2) We note that we need $\partial J_i \subset H \times H$ for each $i \in \sigma_N$.

We assume that (A), (5) *and* (5.23) *hold; then* $\{\partial J_i\}_{i \in N}$ *verify the problem*:

$$(5.28)\qquad \begin{aligned} &\partial J_N(x) = \{0\} \ \text{for every } x \in H \\ &0 \in \partial J_j(x) - S_j^* \partial J_{j+1}(\Lambda_{j+1}(x)) + \partial_x H_j(x, -D_j^* \partial J_{j+1}(\Lambda_{j+1} x)),\ j \in \sigma_{N-1}. \end{aligned}$$

(Riccati equation)

3) From (5.28) and Proposition 5.4 we determine $\partial J_i \subset H \times H$, $i \in \sigma_{N-1}$.

4) We give now „a method" to determine J_i, $i \in \sigma_N$: namely if (x, u) is an optimal pair for (P), then we have

$$(5.29)\qquad \begin{cases} J_i(x_i) = \min\{L_i(x_i, u) + J_{i+1}(S_i x_i + D_i u);\ u \in U\},\ i \in \sigma_{N-1} \\ J_N(x) = 0 \ \text{for every } x \in H \end{cases}$$

(This is Bellman's optimality principle [6]).

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Faculty of Mathematics
University of Iași
România

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ON THE GLOBAL GEOMETRIC STRUCTURE OF SIMPLE BODIES

BY

LILIANA RĂILEANU

1. Introduction. The object of this paper is to describe the global
geometric structure of a simple body, including the material connections
and the material structure of such body. The development is based upon
ideas introduced in [7], where the start of the study of a global geometric
theory for these materials in the context of the Noll's extrinsic theory is
given. These ideas are non imbedded within Noll's new mathematical the-
ory of simple material [3], [4] which gives an intrinsic characteriza-
tion of the local properties of the continuous media.

The plan of the paper can be summarized as follows. In Section 2
the notion of simple body is given using Noll's definitions of material ele-
ment and the local symmetry group of a body. Further, to any simple
body is associated the symmetry groups and the modules of intrinsec tensor
fields. The material structures and G_{k_0}-connections of a simple body, stu-
died in section 3, are used to characterize the locally and globally homo-
geneous simple bodies.

The formalism and the mechanical framework are taken from [3].
For the mathematical background and for a definition of terms not de-
fined in the text, we refer to the mathematical preliminaries of [3] and
Part I of [4], Ch II of [2] and Ch III of [6].

2. The geometrie of a materially uniform continuous body. Let $(\mathfrak{B},$
$\mathcal{A}(\mathfrak{B}), \mathcal{P}(\mathfrak{B}))$ be a materially uniform continuous body (see [3]), where
by $\mathcal{A}(\mathfrak{B}), \mathcal{P}(\mathfrak{B})$ is denoted, respectivelly, the classes of the intrinsic con-
figurations and of the admissible cinematical processes of $\mathfrak{B}$; let $(T_x,$
$\Delta(x), \Pi(x), \hat{\Delta}(x), \hat{S}(x), \hat{\rho}(x))$ be the material element of $\mathfrak{B}$ in the ma-
terial point $x \in \mathfrak{B}$ having $g_{T_x} = g_x \subset \mathrm{Aut}(T_x)$ as the symmetry group.
We suppose that $\mathfrak{B}$ satisfies the following condition:

(G) *The symmetry group g_x is a closed subgroup of the Lie group* $\mathrm{Aut}(T_x)$,
$\forall x \in \mathfrak{B}$; *hence g_x is itself a Lie group by Cartan's theorem.*

Because we have $g_y = A_{x,y} \circ g_x \circ A_{x,y}^{-1}$, $\forall x, y \in \mathfrak{B}$, where $A_{x,y}$ is a
material isomorphism of the material elements of $\mathfrak{B}$ in x and y, it follows
that the previous condition is equivalent with

(G′) *The symmetry group g_x is a closed subgroup of* Aut (T_x) *in a material point* $x \in \mathfrak{B}$.

Now, we consider the principal fibre bundle of the linear frames of $\mathfrak{B}$, denoted by $\mathcal{F}(\mathfrak{B})(\mathfrak{B}, Gl(3, R), \pi)$ where $\mathcal{F}(\mathfrak{B}) = \bigcup_{x \in \mathfrak{B}} \mathrm{Invlin}(E_3, T_x)$. Since $\mathcal{F}(\mathfrak{B})$ is an associated fibre bundle to the trivial tangent vector bundle $\mathcal{T}(\mathfrak{B})$, it results that it is also a trivial fibre bundle: hence it exists the differential global cross-sections in this fibre bundle [2]. A fixed global cross-section of $\mathcal{F}(\mathfrak{B})$ will be called the *reference cross-section* of $\mathfrak{B}$ or *reference* of $\mathfrak{B}$. By mechanical signification it is more suitable to take the references for $\mathfrak{B}$ having their values in the set $\mathcal{A}^{-1}(\mathfrak{B}) \subset \mathcal{F}(\mathfrak{B})$ defined by

$$\mathcal{A}^{-1}(\mathfrak{B}) = \bigcup_{x \in \mathfrak{B}} \mathcal{A}^{-1}(T_x), \tag{1}$$

$$\mathcal{A}^{-1}(T_x) = \{A \mid A \in \mathrm{Invlin}(E_3, T_x),\ A = \alpha^{-1},\ \alpha \in A_{T_x}\}. \tag{1′}$$

