En utilisant (1.2), le lemme 2 et un résultat de E. E. Slutsky de [3] p. 282 section 20.6, et en supposant que les séries qui interviennent en (3.14) sont convergentes, on obtient

$$\sum_{i=i_0}^{+\infty} L\left(x\sqrt{\frac{i-1}{i+2}}\right) P\left(\frac{i}{s} < \nu \leqslant \frac{i+1}{s}\right) - \varepsilon \leqslant \liminf_{n\to+\infty} P\, Y_{\nu_n} < x\sqrt{\nu_n}) \leqslant \tag{3.15}$$
$$\leqslant \limsup_{n\to+\infty} P\,(Y_{\nu_n} < x\sqrt{\nu_n}) \leqslant \sum_{i=i_0}^{+\infty} L\left(x\sqrt{\frac{i+2}{i-1}}\right) P\left(\frac{i}{s} < \nu \leqslant \frac{i+1}{s}\right) + 2\varepsilon.$$

En tenant compte de (3.10) et (3.11) de (3.15), on arrive à :

$$(1-\varepsilon)\, L\,(x) \leqslant \liminf_{n\to+\infty} P\,(Y_{\nu_n} < x\sqrt{\nu_n}) \leqslant \limsup_{n\to+\infty} P\,(Y_{\nu_n} < x\sqrt{\nu_n}) \leqslant \tag{3.16}$$
$$\leqslant (1+\varepsilon)\, L\,(x) + 2\varepsilon.$$

Vu que ε est arbitraire, de (3.16) il résulte

$$\lim_{n\to+\infty} P(Y_{\nu_n} < x\sqrt{\nu_n}) = L\,(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{t^2}{2}\right) dt \tag{3.17}$$

et donc le théorème est démontré aussi dans ce cas.

Remarque 2. Le théorème 1 reste vrai si la condition (1.1) est remplacée par

$$\frac{\nu_n}{h(n)} \xrightarrow{P} \nu \;(n \to +\infty) \tag{3.18}$$

où $h\,(n)$ est une fonction arbitraire positive croissante tendant à l'infini pour $n \to +\infty$.

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ON WEAK SOLUTION OF LINEAR QUASI-STATIC VISCOELASTICITY

BY

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1. In [2], [3] we proved existence and uniqueness theorems of the weak solution for the equations of the linear quasi-static viscoelasticity by using a method of succesive approximations. Here we are pointing out some properties of these solutions which result from the method of demonstration mentioned above.

The first five Sections deal with the displacement boundary solution and the last two with the stress boundary solution. In Sec. 2 and 6 we give the definitions of weak viscoelastic solutions $\mathbf{u}_d^{(r)}$ and $\mathbf{u}_s^{(r)}$ and their weak elastic associated solutions $\mathbf{u}_d^{(e)}$ and $\mathbf{u}_s^{(e)}$. In some regards these Sections are a very short precis of [2], [3], strictly necessary for developing the paper. The limitation of $\mathbf{u}_d^{(r)}$ is obtained in Sec. 3 and the approximation of this solution by its elastic associated solution $\mathbf{u}_d^{(e)}$ is given in Sec. 4. Though these two results refer only to the zero displacement data, they hold also in the case of non-homogeneous displacement data on the boundary. We limited ourself to this case for the sake of unity of Sect. 2, 3, 4 and 5, because in Sec. 5 the continuity of the solution $\mathbf{u}_d^{(r)}$ is obtained only for the case of zero displacement data on the boundary. In Sec. 7 we give a bound for the weak solution $\mathbf{u}_s^{(r)}$ and an approximation of it by the weak elastic associated solution, $\mathbf{u}_s^{(e)}$.

