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STRONGLY NON-RIEMANNIAN FINSLER SPACES

BY

MAKOTO MATSUMOTO

We have many important papers of L. Berwald concerned with two-dimensional Finsler spaces [16, § 28]. In these papers he made the best use of the intrinsic frame field (l^i, m^i), where l^i is the normalized support element and m^i the unit vector orthogonal to l^i. In 1957 A. Moór [19] generalized Berwald's frame to three-dimensional Finsler spaces and introduced the intrinsic frame field (l^i, m^i, n^i), where l^i is as above, m the normalized torsion vector and n^i the unit vector orthogonal to l^i an m^i. The present author [13], [15], [16, § 29] developed a detailed theory of three-dimensional Finsler spaces by means of Moór's frame. Recently Moór [20] studied generalized Finsler spaces of four dimensions with use of an intrinsic frame field, although his method can not be applicable to the ordinary theory of Finsler spaces.

In 1975 Prof. R. Miron and the present author [17] gave a method to construct a frame field in a Finsler space of arbitrary dimension which satisfies certain condition. Such a Finsler space was called *strongly non-Riemannian*. At that time we were dubious about the adequacy of our naming, because it seemed to us that the condition was very weak and strongly non-Riemannian Finsler spaces of general dimension might be really non-Riemannian on the ordinary assumptions of the metric.

In the present paper it is shown that our naming was adequate and there exist non-Riemannian Finsler spaces which are not strongly non-Riemannian. For instance, the four-dimensional Finsler spaces with the so-called Randers metric [7], [21], [22], [24] which play the essential role in the theory of electromagnetic field is not Riemannian and not strongly non-Riemannian. According to Theorem 2 of the present paper, such physical spaces cannot be studied by the method to apply our intrinsic frame field.

§ 1. Definition of strongly non-Riemannian Finsler spaces. We consider an n-dimensional Finsler space F^n with the fundamental function $L(x, y)$, $(y^i = \dot{x}^i)$. Throughout the following, we restrict our discussion to a domain of TF^n where L is positive and the fundamental tensor $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j L^2/2$ is positive-definite.

From the fundamental function $L(x, y)$ of F^n we construct tensor fields of $(0, 2\alpha-1)$-type:

$$L_{i_1 i_2 \cdots i_{2\alpha-1}} = \frac{1}{2^\alpha}\dot\partial_{i_1}\dot\partial_{i_2}\cdots\dot\partial_{i_{2\alpha-1}}L^2 .$$

It is noted that these tensors are completely symmetric and, in particular, $L_i = Ll^i$ and $L_{ijk} = C_{ijk}$ in the usual notations.

Let $g^{ij}(x, y)$ be the contravariant fundamental tensor and put

$$\text{(1.1)}\qquad \begin{aligned} &L_{1)i} = L_i ,\\ &L_{\alpha)i} = L_{ij_1 j_2 \cdots j_{2\alpha-3} j_{2\alpha-2}}\, g^{j_1 j_2}\cdots g^{j_{2\alpha-3} j_{2\alpha-2}} , \quad (\alpha \geqq 2). \end{aligned}$$

The second vector $L_{2)i} = C_i (= C_{ijk}g^{jk})$ is nothing but the torsion vector. Thus $L_{3)i}$, $L_{4)i}$, ... may be called the *generalized torsion vectors*.

Definition. *If n-1 covariant vectors $L_{\alpha)i}$, $(\alpha = 1, \ldots, n-1)$, of F^n are linearly independent, then F^n is called strongly non-Riemannian.*

In a strongly non-Riemannian Finsler space F^n a local field of n-frame is defined by orthogonalizing $L_{1)i}, \ldots, L_{n-1)i}$ and adding more one unit vector orthogonal to them [17], [16, § 27].

In case of two-dimensional Finsler spaces we thus obtain the frame (l^i, m^i) of Berwald. Therefore any two-dimensional Finsler space is strongly non-Riemannian.

In case of three-dimensional Finsler spaces the equation $\lambda l_i + \mu C_i = 0$, $(\lambda, \mu) \neq (0, 0)$, implies $\lambda = 0$ and $C_i = 0$ immediately. Thus any three-dimensional Finsler space with $C_i \neq 0$ is strongly non-Riemannian. Reffer to Deicke's theorem [1].