A_{T_x} is the set of the placements of T_x defined in [3], [4]:

$$A_{T_x} = \{k \in \mathrm{Invlin}(T_x, E_3)\ k^*k \in \Delta(x)\}. \tag{2}$$

If k_x is a fixed element in $\mathcal{A}^{-1}(T_x)$, then the mapping

$$k : \mathfrak{B} \to \mathcal{F}(\mathfrak{B}), \qquad y \to A_{x,y} \circ k_x, \tag{3}$$

where $A_{x,y}$ is a material isomorphism, is a reference with values in $\mathcal{A}^{-1}(\mathfrak{B})$. In fact, from (1′) and (2) it results that it is enough to prove that

$$((A_{x,y} \circ k_x)^{-1})^* (A_{x,y} \circ k_x)^{-1} \in \Delta(y).$$

But $((A_{x,y} \circ k_x)^{-1})^* (A_{x,y} \circ k_x)^{-1} = (k_x^{-1} \circ A_{y,x})^* (k_x^{-1} \circ A_{y,x}) = A'_{y,x} \circ (k_x^{-1})^* \circ k_x^{-1} \circ A_{y,x}$. Since $(k_x^{-1})^* \circ k_x^{-1} \in \Delta(x)$, the previous assertion is proved. Obviously, the reference obtained in this way is a differentiable cross-section of $\mathcal{F}(\mathfrak{B})$.

The references of $\mathfrak{B}$ defined by the formula (3) will be called the *uniform references* of $\mathfrak{B}$. A reference k_0 defined by

$$k_0(x) = (\nabla\varphi(x))^{-1}, \tag{4}$$

where φ is a intrinsic configuration of $\mathfrak{B}$, is called an *integrable reference* (or homogeneous reference in [1]) of $\mathfrak{B}$. It is obvious that the values of an integrable reference of $\mathfrak{B}$ are the elements from the set

$$\mathcal{A}_\varphi^{-1}(\quad) = \{(\nabla\varphi(x))^{-1},\ x \in \mathfrak{B}\} \subset \mathcal{A}^{-1}(\mathfrak{B}); \tag{5}$$

such references are not always the uniform references.

It is known [2] that if s is a differential field of local intrinsic configurations of $\mathfrak{B}$, namely a cross-section of the fibre bundle Invlin $(\mathcal{T}(\mathfrak{B}), \mathfrak{B} \times E_3)$, then it does not correspond to an intrinsic configuration φ of $\mathfrak{B}$ such that $s = \nabla\varphi$. From this fact it results that, generally, not any reference of $\mathfrak{B}$ is a homogeneous one.

Now, let k_0 be a fixed uniform reference of $\mathfrak{B}$, and consider the set

$$\mathcal{F}_m^{k_0} \circ (\mathfrak{B}) = \bigcup_{x \in \mathfrak{B}} (x, g_x \circ k_0(x)) \subset \mathcal{F}(\mathfrak{B}) \tag{6}$$

and the projection $\pi_m^{k_0} : F_m^{k_0} \circ (\mathfrak{B}) \to \mathfrak{B}$ given by $\pi_m^{k_0} \circ (x, g \circ k(x)) = x, g \in g_x$. It is obvious that k_0 is a cross-section of the projected space $\pi_m^{k_0} \circ : \mathfrak{F}_m^{k_0}(\mathcal{B}) \to$ $\to \mathfrak{B}$ because $1_{T_x} \in g_x$, $\forall x \in \mathfrak{B}$.

We can give now the following definition:

Definition 2.1. *A materially uniform continuous body* $\mathfrak{B}$ *satisfying the condition* (G) *is called a simple body if and only if it exists an uniform reference* k_0 *of* $\mathfrak{B}$ *such that* $(\mathcal{F}_m^{k_0}(\mathfrak{B}), \mathfrak{B}, \pi_m^{k_0})$ *is a subbundle of the principal fibre bundle* $\mathcal{F}(\mathfrak{B})(\mathfrak{B}, Gl(3, R), \pi)$:

We call this subfibre bundle *the material fibre bundle of the simple body* $\mathfrak{B}$ *related to the reference* k_0 and their elements the *material frames* of $\mathfrak{B}$.

First, we investigate the following problem: which is the relation between the material fibre bundles of a body $\mathfrak{B}$ related to the uniform references k_0, k'_0? The answer is given by

Proposition 2.1: *If* k_0 *and* k'_0 *are two uniform references of* $\mathfrak{B}$ *then it exists a canonical isomorphism of the projected spaces*:

$$\begin{gathered} I_{k_0, k'_0} : \mathcal{F}_m^{k_0} \circ (\mathfrak{B}) \to \mathcal{F}_m^{k'_0}(\mathfrak{B}). \\ g(x) \circ k_0(x) \to g(x) \circ k'_0(x),\quad \forall g(x) \in g_x; \end{gathered} \tag{7}$$

hence I_{k_0, k'_0} *satisfies the relation* $\pi_m^{k_0} = \pi_m^{k'_0} \circ I_{k_0, k'_0}$, *and its restriction to each fibre of* $\mathcal{F}_m^{k_0}(\mathfrak{B})$ *is a bijection. Moreover, if there exist a fibre morphism* $f : \mathcal{T}(\mathfrak{B}) \to \mathcal{T}(\mathfrak{B})$ *over* $1_{\mathfrak{B}}$ *such that*

$$k'_0(x) = f_x \circ k_0(x),\ f_x = f|_{T_x},$$

then we have $I_{k_0, k'_0}(g(x) \circ k_0(x)) = (g(x) \circ f_x \circ (g(x))^{-1}) \circ g(x) \circ k_0(x) = g(x) \circ k'_0(x)$. *Particularly, if* f *is defined by* $f_x = g(x) \in g_x$, $\forall x \in \mathfrak{B}$,

$$\mathcal{F}_m^{k_0}(\mathfrak{B}) = \mathcal{F}_m^{k'_0}(\mathfrak{B}). \tag{7′}$$

The proof is an immediate consequence of the formula (6).

From this proposition it results that, if the materially uniform continuous body is a simple one then for *any* uniform reference of $\mathfrak{B}$, the set defined by (6) is the total space of a subbundle of $\mathcal{F}(\mathfrak{B})$.