In this paper we used the following notations:

— $\mathbf{L}_2(\Omega)$ — the Hilbert space of vector functions $\mathbf{u} \equiv (u_1, u_2, u_3)$ with $u_i \in L_2(\Omega)$, $i = 1, 2, 3$,

— $\mathbf{D}(\Omega)$ — the set of vector functions having continuous partial derivatives of all orders and compact support in Ω,

— $\mathbf{W}_2^k(\Omega)$ — the space of vector functions $\mathbf{u}$ with components u_i having generalized derivatives up to and including k-th order in $L_2(\Omega)$ [8], [7],

— $\mathring{\mathbf{W}}_2^k(\Omega)$ — the closure of $\mathbf{D}(\Omega)$ in $\mathbf{W}_2^k(\Omega)$,

— $\mathbf{W}_2^{-1}(\Omega)$ — the dual space of $\mathbf{W}_2^1(\Omega)$,

Let I be the time interval $[0, T]$, $T < +\infty$.

If $\mathbf{W}(\Omega)$ is one of the above functional spaces we note by $\mathbf{W}(\Omega, I)$ the linear space of measurable mappings of I onto $\mathbf{W}(\Omega)$ satisfying the condition [1], [2], [3]

$$\operatorname*{ess\,sup}_{t\in I} \| \mathbf{u}(t) \|_{W(\Omega)} < +\infty .$$

We also used the definitions of [7] for domains Ω of type $\mathfrak{N}^{(k),\alpha}$.

2. Let Ω be a three-dimensional domain of type $\mathfrak{N}^{(0),1}$ [7], and Γ its boundary, and let $\mathbf{x} \equiv (x_1, x_2, x_3)$ be cartesian orthogonal coordinates of the point $\mathbf{x}$.

Consider a viscoelastic solid defined by the constitutive relations

$$\sigma_{ij}(\mathbf{x}, t) = C_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x}, t) + \int_0^t G_{ijkl}(\mathbf{x}, t-\tau)\, u_{k,l}(\mathbf{x}, \tau)\, d\tau \tag{2.1}$$

occupying the domain Ω in the time interval $I = [0, T]$, $0 < T < \infty$. Here $\sigma_{ij}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t) \equiv (u_1(\mathbf{x}, t), u_3(\mathbf{x}, t), u_2(\mathbf{x}, t))$ denote the cartesian components of the stress tensor and the displacement vector, respectively, at the point $\mathbf{x} \in \Omega$ and time $t \in I$.

Throughout this paper, all the suffixes range over the values 1, 2, 3 and the usual convention of summing over repeated indices is adopted.

Suppose *the elasticities* $C_{ijkl}(\mathbf{x})$ and the ***relaxation moduli*** $G_{ijkl}(\mathbf{x}, t)$ satisfy the following conditions:

a) $C_{ijkl}(\mathbf{x})$ are essentially bounded functions on Ω.

b) $G_{ijkl}(\mathbf{x}, t)$ are essential bounded functions on Ω for every fixed $t \in I$ and uniformly continuous of $t \in I$ with respect to $\mathbf{x} \in \Omega \cup \Gamma$,

$C_{ijkl}(\mathbf{x})$ and $G_{ijkl}(\mathbf{x}, t)$ satisfy the usual symmetry conditions.

If $\mathbf{C}(\mathbf{x})$ and $\mathbf{G}(\mathbf{x}, t)$ are the symmetric 6×6 matrices of components $C_{ijkl}(\mathbf{x})$ and $G_{ijkl}(\mathbf{x}, t)$, we put

$$\operatorname*{ess\,sup}_{x\in\Omega} \{\operatorname{tr} \mathbf{C}^2(\mathbf{x})\}^{1/2} = \mathbf{A} < \infty, \quad \operatorname*{ess\,sup}_{\substack{x\in\Omega \\ t\in I}} \{\operatorname{tr} \mathbf{G}^2(\mathbf{x}, t)\}^{1/2} = B < \infty . \tag{2.2}$$

The material defined by the constitutive relations

$$\sigma^{(e)}_{ij}(\mathbf{x}, t) = C_{ijkl}(\mathbf{x})\, u_{k,l}(\mathbf{x}, t) \tag{2.3}$$

is called the ***elastic material associated to the viscoelastic material*** defined by (2.1).