§ 2. Condition for F^n not to be strongly non-Riemannian. We treat the first of generalized torsion vectors $L_{3)i}$ of an $n\,(\geqq 4)$-dimensional Finsler space F^n. From $C_i = g^{ab}C_{abi}$ and $\dot\partial_k g^{ab} = -2C_k^{ab} (= -2C_{ijk}\, g^{ai}g^{bj})$ it follows that

$$\text{(2.1)}\qquad \dot\partial_j C_i = g^{ab}\dot\partial_j C_{abi} - 2D_{ij} ,$$

where $D_{ij} = C_{abi}C_j^{ab}$, and from $C_j^{ab} = g^{ac}g^{bd}C_{jcd}$, we obtain

$$\dot\partial_k C_j^{ab} = g^{ac}g^{bd}\dot\partial_k C_{jcd} - 2C_k^{ac}C_j{}^b{}_c - 2C_j^{ac}C_k{}^b{}_c .$$

Therefore, from the definition of D_{ij} it follows that

$$\dot\partial_k D_{ij} = C_j^{ab}\dot\partial_k C_{abi} + C_i^{ab}\dot\partial_k C_{jab} - 4C_b{}^a{}_i C_j{}^b{}_c C_k{}^c{}_a .$$

Thus, differentiating (2.1) by y^k, we have

$$\text{(2.2)}\qquad \dot\partial_k\dot\partial_j C_i = -2\mathfrak{S}_{(ijk)}\{C_i^{ab}\dot\partial_j C_{kab}\} + g^{ab}\dot\partial_k\dot\partial_j C_{iab} + 8C_i{}^a{}_b C_j{}^b{}_c C_k{}^c{}_a ,$$

where the notation $\mathfrak{S}_{(ijk)}\{\ldots\}$ denotes, for instance,

$$\mathfrak{S}_{(ijk)}\{A^h{}_i B_{hjk}\} = A^h{}_i B_{hjk} + A^h{}_j B_{hki} + A^h{}_k B_{hij} .$$

Consequently the contraction of (2.2) by g^{jk} yields the following form of the generalized torsion vector $L_{3)i} = g^{ab}g^{jk}\dot\partial_k\dot\partial_j C_{abi}/2$:

$$\text{(2.3)}\qquad 2L_{3)i} = g^{jk}\dot\partial_k\dot\partial_j C_i + 4C^{abc}\dot\partial_i C_{abc} + 2g^{jk}C_i^{ab}\dot\partial_j C_{kab} - 8C_i^{ab}D_{ab} .$$

We now pay attention to the important tensor field

$$T_{hijk} = LC_{hij}|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij} .$$

Here we denote by $|_k$ the covariant differentiation by y^k. That is, for a ensor field X_{ij} of $(0, 2)$-type we have

$$X_{ij}|_k = \dot\partial_k X_{ij} - X_{aj}C_i{}^a{}_k - X_{ia}C_j{}^a{}_k .$$

The tensor field T_{hijk} was first introduced in 1972 almost simultaneously by H. Kawaguchi [9] and the present author [12]. It is noted that T_{hijk} is completely symmetric and $T_{hijk}\,y^k = 0$.

Putting $D = D_{ij}g^{ij}$ and $T_{ij} = T_{hijk}g^{hk}$, it is easily derived from the definition of T_{hijk} that

$$\text{(2.4)}\qquad LC^{abc}\dot\partial_i C_{abc} = T_{iabc}C^{abc} + 3LC_i^{ab}D_{ab} - Dl_i ,$$

$$\text{(2.5)}\qquad Lg^{jk}C_i^{ab}\dot\partial_j C_{kab} = T_{ab}C_i^{ab} + LC^a D_{ai} + 2LC_i^{ab}D_{ab} .$$

The symmetric tensor T_{ij} is written in the form

$$\text{(2.6)}\qquad T_{ij} = LC_i|_j + l_i C_j + l_j C_i .$$

Since $l_i|_j = L^{-1}h_{ij}$, it follows from (2.6) that

$$T_{ij}|_k = LC_i|_j|_k + L^{-1}(h_{ik}C_j + h_{jk}C_i) + L^{-1}\mathfrak{S}_{(ijk)}\{l_i T_{jk} - 2C_i l_j l_k\},$$

which yields immediately

$$\text{(2.7)}\qquad T_{ij}|_k\, g^{jk} = LC_i|_j|_k\, g^{jk} + L^{-1}((n-2)C_i + Tl_i),$$

where we put $T = T_{jk}g^{jk}$.