Let k_0 be a uniform reference of $\mathfrak{B}$ and $\mathcal{F}_m^{k_0} \circ (\mathfrak{B})$ be its corresponding material fibre bundle. From the definition of a principal fibre bundle [2], [6] it follows that there exists a closed subgroup G_{k_0} of Aut (E_3) which satisfies the following conditions:

1. G_{k_0} acts freely on the right on $\mathcal{F}_m^{k_0} \circ (\mathfrak{B})$;

2. $\mathfrak{B}$ is the quotient space of the set $\mathcal{F}_m^{k_0}\circ(\mathfrak{B})$ by the equivalence relation induced by G_{k_0} and the canonical projection $\pi_m^{k_0}$ is a differentiable mapping;

3. $\mathcal{F}_m^{k_0}\circ$ is locally trivial, namely every point $x\in\mathfrak{B}$ has an open neighbourhood U and a diffeomorphism $h_U:(\pi_m^{k_0})^{-1}(\mathcal{U})\to\mathfrak{B}\times G_{k_0}$ given by $h_U(u)=(\pi_m^{k_0}(u),\varphi_U(u))$ where φ_U is a mapping from $(\pi_m^{k_0})^{-1}(U)$ on G_{k_0} which satisfies the condition:

$$\varphi_{\mathcal{U}}(ua)=\varphi_{\mathcal{U}}(u)\,a,\quad \forall u\in(\pi_m^{k_0})^{-1}(\mathcal{U}),\quad \forall a\in G_{k_0}.$$

It is also known from the theory of the fibre bundles [2], [6] that the restriction $h_x=h_U\,|\,(\pi_m^{k_0})^{-1}(x)$ is an isomorphism of the groups $\{g(x)\circ k_0(x)\,|\,g(x)\in g_x\}$ and G_{k_0}. Hence G_{k_0} is diffeomorphical with g_x, $\forall x\in\mathcal{B}$ and for this fact it is called *symmetry group of a materially uniform simple body* $\mathfrak{B}$ *related to the uniform reference* k_0.

Let now k_0' be another uniform reference of $\mathfrak{B}$; then, by formula (3), we have $k_0(y)=A_{x,y}\circ k_x^0$, $k_0'(y)=A_{x,y}\circ k_x^0$, where k_x^0, k_x^0 are fixed elements in $\mathcal{A}^{-1}(T_x)$. There exists then an isomorphism of the space E_3, $a_{k_0,k_0'}$ such that $k_x^0=k_x^0\circ a_{k_0,k_0'}$, and we have: $k_0'(y)=k_0(y)\circ a_{k_0,k_0'}$; hence $g(x)\circ k_0'(x)=g(x)\circ k_0(x)\circ a_{k_0,k_0'}$ $\forall x\in\mathfrak{B}$ and $\varphi_U(g(x)\circ k_0'(x))=a^{-1}_{k_0,k_0'}\circ$ $\circ\,\varphi_U(g(x)\circ k_0(x))\circ a_{k_0,k_0'}$ $\forall x\in U\subset\mathfrak{B}$. Consequently we get:

Proposition 2.2: *The symmetry group of a simple body* $\mathfrak{B}$ *related to the uniform references* k_0, k_0' *are the conjugate Lie subgroups in* Aut (E_3): $G_{k_0'}=a^{-1}_{k_0,k_0'}\circ G_{k_0}\circ a_{k_0,k_0'}$.

The vector bundle with E_3 as typical fibre, associated to the principal fibre bundle $\mathcal{F}_m^{k_0}(\mathfrak{B})$ is a reduction of the tangent vector bundle $\mathcal{T}(\mathfrak{B})$ to the group G_{k_0}[2]; we denote this vector bundle by $\mathcal{T}_m^{k_0}(\mathfrak{B})$ and it will be called the *material tangent fibre bundle related to the uniform reference* k_0. Analogous we obtain the *material tensorial fibre bundle of* $\mathfrak{B}$ *related to* k_0 denoted by $\mathcal{T}_{s,m}^{r,k_0}(\mathfrak{B})$; obviously $\mathcal{T}_{0,m}^{1,k_0}(\mathfrak{B})=\mathcal{T}_m^{k_0}(\mathfrak{B})$.

We remark now that a differential cross-section s of the material fibre bundle $\mathcal{F}_m^{k_0}(\mathfrak{B})$ has the following property: $s(y)\circ(s(x))^{-1}$ is a material isomorphism from T_x at T_y, $\forall x,y\in\mathfrak{B}$.

In fact, $s(y)\circ(s(x))^{-1}=(g(y)\circ k_0(y))\circ(g(x)\circ k_0(x))^{-1}=g(y)\circ$ $\circ\,k_0(y)\circ(k_0(x))^{-1}\circ(g(x))^{-1}$, and the assertion results using (3) and the properties of the symmetry groups g_x, g_y of $\mathfrak{B}$.

The cross-sections of the material tangent fibre bundle $\mathcal{T}_m^{k_0}(\mathfrak{B})$ are obtained by the composition of the cross-sections of the fibre bundle $\mathcal{F}_m^{k_0}(\mathfrak{B})$ with the projection $\pi_{y_0}:\mathcal{F}_m^{k_0}(\mathfrak{B})\to\mathcal{T}_m^{k_0}(\mathfrak{B})$; this projection satisfies the property:

$$\pi_m^{k_0}=\bar{\pi}_m^{k_0}\circ\pi_{y_0}\ (\bar{\pi}_m^{k_0}:\mathcal{T}_m^{k_0}(\mathfrak{B})\to\mathfrak{B})$$

and it is given by

$$\pi_{y_0}(\bar{\alpha})=\bar{\varphi}_U(x,\varphi_x^{-1}(\bar{\alpha})\circ y_0), \tag{8}$$