Assuming the quadratic form of the elastic strain energy is positive definite, from Korn's inequality it follows [5], [1]:

$$\int_\Omega C_{ijkl}(\mathbf{x})\, u_{i,j}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)\, dx \geqslant a \int_\Omega \sum_{i,j=1} u^2_{i,j}(\mathbf{x}, t)\, dx, \tag{2.4}$$

where the constant $a > 0$ does not depend on t.

The system of equilibrium equations of viscoelastic body defined by (2.1) is

$$-\sigma_{ij,j}(\mathbf{x}, t) = F_i(\mathbf{x}, t), \tag{2.5}$$

where $\mathbf{F}(\mathbf{x}, t) \equiv (F_1(\mathbf{x}, t), F_2(\mathbf{x}, t), F_3(\mathbf{x}, t))$ is the body force.

Definition. *Let* $\mathbf{F}(\mathbf{x}, t) \in \mathbf{W}_2^{-1}(\Omega, I)$. *The function* $\mathbf{u}_d^{(r)} \in \mathring{\mathbf{W}}_2^1(\Omega, I)$ *is the weak solution of the displacement boundary value problem of the quasistatic linear viscoelasticity with zero displacement data, if for any* $\varphi(\mathbf{x}) \in \mathbf{D}(\Omega)$ *and* $t \in I$ *we have* [1], [2]

$$\int_\Omega \sigma_{ij}(\mathbf{x}, t)\, \varphi_{i,j}(\mathbf{x})\, dx = \int_\Omega F_i(\mathbf{x}, t)\, \varphi_i(\mathbf{x})\, dx. \tag{2.6}$$

3. Let

$$\mathbf{u}_d^{(e)} = \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(n)}, \ldots \tag{3.1}$$

be a sequence of functions of $\mathring{\mathbf{W}}_2^1(\Omega, I)$ defined by 1° and 2°:

1°. $\mathbf{u}_d^{(e)}$ is the weak elastic solution associated to the weak solution $\mathbf{u}_d^{(r)}$ [4]. This means that $\mathbf{u}_d^{(e)}$ is the weak solution of the system

$$-\sigma^{(e)}_{ij,j}(\mathbf{x}, t) = F_i(\mathbf{x}, t), \quad \mathbf{u}(\mathbf{x}, t)\big|_\Gamma = 0, \tag{3.2}$$

i.e. for any $\varphi \in \mathbf{D}(\Omega)$, we have (2.6) with $\sigma^{(e)}_{ij}(\mathbf{x}, t)$ instead of $\sigma_{ij}(\mathbf{x}, t)$

2°. $\mathbf{u}^{(n)}$, $n \geqslant 2$ are the weak solutions of the system

$$-[C_{ijkl}(\mathbf{x})\, u^{(n)}_{k,l}(\mathbf{x}, t)]_{,j} = F_i(\mathbf{x}, t) + \left[\int_0^t G_{ijkl}(\mathbf{x}, t-\tau)\, u^{(n-1)}_{k,l}(\mathbf{x}, \tau)\, d\tau\right]_{,j}, \tag{3.3}$$

$$\mathbf{u}^{(n)}(\mathbf{x}, t)\big|_\Gamma = 0,$$

i.e. for any $\varphi \in D(\Omega)$ we have

$$\int_\Omega C_{ijkl}(\mathbf{x})\, u^{(n)}_{k,l}(\mathbf{x}, t)\, \varphi_{i,j}(\mathbf{x})\, dx = \int_\Omega \Big[F_i(\mathbf{x}, t)\, \varphi_i(\mathbf{x}) - \int_0^t G_{ijkl}(\mathbf{x}, t-\tau)\, u^{(n-1)}_{k,l}(\mathbf{x}, \tau)\, \varphi_{i,j}(\mathbf{x})\, d\tau \Big] dx. \tag{3.4}$$