On the other hand we have

$$C_i|_j|_k = \dot\partial_k C_i|_j - C_a|_j C_i{}^a{}_k - C_i|_a C_j{}^a{}_k .$$

Then, in virtue of (2.6) the quantity $C_i|_j|_k\, g^{jk}$ appearing in (2.7) is written as

$$C_i|_j|_k\, g^{jk} = g^{jk}\dot\partial_k\dot\partial_j C_i - 2L^{-1}(C_i^{ab}T_{ab} + C^a T_{ai} - C^2 l_i) - C^a D_{ai} - C_{iab}C^a C^b ,$$

where $C^2 = g^{ij}C_i C_j$. Thus (2.7) is rewritten in the form

$$\text{(2.8)}\qquad \begin{aligned} Lg^{jk}\dot\partial_k\dot\partial_j C_i = {}& T^a{}_i|_a + 2C_i^{ab}T_{ab} + LC^a D_{ai} + LC_{iab}C^a C^b + \\ &+ 2C^a T_{ai} - L^{-1}(n-2)C_i - 2(C^2 + L^{-1}T)\,l_i . \end{aligned}$$

Finally, substituting from (2.4), (2.5) and (2.8) into (2.3), we establish the expression of $L_{3)i}$ as follows:

$$2L_{3)i} = L^{-1} T^a{}_{i|a} + 2L^{-1} C^a T_{ai} + 4L^{-1} C^{abc} T_{abci} + 3C^a D_{ai} +$$
$$(2.9) \qquad + (4L^{-1} T_{ab} + 8D_{ab} + C_a C_b) C_i{}^{ab} - L^{-2} (n-2) C_i -$$
$$- L^{-1} (2C^2 + 4D + L^{-1} T) l_i .$$

Because $L_{1)i} = Ll_i$ and $L_{2)i} = C_i$ appear in the last two terms of (2.9), the expression (2.9) of $L_{3)i}$ leads us to the following theorem:

Theorem 1. *Putting*

$$M_i = L^{-1} T^a{}_{i|a} + 2L^{-1} C^a T_{ai} + 4L^{-1} C^{abc} T_{abci} + 3C^a D_{ai} +$$
$$+ (4L^{-1} T_{ab} + 8D_{ab} + C_a C_b) C_i{}^{ab} .$$

(1) a four-dimensional Finsler space is strongly non-Riemannian, iff three vectors l_i, C_i and M_i are linearly independent; (2) a Finsler space of dimension $n \geqq 5$ is not strongly non-Riemannian, if l_i, C_i and M_i are linearly dependent.

The equation $\lambda l_i + \mu C_i + \nu M_i = 0$, $(\lambda, \mu, \nu) \neq (0, 0, 0)$, showing the linear dependence of l_i, C_i and M_i leads us immediately to $C_i = 0$ or

$$(2.10) \qquad M_i + L^{-2} T l_i = \gamma C_i ,$$

where γ is a scalar, because $T^a{}_{i|a} l^i = - L^{-1} T$ implies $\lambda + \nu(-L^{-1} T) = 0$.

§ 3. Finsler spaces with (α, β)-metric. In 1941 G. Randers [24] introduced the four-dimensional Finsler space with the fundamental function $L(x, y) = \alpha(x, y) + \beta(x, y)$, where $\alpha^2(x, dx) = a_{ij}(x) dx^i dx^j$ is a Riemannian metric and $\beta(x, dx) = b_i(x) dx^i$ is a differential 1-form. After then many physicists have been concerned with this Finsler space in various points of view. See [2], [7], [8], [10], [21], [22]. Recently we have several papers concerned with Randers' Finsler spaces from the standpoint of Finsler geometry. See [5], [6], [14], [25]. In 1972 the present author generalized this metric and introduced the concept of Finsler space with (α, β)-metric or generalized Randers space. We have some papers concerned with such spaces [3], [4], [23], [16, § 30].

The (α, β)*-metric* $L(\alpha, \beta)$ is by definition a function of the above α and β which is positively homogeneous of degree one in α, β.