where $\alpha\in(\pi_m^{k_0})^{-1}(x)$, $x\in\mathfrak{B}$, $\varphi_x=\varphi_U\,|(\pi_m^{k_0})^{-1}(x)$, $\bar{\varphi}_U:(\bar{\pi}_m^{k_0})^{-1}(U)\to U\times E_3$ is a trivialization of $\mathcal{F}_m^{k_0}(\mathfrak{B})$ and y_0 is a fixed point in E_3. We show now that every cross-section $\bar{s}=\pi_{y_0}\circ s$ of the fibre bundle $\mathcal{T}_m^{k_0}(\mathfrak{B})$, s being a cross-section of $\mathcal{F}_m^{k_0}(\mathfrak{B})$, has the following property:

$$\bar{s}(y)=A_{x,y}(\bar{s}(x)). \tag{9}$$

where $A_{x,y}=s(y)\circ(s(x))^{-1}$ is a material isomorphism from T_x at T_y. In fact $(s(y)\circ(s(x))^{-1})(s(x))=(s(y)(s(x))^{-1}(\pi_{y_0}(s(x)))=s(y)((s(x))^{-1}(\bar{\varphi}_x(\varphi_x^{-1}$ $(s(x)).y_0)))=\bar{\varphi}_y(\varphi_y^{-1}(s(y))\,y_0)$.
Formula (9) points out that the cross-sections of $\mathcal{T}_m^{k_0}(\mathfrak{B})$ are intrinsic ones, using the terminology of [7]. Let $\mathcal{X}_m^{k_0}(\mathfrak{B})$ be the set of such cross-sections of C^∞ class. It is easy to see that $\mathcal{X}_m^{k_0}(\mathfrak{B})$ has a linear space structure which, locally, has the dimension 3. It results also that a C^∞ cross-section of $\mathcal{T}_{s,m}^{r,k_0}(\mathfrak{B})$ satisfies a property analogous to (9) where $A_{x,y}$ is, in this case, the transformation induced by the material isomorphism $s(y)\circ(s(x))^{-1}$ on the linear space $T_s^r(T_x)$.

Remark 1. If a tangent vector field on $\mathfrak{B}$, s, has the property

$$(k_0(x))^{-1}(s(x))=(k_0(y))^{-1}(s(y)),\ \forall x,y\in\mathcal{B}, \tag{10}$$

(this is a *material constant* vector field in the terminology of [1]), then

$$k_0(y)(k_0(x))^{-1}(s(x))=s(y). \tag{10'}$$

Because k_0 is a uniform reference, it results that every material constant vector field s on $\mathfrak{B}$ satisfies (9) and, moreover, it is a cross-section of the material fibre bundle $\mathcal{F}_m^{k_0}(\mathfrak{B})$. The reciprocal assertion is not true.

Denoting by $\mathcal{X}_{m,c}^{k_0}(\mathfrak{B})$ the set of the material constant vector fields, from Remark 1 it follows that this set has a real three dimensional Lie algebra structure. Moreover we obtain:

Proposition 2.3: *The fibre bundle* $\mathcal{F}_m^{k_0}(\mathfrak{B})$ *is a trivial fibre bundle if, and only if,* $\mathcal{X}_{m,c}^{k_0}(\mathfrak{B})=\mathcal{X}_m^{k_0}(\mathfrak{B})$.

This proposition permits a simple mechanical interpretation of the triviality of the fibre bundle $\mathcal{F}_m^{k_0}(\mathfrak{B})$. This interpretation is connected with the homogeneity of $\mathfrak{B}$. To this end, we observe that we may associate to the simple body $\mathfrak{B}$ the following five projected spaces, two of them being the principal fibre bundles: $\mathcal{F}(\mathfrak{B})$, $\mathcal{A}^{-1}(\mathfrak{B})$, $\mathcal{A}_\varphi^{-1}(\mathfrak{B})$, $\varphi\in\mathcal{A}(\mathfrak{B})$, $\mathcal{F}_m^{k_0}(\mathfrak{B})$ and,

$$\mathcal{A}_o^{-1}(\mathfrak{B})=\bigcup_{x\in\mathfrak{B}}\mathcal{A}_\varphi^{-1}(T_x), \tag{11}$$

where

$$\mathcal{A}_o^{-1}(T_x)=\{A,\ A\in\mathrm{Invlin}(E_3,T_x),\ A=(\nabla\varphi(x))^{-1},\ \forall\varphi\in\mathcal{A}(\mathfrak{B})\}. \tag{11'}$$

$\mathcal{A}^{-1}(\mathfrak{B})$ will be called the set of the holonomical frames of the simple body $\mathfrak{B}$.

First, we show that $\mathcal{F}_m^{k_0}(\mathfrak{B})\subseteq\mathcal{A}^{-1}(\mathfrak{B})$.

In fact, let $x \in \mathfrak{B}$, $g(x) \circ k_0(x) \in \mathcal{F}^{k_0}_m(\mathfrak{B})$, $g(x) \in g_x$: then we have: $(g(x) \circ k_0(x))^{-1} = (k_0(x))^{-1} \circ (g(x))^{-1} \in \Delta(x)$ because $((k_0(x))^{-1} (g(x))^{-1})^* \circ ((k_0(x))^{-1} \circ (g(x))^{-1} = ((g(x))^{-1})^* \circ ((k_0(x))^{-1*} \circ (k_0(x))^{-1} (g(x))^{-1}$; because $((k_0(x))^{-1})^* \circ (k_0(x))^{-1} \in \Delta(x)$ and $k_0(x) \in \mathcal{A}^{-1}(T_x)$ $\forall x \in \mathfrak{B}$, it results that the above element belongs to $\Delta(x)$ and hence $g(x) \circ k_0(x) \in \in \mathcal{A}^{-1}(T_x)$.

Remark 2. If the reference k_0 given by $k_0(x) = (\nabla\varphi(x))^{-1}$ with $\varphi \in \mathcal{A}(\mathfrak{B})$ is a uniform one, then $\mathcal{A}^{-1}_{\varphi}(\mathfrak{B}) \subset \mathcal{F}^{k_0}_m(\mathfrak{B})$.