In [2] we have proved that the sequence (3.1) converges in $\mathring{\mathbf{W}}_2^1(\Omega, I)$ to the weak solution of the displacement boundary value problem of quasistatic linear viscoelasticity with zero displacement data. If $\mathbf{u}_d^{(r)}$ is this solution, then

$$\lim_{n\to\infty} \mathbf{u}^{(n)} = \mathbf{u}_d^{(r)}. \tag{3.5}$$

We shall obtain now a bound for the solution $\mathbf{u}_d^{(v)}$. First, for every $t \in I$, we have the limitation

$$a \, \| \mathbf{u}_d^{(e)}(t) \|_{\mathring{W}_2^1(\Omega)} \leqslant \| \mathbf{F}(t) \|_{W_2^{-1}(\Omega)}, \quad t \in I \tag{3.6}$$

and therefore

$$a \, \| \mathbf{u}_d^{(e)} \|_{\mathring{W}_2^1(\Omega, I)} \leqslant \| F \|_{W_2^{-1}(\Omega, I)}, \tag{3.7}$$

for the weak associated elastic solution $\mathbf{u}_d^{(e)}$ [1], [6], [7]. From the Lemma in Sec. 2 of [2], the definition of the norms in $\mathring{\mathbf{W}}_2^1(\Omega, I)$, $\mathring{\mathbf{W}}_2^{-1}(\Omega, I)$, and (3.6) for the weak solution $\mathbf{u}^{(n)}$ of the system (3.3) we have

$$a \, \| \mathbf{u}^{(n)}(t) \|_{\mathring{W}_2^1(\Omega)} \leqslant B \int_0^t \| \mathbf{v}_{(\tau)}^{(n-1)} \|_{\mathring{W}_2^1(\Omega)} \, d\tau + \| \mathbf{F} \|_{W_2^{-1}(\Omega, I)} \tag{3.8}$$

for every $t \in I$ and $n \geqslant 2$ [2]. Now by using (3.7), (3.8) we obtain

$$\| \mathbf{u}^{(n)} \|_{\mathring{W}_2^1(\Omega, I)} \leqslant \frac{1}{a} \left[\frac{D^{n-1}}{(n-1)!} + \frac{D^{n-2}}{(n-2)!} + \ldots + \frac{D}{1!} + 1 \right] \| \mathbf{F} \|_{W_2^{-1}(\Omega, I)}, \tag{3.9}$$

and therefore, as a consequence of (3.5), it results:

$$\| \mathbf{u}_d^{(v)} \|_{\mathring{W}_2^1(\Omega, I)} \leqslant \frac{1}{a} e^{D} \| \mathbf{F} \|_{W_2^{-1}(\Omega, I)} \tag{3.10}$$

where $D = (BT + a)/a$.

The inequality (3.10) expresses a limitation for the weak solution of the displacement boundary value problem of quasi-static linear viscoelasticity in terms of the intrinsic characteristics of viscoelastic material a and B, and the length T of the time interval. Also the inequality (3.10) shows that the solution $\mathbf{u}_d^{(v)}$ depends continuously upon body force $\mathbf{F}$ an the time interval I; indeed for every $\varepsilon > 0$, it results that $\| \mathbf{u}_d^{(v)} \|_{\mathring{W}^1(\Omega, I)} < \varepsilon$ as soon as $\| F \|_{W_2^{-1}(\Omega, I)} < \varepsilon$.

4. Consider non the sequence

$$\mathbf{u}^{(e)} = \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}, \ldots \tag{4.1}$$

where $\mathbf{v}^{(n)} = \mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}$, $n = 2, 3, \ldots$, $\mathbf{u}^{(n)}$ being the general term of the sequence (3.1); i.e. satisfying 1° and 2°.

From (3.3) it follows that $\mathbf{v}^{(n)}$, $n \geqslant 2$ is the weak solution of the system

$$-[C_{ijkl}(\mathbf{x}) v_{k,l}^{(n)}(\mathbf{x}, t)]_{,j} = \int_0^t [G_{ijkl}(\mathbf{x}, t-\tau) v_{k,l}^{(n-1)}(\mathbf{x}, \tau)]_{,j} \, d\tau, \quad \mathbf{v}^{(n)}(\mathbf{x}, t) \Big|_{\Gamma} = 0 \tag{4.2}$$

From (3.6), (3.7) and Lemma in Sec. 2 of [2] we have, for any $t \in I$ and $n \geqslant 2$,

$$\| \mathbf{v}(t) \|_{\mathring{W}_2^1(\Omega, I)} \leqslant M \int_0^t \| v_{(\tau)}^{(n-1)} \|_{\mathring{W}_2^1(\Omega)} \, d\tau, \tag{4.3}$$

where $M = B/a$.