The purpose of the present section is to show

Theorem 2. *Any Finsler space of dimension $n \geqq 4$ with (α, β)-metric is not strongly non-Riemannian.*

As a result of Theorem 2, it may happen a wrong situation in physical application of Finsler geometry, because any four-dimensional Finsler space with the Randers metric is not strongly non-Riemannian, so that we can not study the space applying the field of 4-frame in our sense.

The proof of Theorem 2 is straightforward; we don't refer to Theorem 1. In fact, putting $F(\alpha, \beta) = L^2(\alpha, \beta)/2$, we shall find $2L_{3)j} = g^{hi} g^{kl} \dot\partial_h \dot\partial_i \dot\partial_k \dot\partial_l \dot\partial_j F$. First it is seen that

$$(3.1) \qquad \alpha_i (= \dot\partial_i \alpha) = \alpha^{-1} Y_i , \quad \beta_i (= \dot\partial_i \beta) = b_i$$

where $Y_i = a_{ij} y^j$. Further, if the derivatives of α by y^i are denoted by putting subscripts, we obtain

$$(3.2) \qquad \alpha_{ij} = \alpha^{-1} (a_{ij} - \alpha^{-2} Y_i Y_j),$$

$$(3.3) \qquad \alpha_{ijk} = - \alpha^{-3} \mathfrak{S}_{(ijk)} \{a_{ij} Y_k\} + 3\alpha^{-5} Y_i Y_j Y_k ,$$

$$(3.4) \qquad \alpha_{hijk} = 3\alpha^{-5} \mathfrak{S}_6 \{a_{hi} Y_j Y_k\} - \alpha^{-3} \mathfrak{S}_{(ijk)} \{a_{hi} a_{jk}\} - 15\alpha^{-7} Y_h Y_i Y_j Y_k ,$$

$$(3.5) \qquad \alpha_{hijkl} = -15\alpha^{-7} \mathfrak{S}_{10} \{a_{hi} Y_j Y_k Y_l\} + 3\alpha^{-5} \mathfrak{S}_{15} \{Y_h a_{ij} a_{kl}\} + 105\alpha^{-9} Y_h Y_i Y_j Y_k Y_l ,$$

where $\mathfrak{S}_{10} \{a_{hi} Y_j Y_k Y_l\}$, for instance, denotes the summation of ten terms which are obtained from $a_{hi} Y_j Y_k Y_j$ by symmetrization, similar to $\mathfrak{S}_{(ijk)} \{\ldots\}$.

Next we have

$$\dot\partial_i F = F_\alpha \cdot \alpha_i + F_\beta \cdot \beta_i ,$$

$$\dot\partial_j \dot\partial_i F = F_{\alpha\alpha} \cdot \alpha_i \alpha_j + F_{\alpha\beta} (\alpha_i \beta_j + \alpha_j \beta_i) + F_{\beta\beta} \cdot \beta_i \beta_j + F_\alpha \cdot \alpha_{ij} ,$$

where subscripts α, β of F denote the differentiations by α, β respectively. After complicated computation it is seen that $2L_{3)j}$ is written as the linear combination of α_j, β_j and the following thirteen vectors:

$$(3.6) \qquad \gamma_j = \alpha^a \alpha_{aj} , \quad \delta_j = \beta^a \alpha_{aj} , \quad \gamma^a \alpha_{aj} , \quad \delta^a \alpha_{aj} ,$$

$$(3.7) \qquad g^{ab} \alpha_{abj} , \quad \alpha^a \alpha^b \alpha_{abj} , \quad \alpha^a \beta^b \alpha_{abj} , \quad \beta^a \beta^b \alpha_{abj} ,$$

$$(3.8) \qquad g^{ab} \alpha_{abc} \alpha^c{}_j , \quad g^{ab} \alpha_{acj} \alpha^c{}_b ,$$

$$(3.9) \qquad g^{ab} \alpha_{abcj} \alpha^c , \quad g^{ab} \alpha_{abcj} \beta^c ,$$

$$(3.10) \qquad g^{ab} g^{cd} \alpha_{abcdj} ;$$

where the raising of subscripts is done by g^{ij}.

It is known [11] that g^{ij} is written in the form

$$(3.11) \qquad g^{ij} = p^{-1} a^{ij} - s_0 B^i B^j - s_1 (B^i y^j + B^j y^i) - s_2 y^i y^j ,$$

where a^{ij} is the reciprocal of the Riemannian fundamental tensor a_{ij}, $B^i = a^{ij} b_j$ and p, s_0, s_1, s_2 are certain scalars.