In fact $1_x \in g_x$ and since $(\nabla\varphi(x))^{-1} = 1_x \circ (\nabla\varphi(x))^{-1} = 1_x \circ k_0(x)$ the remark is easily deduced. So, between these five there exists the following inclusion relations:

(12)
$$\mathcal{F}(\mathfrak{B}) \supset \mathcal{A}^{-1}(\mathfrak{B}) \supseteq \mathcal{A}^{-1}_{\varphi}(\mathfrak{B}) \supseteq \mathcal{A}^{-1}_{\varphi}(\mathfrak{B})$$
$$\breve{\mathcal{F}}^{k_0}_m(\breve{\mathfrak{B}})$$

We can also associate to every open set $U \subset \mathfrak{B}$ the corresponding five projected spaces $\mathcal{F}(U)$, $\mathcal{A}^{-1}(U)$, $\mathcal{A}^{-1}_0(U)$, $\mathcal{F}^{k_0}_m(U)$ and $\mathcal{A}^{-1}_{\varphi|U}(U)$ which satisfy to the similar relations as the previous spaces.

Definition 2.2. *A simple body $\mathfrak{B}$ is called a locally homogeneous body if each material point $x \in \mathfrak{B}$ has a open set $U \ni x$ and an intrinsic configuration $\varphi \in \mathcal{A}(\mathfrak{B})$ such that the reference k_0 given by $k_0(x) = (\nabla\varphi|_U(x))^{-1}$, $\forall x \in U$ is a uniform reference over U.*

From Remark 2 it results that, if $\mathfrak{B}$ is a locally homogeneous body, then each $x \in \mathfrak{B}$ has an open set U and an intrinsic configuration $\varphi \in \mathcal{A}(\mathfrak{B})$ such that $\mathcal{F}^{k_0}_m(U) \supset \mathcal{A}^{-1}_{\varphi|U}(U)$ where k_0 is the reference from the definition 1.2.

Definition 2.3. *A simple body $\mathfrak{B}$ is called a globally homogeneous body if there exists an intrinsic configuration $\varphi \in \mathcal{A}(\mathfrak{B})$, such that the global reference $k_0 : x \to k_0(x) = (\nabla\varphi(x))^{-1}$ $\forall x \in \mathfrak{B}$ is a uniform reference of $\mathfrak{B}$.*

Remark 3. From the Definitions 1.2 and 1.3 one deduces that a globally homogeneous simple body is a locally homogeneous body and the fibre bundle $\mathcal{F}^{k_0}_m(\mathfrak{B})$ is trivial (k_0 from the definition 1.3). (The converse assertion is not always true). Moreover, for such body we have $\mathcal{F}^{k_0}_m(\mathfrak{B}) \supset \supset \mathcal{A}^{-1}_{\varphi}(\mathfrak{B})$

3. Material structure of a simple body: material connections. First, we remark that according to the definition of a G-structure [5], [6] the fibre bundle $\mathcal{F}^{k_0}_m(\mathfrak{B})$ defines a G_{k_0}-structure on the manifold $\mathfrak{B}$, for all uniform references k_0 of $\mathfrak{B}$; this structure will be called the *material structure of related to the reference* k_0. Using the result from Prop. 1.2 we obtain:

Proposition 3.1. *The material structures of a simple body related to the references k_0, k'_0 are the conjugate equivalent structures in the sense of* [6] *pag. 314 for all uniform references k_0, k'_0 of $\mathfrak{B}$.*

Remark 1. From the previous proposition it follows that an assertion valid for a material structure of $\mathfrak{B}$ is also valid for any other material structure of $\mathfrak{B}$. Consequently we may speak about a material structure of a simple body $\mathfrak{B}$ if other explanation is not necessary.

Further we investigate the basic problems of the theory of G-structures for a material structure of $\mathfrak{B}$ namely the integrability and linear connections compatibles with a G-structure [5].

We remark, firstly, that the material frames are the admissible frames of a material structure on $\mathfrak{B}$. It is known [5] that a G-structure on a manifold $\mathfrak{B}$ is called integrable if there exists an atlas of the differential structure of $\mathfrak{B}$ such that the natural frames defined by the coordinate maps are the admissible frames of given G-structure. Then we obtain:

Proposition 3.2. *Every globally homogeneous simple body $\mathfrak{B}$ has a uniform global reference k_0 such that its corresponding G_{k_0}-structure is integrable and the fibre bundle $\mathcal{F}^{k_0}_m(\mathfrak{B})$ is trivial.*

Proof. The reference k_0 from the Def. 1.3 satisfies, obviously the required conditions.

The integrability problem leads to the definition of the *structure tensor* (*Chern-Bernard*) of a G-structure such that the vanishing of this tensor is a necessary condition for the integrability of the G-structure [5].

A G-structure is called *almost integrable* if its structure tensor vanish.

We construct now the structure tensor of a material structure of $\mathfrak{B}$. To this, end let $\overline{G}_{k_0}$ be the Lie algebra of G_{k_0}; we have $\overline{G}_{k_0} \subset \mathrm{Lin}(E_3) \simeq \simeq E_3 \otimes E^*_3$. Let us consider the following sequence of mappings:

(1)
$$\overline{G}_{k_0} \otimes E^*_3 \xrightarrow{i} E_3 \otimes E^*_3 \otimes E^*_3 \xrightarrow{A} E_3 \otimes E^*_3 \wedge E^*_3$$

where i is the inclusion map and A is the alternation operator, and let

(2)
$$\partial = A \circ i.$$

The fundamental 1-form of the G_{k_0}-structure of $\mathfrak{B}$, ω_{k_0}, is defined on $\mathcal{F}^{k_0}_m(\mathfrak{B})$ having its values in E_3 by:

