.2).

Theorem 5.1. *We assume that (A), (β) and (5.23) hold; let $(x, u) \in l^2(\sigma_N; H) \times l^2(\sigma_{N-1}; U)$ be an optimal pair for (P). Then*

$$u_j \in \partial_p H_j(x_j, -D_j^* \partial J_{j+1}(\Lambda_{j+1} x_j)),\ j \in \sigma_{N-1}. \tag{5.27}$$

Proof. From Optimality theorem we obtain:

$$\begin{cases} x_{j+1} = S_j x_j + D_j u_j,\ j \in \sigma_{N-1},\ x_0 = \hat{x}_0,\ p_N = 0, \\ p_j = S_j^* p_{j+1} - q_j,\ j \in \sigma_{N-1},\quad (q_j, D_j^* p_{j+1}) \in \partial L_j(x_j, u_j),\ j \in \sigma_{N-1} \end{cases}$$

Hence $u_j \in \partial_p H_j(x_j, D_j^* p_{j+1})$, $j \in \sigma_{N-1}$, and $-p_j \in \partial J_j(x_j)$, $j \in \sigma_{N-1}$. Then $u_j \in \partial_p H_j(x_j, -D_j^* \partial J_{j+1}(x_{j+1}))$, $j \in \sigma_{N-1}$, where $J_N(h) = 0$ for every $h \in H$.

Since $x_{j+1} = S_j x_j + D_j u_j$, $j \in \sigma_{N-1}$, it follows $x_{j+1} - S_j x_j \in D_j \partial_p H_j(x_j, -D_j^* \partial J_{j+1}(x_{j+1}))$, $j \in \sigma_{N-1}$. From Proposition 5.4 this equation has a solution $x_{j+1} \in \Lambda_{j+1} x_j$, $x_{j+1} \in D(J_{j+1})$, $j \in \sigma_{N-1}$, and this completes the proof.

Remarks. 1) If J_i, $i \in \sigma_N$, are known, then we obtain by Proposition 5.4, Λ_j, $j = 1, 2, \ldots, N$ and we determine u_0 by (5.27). Then we obtain $x_1 = S_0 \hat{x}_0 + D_0 u_0$ and we determine u_1 by (5.27) etc.

2) We note that we need $\partial J_i \subset H \times H$ for each $i \in \sigma_N$.

We assume that (A), (3) and (5.23) hold; then $\{\partial J_i\}_{i \in N}$ verify the problem:

$$(5.28) \quad \begin{aligned} &\partial J_N(x) = \{0\} \text{ for every } x \in H \\ &0 \in \partial J_j(x) - S_j^* \partial J_{j+1}(\Lambda_{j+1}(x)) + \partial_x H_j(x, -D_j^* \partial J_{j+1}(\Lambda_{j+1} x)),\ j \in \sigma_{N-1}. \end{aligned}$$

(Riccati equation)

3) From (5.28) and Proposition 5.4 we determine $\partial J_i \subset H \times H$, $i \in \sigma_{N-1}$.

4) We give now „a method" to determine J_i, $i \in \sigma_N$: namely if (x, u) is an optimal pair for (P), then we have

$$(5.29) \quad \begin{cases} J_i(x_i) = \min\{L_i(x, u) + J_{i+1}(S_i x_i + D_i u);\ u \in U\},\ i \in \sigma_{N-1}, \\ J_N(x) = 0 \text{ for every } x \in H \end{cases}$$

(This is Bellman's optimality principle [6]).

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Faculty of Mathematics
University of Iaşi
România

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ON THE GLOBAL GEOMETRIC STRUCTURE OF SIMPLE BODIES

BY

LILIANA RĂILEANU

1. Introduction. The object of this paper is to describe the global geometric structure of a simple body, including the material connections and the material structure of such body. The development is based upon ideas introduced in [7], where the start of the study of a global geometric theory for these materials in the context of the Noll's extrinsic theory is given. These ideas are non imbedded within Noll's new mathematical theory of simple material [3], [4] which gives an intrinsic characterization of the local properties of the continuous media.

The plan of the paper can be summarized as follows. In Section 2 the notion of simple body is given using Noll's definitions of material element and the local symmetry group of a body. Further, to any simple body is associated the symmetry groups and the modules of intrinsec tensor fields. The material structures and G_{E_0}-connections of a simple body, studied in section 3, are used to characterize the locally and globally homogeneous simple bodies.

The formalism and the mechanical framework are taken from [3]. For the mathematical background and for a definition of terms not defined in the text, we refer to the mathematical preliminaries of [3] and Part I of [4], Ch II of [2] and Ch III of [6].

2. The geometric of a materially uniform continuous body. Let $(\mathfrak{B}, \mathcal{A}(\mathfrak{B}), \mathcal{P}(\mathfrak{B}))$ be a materially uniform continuous body (see [3]), where by $\mathcal{A}(\mathfrak{B})$, $\mathcal{P}(\mathfrak{B})$ is denoted, respectivelly, the classes of the intrinsic configurations and of the admissible cinematical processes of $\mathfrak{B}$; let $(T_x, \Delta(x), \Pi(x), \hat{\Delta}(x), \hat{S}(x), \hat{\varsigma}(x))$ be the material element of $\mathfrak{B}$ in the material point $x \in \mathfrak{B}$ having $g_{T_x} = g_x \subset \mathrm{Aut}(T_x)$ as the symmetry group. We suppose that $\mathfrak{B}$ satisfies the following condition:

(G) *The symmetry group g_x is a closed subgroup of the Lie group* Aut (T_x), $\forall x \in \mathfrak{B}$; *hence g_x is itself a Lie group by Cartan's theorem.*

Because we have $g_y = A_{x,y} \circ g_x \circ A_{x,y}^{-1}$, $\forall x, y \in \mathfrak{B}$, where $A_{x,y}$ is a material isomorphism of the material elements of $\mathfrak{B}$ in x and y, it follows that the previous condition is equivalent with