It is first shown from (3.1) and (3.11) that

$$(3.12) \qquad \begin{aligned} \alpha^i &= \alpha^{-1} (p^{-1} - s_1 \beta - s_2 \alpha^2) y^i - \alpha^{-1} (s_0 \beta + s_1 \alpha^2) B^i , \\ \beta^i &= -(s_1 b^2 + s_2 \beta) y^i + (p^{-1} - s_0 b^2 - s_1 \beta) B^i , \end{aligned}$$

where $b^2 = a^{ij} b_i b_j$. Further it follows from (3.2) that

$$(3.13) \qquad y^i \alpha_{ij} = 0, \quad B^i \alpha_{ij} = \alpha^{-1} m_j ,$$

where $m_j = b_j - \alpha^{-2} \beta Y_j$. Therefore two vectors γ_j and δ_j of (3.6) are written as linear combinations of b_j and Y_j. Further it is seen from (3.11) that γ^i and δ^i are written in the forms

$$\gamma^i = \gamma B^i + \gamma' y^i, \qquad \delta^i = \delta B^i + \delta' y^i,$$

so that (3.13) leads us to

$$\gamma^a \alpha_{aj} = \alpha^{-1} \gamma m_j, \qquad \delta^a \alpha_{aj} = \alpha^{-1} \delta m_j .$$

Consequently four vectors of (3.6) belong to the 2-section (b_j, Y_j) spanned by b_j and Y_j.

Next we obtain from (3.3)

$$a^{ab} \alpha_{abj} = -(n-1)\alpha^{-3} Y_j, \qquad y^a y^b \alpha_{abj} = 0,$$

$$B^a y^b \alpha_{abj} = -\alpha^{-1} m_j, \qquad B^a B^b \alpha_{abj} = (3\alpha^{-5} \beta^2 - \alpha^{-3} b^2) Y_j - 2\alpha^{-3} \beta b_j .$$

Hence it is easy to show that the first vector of (3.7) belongs to the 2-section (b_j, Y_j). By means of (3.12) the analogous fact holds good as to the remaining three vectors of (3.7).

The similar way is applied to show that five vectors of (3.8), (3.9) and (3.10) also belong to the 2-section (b_j, Y_j). For example, it is easily verified that

$$a^{ab} \alpha_{abcj}, \qquad y^a y^b \alpha_{abcj}, \qquad B^a y^b \alpha_{abcj}, \qquad B^a B^b \alpha_{abcj}$$

are written as linear combinations of a_{cj}, $b_c b_j$, $b_c Y_j + b_j Y_c$ and $Y_c Y_j$, so that $g^{ab} \alpha_{abcj}$ are written in a similar form. Thus two vectors of (3.9) belong to the 2-section (b_j, Y_j).

Consequently it is proved that the generalized torsion vector $L_{2)j}$ belongs to the 2-section (b_j, Y_j). On the other hand, it has been known [11], [16, § 30] that both of l_j and C_j belong to the same 2-section. Therefore Theorem 2 is established.

§ 4. *C*-reducible Finsler spaces. In 1972 the present author [11], [16, § 30] found Finsler spaces with specially simple C_{ijk}. That is, a Finsler space F^n of dimension $n \geqq 3$ is called *C-reducible*, if C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j), \tag{4.1}$$

where C_i is the torsion vector and $h_{ij} = g_{ij} - l_i l_j$ is usually called the angular metric tensor. It is remarked that a *C*-reducible Finsler space with $C_i = 0$ is Riemannian, which is obvious without reference to Deicke's theorem [1].

Recently C. Shibata and the present author [18] showed interesting results as to *C*-reducible Finsler spaces with R_{hijk} of special forms. It has been shown [11] that a Finsler space with (α, β)-metric is *C*-reducible, iff the fundamental function $L(\alpha, \beta)$ is of the Randers type $(\alpha + \beta)$ or the Kropina type (α^2/β). It is obvious that the indicatrix of a Finsler space with (α, β)-metric of the Kropina type is rather strenge.