(3)
$$(\omega_{k_0}(z))(v_z)) = z^{-1} d_z \pi^{k_0}_m(v_z)$$

where $z \in \mathcal{F}^{k_0}_m(\mathfrak{B})$, $v_z \in T_z$, (T_z is the tangent space of $\mathcal{F}^{k_0}_m(\mathfrak{B})$ at the point z) and $d_z \pi^{k_0}_m : T_z(\mathcal{F}^{k_0}_m(\mathfrak{B}) \to T_{\pi^{k_0}_m(z)}(\mathfrak{B})$ is the linear tangent mapping induced by $\pi^{k_0}_m$ (see [2], [6]). Choosing a linear map $\Phi_z : E_3 \to T_z$ such that $\omega_{k_0}(z) \circ \Phi_z = 1_{E_3}$ we obtain:

(4)
$$d\omega_{k_0}(z) \circ (\Phi_z \wedge \Phi_z) \in \mathrm{Lin}(E_3 \wedge E_3, E_3) \simeq E_2 \otimes (E^*_3 \wedge E^*_3).$$

Denoting by $K^{k_0}_m = (E_3 \otimes E^*_3 \wedge E^*_3)/\partial(\overline{G}_{k_0} \otimes E^*_3)$ the quotient space and by $p^{k_0}_m : E_3 \otimes E^*_3 \wedge E^*_3 \to K^{k_0}_m$ the canonical projection, it is possible to show that the element $p^{k_0}_m(d\omega_{k_0}(z)(\Phi_z \wedge \Phi_z))$ is independent on the choice of the mapping Φ_z such that we may define the required structure tensor:

(5)
$$S^{k_0}_m : \mathcal{F}^{k_0}_m(\mathfrak{B}) \to K^{k_0}_m$$
$$z \to p^{k_0}_m(d\omega_{k_0}(z))(\Phi_z \wedge \Phi_z)).$$

It is remarkable that this tensor may be expressed by the torsion tensor of a special linear connection in $\mathfrak{B}$.

Definition 3.1. *An infinitesimal connection in the principal fibre bundle $\mathcal{F}^{k_0}_m(\mathfrak{B})$ is called a G_{k_0}-connection in $\mathfrak{B}$.*

Remark 1. The connection from Def. 2.1 is an affine one namely it is a connection in the fibre bundle of the affine frames, $A(\mathfrak{B})$ [2] having the property that the field φ from the definition of this connection is the field of the identity transformations of the tangent spaces at the points of $\mathfrak{B}$. Because between the set of these connections and the set of the linear connections on the fibre bundle $\mathcal{F}(\mathfrak{B})$ there exists a one-to-one correspondence ([2] pag. 129), it follows that a G_{k_0}-connection may be considered as a linear connection in $\mathcal{F}^{k_0}_m(\mathfrak{B})$.

Now, since the body is a metrisable space, it results that it is a paracompact space and, using a classical result of the connection's theory on the real manifolds [2], we have:

Proposition 3.2. *For every uniform reference k_0 of $\mathfrak{B}$, there exists G_{k_0}-connections in $\mathfrak{B}$.*

Denoting by $i^{k_0}_m : \mathcal{F}^{k_0}_m(\mathfrak{B}) \to \mathcal{F}(\mathfrak{B})$ the inclusion map it is known that this map induces an injection $(i^{k_0}_m)^{\#}$ if the set of infinitesimal connections in $\mathcal{F}^{k_0}_m(\mathfrak{B})$ (G_{k_0}-connections) at the set of the infinitesimal connections in $\mathcal{F}(\mathfrak{B})$ (linear connections in $\mathfrak{B}$ [2]). A linear connection of $\mathfrak{B}$ which is the range by $(i^{k_0}_m)^{\#}$ of a G_{k_0}-connection is called a *linear connection compatible with the G_{k_0}-structure on* $\mathfrak{B}$ or, shortly *a material connection on $\mathfrak{B}$ related to k_0*.

From Prop. 3.2 it results that a simple body $\mathfrak{B}$ admits always material connections.

We recall now that an infinitesimal connection in a principal fibre bundle induces an infinitesimal connection in each fibre bundle associated to it, such that we obtain the following result: *every material connection on $\mathfrak{B}$ related to a uniform reference k_0 induces a connection on the vector bundle associated to $\mathcal{F}^{k_0}_m(\mathfrak{B}) : \mathcal{T}^{k_0}_m(\mathfrak{B})$, $\mathcal{T}^{r,k_0}_{s,m}(\mathfrak{B})$ which will be called also a material connection on* $\mathfrak{B}$. Let ∇_{k_0} be the material connection in the tangent material fibre bundle $\mathcal{T}^{k_0}_m(\mathfrak{B})$. But a linear connection on $\mathfrak{B}$ may be given (see [2] Prop. 7.5 pag. 143) by a mapping:

$$\nabla : \mathcal{X}(\mathfrak{B}) \times \mathcal{X}(\mathfrak{B}) \to \mathcal{X}(\mathfrak{B}),$$

$\mathcal{X}(\mathfrak{B})$ being the modulus of the tangent vector fields of $\mathfrak{B}$ (namely the cross-sections of $\mathcal{T}(\mathfrak{B})$), satisfying the properties:

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \qquad \nabla_{fX} Y = f\nabla_X Y,$$

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \qquad \nabla_X fY = f\nabla_X Y + (Xf)\, Y,$$

$X, Y, Z \in \mathcal{X}(\mathfrak{B})$, f is a real function on $\mathfrak{B}$ and $\nabla_X Y = \nabla(X, Y)$; moreover $\nabla_X Y$ is the covariant derivative of the field Y in the direction of X associated to the given linear connection.

The torsion T and the curvature R of this connection are given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in \mathcal{X}(\mathfrak{B}). \tag{6}$$

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \mathrm{Lin}(\mathcal{X}(\mathfrak{B}), \mathcal{X}(\mathfrak{B})). \tag{7}$$

Obviously, we have:

$$T(fX, Y) = T(X, fY) = fT(X, Y). \tag{6'}$$

$$R(fX, Y) = R(X, fY) = fR(X, Y). \tag{7'}$$

Proposition 3.3. *Let k_0 be a uniform reference of $\mathfrak{B}$. An affine connection ∇ on $\mathfrak{B}$ is a material connection related to k_0 if and only if the following condition is satisfied:*

$$\nabla_X Y = 0 \quad \forall X \in \mathcal{X}(\mathfrak{B}),\ Y \in \mathcal{X}^{k_0}_m(\mathfrak{B}). \tag{8}$$

or, equivalently, every tangent intrinsec vector field Y is parallel related to the connection ∇, $\forall X \in \mathcal{X}(\mathfrak{B})$.

Proof. a) Let ∇_X be a material connection related to k_0. This connection defines a parallel displacement ([2]) in every fibre bundle associated to $\mathcal{F}(\mathfrak{B})$; particularly, we obtain such a displacement on $\mathcal{T}(\mathfrak{B})$: if c: $[0, 1] \to \mathfrak{B}$ is a piecewise differential curve in $\mathfrak{B}$, and $x_0 \in T_{c(0)}$, then there exists a unique vector $X_t \in T_{c(0)}$. The map

$$X_0 \to X_{c(t)} \tag{9}$$

is a linear mapping such that the range of a linear frame by this map is also a linear frame. Then we have:

$$(\nabla_X Y)(x) = \left(\lim_{t\to 0} \frac{(\hat{Y}_t - Y_{c(0)})}{t}\right) t = 0,$$

where $x \in \mathfrak{B}$, c is the integral curbure of Y defined by x, $Y_t = Y(c(t))$ and Y_t is the range by the map (9) of Y_t. If $Y \in \mathcal{X}^{k_0}_m(\mathfrak{B})$ then from (9) we get $Y_t = A_{c(0),c(t)} \hat{Y}_0$ and since the parallel displacement defined by ∇_X is given by material isomorphisms, we have: $Y_t = A_{c(0),c(t)} \cdot A^{-1}_{c(0),c(t)} Y_0 = Y_0$ and $(\nabla_X Y)(x) = 0$.

b) Let ∇_X be an affine connection on $\mathfrak{B}$ satisfying the condition (8). Then the parallel displacement induced by ∇_X applies the frames from $\mathcal{F}^{k_0}_m(\mathfrak{B})$ on the frames from the same set such that it applies the fibres of $\mathcal{T}^{k_0}_m(\mathfrak{B})$ on the fibres of the same space. Hence ∇ is the extension of a connection in $\mathcal{F}^{k_0}_m(\mathfrak{B})$.

Corollary 1. *The linear transformations between the tangent spaces at the points of $\mathfrak{B}$ induced by a material connection on $\mathfrak{B}$ are material isomorphisms.*

Corollary 2. *The curvature and the torsion of a material connection on $\mathfrak{B}$ satisfies the following relations:*

$$T\,|_{\mathcal{X}^{k_0}_m(\mathfrak{B}) \times \mathcal{X}^{k_0}_m(\mathfrak{B})}(X, Y) = T^{k_0}_m(X, Y) = -[X, Y]. \tag{10}$$

$$R\,|_{\mathcal{X}^{k_0}_m(\mathfrak{B})}(X, Y) = 0 \qquad \forall X, Y \in \mathcal{X}(\mathfrak{B}). \tag{11}$$

It is known ([2], pag. 92) that a connection in a principal fibre bundle $P(M, G, \pi)$ is a flat connection if every point $x \in M$, has a neighbourhood U such that the connection induced in $\pi^{-1}(U)$ is isomorphic with the canonical flat connection in $U \times G$, namely with the connection having as horizontal spaces in $u = (x, a)$ $U \times G$ the tangent space of $M \times \{a\}$ at u.

It is also known that a connection in a principal fibre bundle is a flat connection if and only if its curvature form vanishes. Particularly, an affine connection on $\mathfrak{B}$ is a flat connection if and only if the torsion and the curvature of its corresponding linear connection vanish. Hence: a material connection on the simple body $\mathfrak{B}$ is a flat connection if and only if the torsion and the curvature tensors of U vanish.

Proposition 3.4: *Any locally homogeneous simple body $\mathfrak{B}$ admits a flat material connection.*

In fact from Def. 1.2 it follows that every material point has a neighbourhood U and an intrinsic configuration φ, such that the reference $k_0(x) = (\nabla\varphi(x))^{-1}$ is a uniform reference of U. So, $\forall x \in U$, the material frame $k_0(x)$ is a natural frame and, from (10), it results that the torsion tensor $T_{k_0}(U)$ of the linear connection in U associated to the material connection related to k_0 vanishes for the fields of the previous frame. From (10′) it results that $T_{k_0}(U)$ vanish for all fields of $\mathcal{X}(U)$ and, using the globalisation theorem of a local connection it results that $T_{k_0}(\mathfrak{B})$ vanishes.

Analogous, from (11′) we deduce that the curvature tensor $R_{k_0}(U)$ of the material connection related to k_0 vanishes on $\mathcal{X}^{k_0}_m(\mathfrak{B})$ and, because the natural frame over U is a material one, from (11′) it is deduced that R_{k_0} vanishes on every neighbourhood U: using the globalisation theorem, it results that R_{k_0} vanishes.

Now, let $z \in \mathcal{F}^{k_0}_m(\mathfrak{B})$; hence $z \in \mathrm{Invlin}(E_3, T_x)$, $x = \pi^{k_0}_m(z)$. We construct the following map:

$$
\overline{T}_{k_0} : \mathcal{F}^{k_0}_m(\mathfrak{B}) \to E_3 \otimes E_3^* \wedge E_3^* \simeq \mathrm{Lin}(E_3 \wedge E_3, E_3), \tag{12}
$$

$$
z \to \overline{T}_{k_0}(z) : (v, w) \to z^{-1}[T_{k_0}(X, Y)(x)],
$$

where $X \in \mathcal{X}(\mathfrak{B})$ is a differential vector field on $\mathfrak{B}$ such that $k_0(v) = X(x)$ and $Y \in \mathcal{X}(\mathfrak{B})$, $Y(x) = k_0(w)$.