In the present section we are concerned with a *C*-reducible Finsler space of dimension $n \geqq 4$ which is not strongly non-Riemannian. For such

a space it is known [16, § 30] that the tensor T_{hijk} is of specially simple form

$$T_{hijk} = \frac{LC^2}{n^2-1}(h_{hi} h_{jk} + h_{hj} h_{ki} + h_{hk} h_{ij}), \tag{4.2}$$

where $C^2 = C_i|_j g^{ij}$. Therefore the tensors appearing in the form of M_i of Theorem 1 are written as follows:

$$C^{abc} T_{abci} = \frac{3LC^2}{n^2-1} C_i, \qquad T_{ai} = \frac{LC^2}{n-1} h_{ai},$$

$$C^a T_{ai} = \frac{LC^2}{n-1} C_i, \qquad T_{ab} C_i^{ab} = \frac{LC^2}{n-1} C_i,$$

$$T^a{}_i|_a = \frac{L}{n-1} C^2|_a h^a{}_i - C^2 l_i .$$

Since the scalar T is equal to LC^2, we have $C^2|_a = L^{-1} T|_a - L^{-2} Tl_a$. Because T is positively homogeneous of degree -1 in y^i, we have $T|_a h_a{}^i = T|_i + L^{-1} Tl_i$. Thus

$$T^a{}_i|_a = \frac{1}{n-1} T|_i - \frac{n-2}{n-1} L^{-1} Tl_i .$$

Further we obtain from (4.1)

$$C_a C_b C_i^{ab} = \frac{3C^2}{n+1} C_i, \qquad D_{ij} = \frac{1}{(n+1)^2}(2C^2 h_{ij} + (n+5) C_i C_j),$$

$$C^a D_{ai} = \frac{(n+7) C^2}{(n+1)^2} C_i, \qquad D_{ab} C_i^{ab} = \frac{(5n+17) C^2}{(n+1)^3} C_i .$$

Consequently (2.10) is written in the form

$$LT|_i + Tl_i = \delta C_i, \tag{4.3}$$

where δ is a certain scalar.

If we pay attention to the well-known equation $C_i = \partial_i \log \sqrt{g}$, (4.3) is rewritten in more interesting form as follows:

$$\dot{\partial}_i (LT) = \delta \dot{\partial}_i \log \sqrt{g}. \tag{4.4}$$

As a conclusion we have

Theorem 3. *A C-reducible Finsler space F^n of dimension $n \geqq 4$ is not strongly non-Riemannian, if (4.3) or (4.4) holds good, where L is the fundamental function, $T = T_{hijk} g^{hi} g^{jk}$ and $g = \det(g_{ij})$. The condition is necessary and sufficient for F^n not to be strongly non-Riemannian.*

In particular we are interested in the Randers metric $L = \alpha + \beta$. It is obvious that this metric satisfies the conditions that $L > 0$ and the fundamental tensor is positive-definite, under some weak assumptions. It is known [14] that the scalar LT in (4.4) is given by

$$LT = -\frac{n^2-1}{4L\alpha}(\alpha^2 b^2 + \beta^2 + 2\alpha\beta).$$

which implies

$$\dot{\partial}_i(LT) = \frac{n^2-1}{4L^2\alpha}(\alpha^2 b^2 - \beta^2 - 2\alpha\beta - 2\alpha^2)(b_i - \alpha^{-2}\beta Y_i).$$

On the other hand, it is also known [14] that

$$C_i = \frac{n+1}{2L}(b_i - \alpha^{-2}\beta Y_i).$$

Therefore (4.4) holds good, where $\delta = \dfrac{n-1}{2L\alpha}(\alpha^2 b^2 - \beta^2 - 2\alpha\beta - 2\alpha^2)$.

§ 5. Conclusions. It may be an inconvenient condition for physicists that our frames are useless in studying four-dimensional Finsler spaces FR^4 with the Randers metric, but geometrician's purpose of studying Finsler spaces with (α, β)-metric originally consists in considering Finsler spaces which are quite similar to Riemannian spaces in a sense, and then Theorem 2 ensures the similarity. Of course, there are other methods to introduce some intrinsic frame field in FR^4.

To construct simple and concrete examples of strongly non-Riemannian Finsler spaces of four dimensions, it will be sufficient to introduce Finsler spaces with some (α, β, γ)-metric, where α, β are as above and γ is more one differential 1-form independent of β; it is another matter whether such Finsler spaces will be useful or not for physicists.

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Institute of Mathematics
Yoshida College,
Kyoto University,
Japan