In the theory of G-structures (e.g. [6]) it is proved that

$$
p^{k_0}_m \circ \overline{T}_{k_0} = S^{k_0}_m, \tag{13}
$$

where $S^{k_0}_m$ is the structure tensor of the G_{k_0}-structure given by (5). Now we shall point out the mechanical interpretation of the almost integrability of a G_{k_0}-structure of a simple body $\mathfrak{B}$.

If $\overline{T}_{k_0} = 0$ we hage, obviously, that $S^{k_0}_m = 0$. But $\overline{T}_{k_0} = 0$ if and only if $T^{k_0}_m(z) = 0\ \forall z \in \mathcal{F}^{k_0}_m(\mathfrak{B})$ and this fact is true if and only if $T_{k_0}(z)(v, w) = 0\ \forall z \in \mathcal{F}^{k_0}_m(\mathfrak{B})$, $\forall v, w \in E_2$, namely $z^{-1}(T_{k_0}(X, Y)(x)) = 0\ \forall z \in \mathcal{F}^{k_0}_m(\mathfrak{B})$, $\forall x \in \mathfrak{B}$, $\forall v, w \in E_3$, hence $T_{k_0} = 0$ on $\mathcal{X}(\mathfrak{B}) \times \mathcal{X}(\mathfrak{B})$.

So, $T_{k_0} = 0$ if and only if the connection Δ_{k_0} is a symmetric one. From this, we can state.

Proposition 3.5. *On every locally homogeneous simple body there exists an almost integrable material structure.*

In fact, from the proof of Prop. 3.4 it follows that it exists a material connection on $\mathfrak{B}$ such that its torsion tensor vanishes and hence from the previous considerations, it is deduced that the corresponding G_{k_0}-structure is an almost integrable one.

Definition 3.2. *A uniform reference k_0 having the property:*

$$
\nabla_{k_0} T_{k_0}(X, Y) = 0 \quad \forall X, Y \in \mathcal{X}(\mathfrak{B}) \tag{14}
$$

is called a reference with a constant distribution of the inhomogeneities [1].

From Def. 2.2 and Prop. 3.3, it follows that $T_{k_0}(X, Y) \in \mathcal{X}^{k_0}_m(\mathfrak{B})$ $\forall X, Y \in \mathcal{X}(\mathfrak{B})$, and using (13) we obtain:

Proposition 3.6. *If $X, Y \in \mathcal{X}^{k_0}_m(\mathfrak{B})$ and k_0 is a reference with a constant distribution of the inhomogeneities, then $[X, Y] \in \mathcal{X}^{k_0}_m(\mathfrak{B})$, hence, in this case, $\mathcal{X}^{k_0}_m(\mathfrak{B})$ is a Lie subalgebra of the Lie algebra of the tangent vector fields of $\mathfrak{B}$.*

Generally, this subalgebra has not a finite dimension. Using Prop. 1.2 we obtain

Proposition 3.7. *If k_0 is a reference with a constant distribution of the inhomogeneities and $\mathcal{F}^{k_0}_m(\mathfrak{B})$ is a trivial fibre bundle, then $\mathcal{X}^{k_0}_m(\mathfrak{B})$ is a 3-dimensional real Lie algebra.*

Prop. 3.7 describes the situation from [1]. In the hypothesis of this proposition the Riemann curvature of the material connection of $\mathfrak{B}$ related to k_0 vanishes; then every field of tangent vectors on $\mathfrak{B}$ may be written locally as a linear combination of intrinsic fields having as coeficients the scalar mapping on $\mathfrak{B}$. So, G. F. Belinfante reduces in [1] the study of the structure of the field set of material constant inhomogenetities to find all real 3-dimensional Lie algebras; moreover the classification of this algebra is reduced to the study of the dimension of its centres and of its derived ideals.

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Faculty of Mathematics
University of Iaşi
România

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STRONGLY NON-RIEMANNIAN FINSLER SPACES

BY

MAKOTO MATSUMOTO

We have many important papers of L. Berwald concerned with two-dimensional Finsler spaces [16, § 28]. In these papers he made the best use of the intrinsic frame field (l^i, m^i), where l^i is the normalized support element and m^i the unit vector orthogonal to l^i. In 1957 A. Moór [19] generalized Berwald's frame to three-dimensional Finsler spaces and introduced the intrinsic frame field (l^i, m^i, n^i), where l^i is as above, m^i the normalized torsion vector and n^i the unit vector orthogonal to l^i and m^i. The present author [13], [15], [16, § 29] developed a detailed theory of three-dimensional Finsler spaces by means of Moór's frame. Recently Moór [20] studied generalized Finsler spaces of four dimensions with use of an intrinsic frame field, although his method can not be applicable to the ordinary theory of Finsler spaces.

In 1975 Prof. R. Miron and the present author [17] gave a method to construct a frame field in a Finsler space of arbitrary dimension which satisfies certain condition. Such a Finsler space was called *strongly non-Riemannian*. At that time we were dubious about the adequacy of our naming, because it seemed to us that the condition was very weak and strongly non-Riemannian Finsler spaces of general dimension might be really non-Riemannian on the ordinary assumptions of the metric.

In the present paper it is shown that our naming was adequate and there exist non-Riemannian Finsler spaces which are not strongly non-Riemannian. For instance, the four-dimensional Finsler spaces with the so-called Randers metric [7], [21], [22], [24] which play the essential role in the theory of electromagnetic field is not Riemannian and not strongly non-Riemannian. According to Theorem 2 of the present paper, such physical spaces cannot be studied by the method to apply our intrinsic frame field.

§ 1. Definition of strongly non-Riemannian Finsler spaces. We consider an n-dimensional Finsler space F^n with the fundamental function $L(x, y)$, $(y^i = \dot{x}^i)$. Throughout the following, we restrict our discussion to a domain of TF^n where L is positive and the fundamental tensor $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j L^2/2$ is positive-definite.