From (4.3) taking into account (3.6) and (3.7), we get

$$\| \mathbf{v}^{(n)} \|_{\mathring{W}_2^1(\Omega, I)} \leqslant \frac{1}{a} \frac{(MT)^{n-1}}{(n-1)!} \| \mathbf{F} \|_{\mathring{W}_2^1(\Omega, I)}, \tag{4.4}$$

for any $n \geqslant 1$. Now let us estimate the quantity $\| \mathbf{u}^{(n)} - \mathbf{u}_d^{(e)} \|_{\mathring{W}_2^1(\Omega, I)}$.

By using (4.4) we obtain successively

$$\begin{aligned} \| \mathbf{u}^{(n)} - \mathbf{u}_d^{(e)} \|_{\mathring{W}_2^1(\Omega, I)} &\leqslant \| \mathbf{v}^{(n)} \|_{\mathring{W}_2^1(\Omega, I)} + \| \mathbf{v}^{(n-1)} \|_{\mathring{W}_2^1(\Omega, I)} + \ldots \\ + \| \mathbf{v}^{(2)} \|_{\mathring{W}_2^1(\Omega, I)} &\leqslant \frac{1}{a} \left[\frac{(MT)^{n-1}}{(n-1)!} \frac{(MT)^{n-2}}{(n-2)!} + \ldots + \frac{MT}{1!} \right] \| \mathbf{F} \|_{W_2^{-1}(\Omega, I)}. \end{aligned} \tag{4.5}$$

Hereof, because of (3.5) it results

$$\| \mathbf{u}_d^{(v)} - \mathbf{u}^{(e)} \|_{\mathring{W}_2^1(\Omega, I)} \leqslant \frac{1}{a} (e^{MT} - 1) \| \mathbf{F} \|_{W_2^{-1}(\Omega, I)}, \tag{4.6}$$

which gives an estimation for the difference between the weak solution of the displacement boundary value problem of the quasi-static linear viscoelasticity and its weak elastic associated solution with zero displacement data. The above relation lends itself to the following mechanical interpretations:

1°. If the „viscosity" B of the viscoelastic material is sufficiently small, the weak viscoelastic solution $\mathbf{u}_d^{(v)}$ can be sufficiently well approximated in the norm of $\mathbf{W}_2^1(\Omega, I)$ by its weak elastic associated solution $\mathbf{u}_d^{(e)}$ on a given time interval I.

2°. For a viscoelastic material with a given „viscosity" B, the viscoelastic weak solution $\mathbf{u}_d^{(v)}$ can be sufficiently well approximated in the norm of $\mathbf{W}(\Omega, I)$ by its weak elastic associated solution $\mathbf{u}_d^{(e)}$ as soon as the length T of the time interval I is sufficiently small.

5. In this section we shall prove the continuity of the weak solution $\mathbf{u}_d^{(v)}$. Firstly, we prove the

Lemma. *Let* $\mathcal{G}(x, t)$ *be the vector function of components*

$$\mathcal{G}_i(x, t) = \left[\int_0^t G_{ijkl}(\mathbf{x}, t-\tau) \, u_{k,l}(\mathbf{x}, \tau) \, d\tau \right]_{,j}. \tag{5.1}$$

If $G_{ijkl}(\mathbf{x}, t) \in C^1(\Omega)$ *for all* $t \in I$, *the partial derivatives of* $G_{ijkl}(\mathbf{x}, t)$ *are essentially bounded functions on* Ω *for every fixed* $t \in I$, *uniformly continuous of* $t \in I$ *with respect to* $\mathbf{x} \in \Omega \cup \Gamma$, $\mathbf{u}(x, t) \in \mathbf{W}_2^2(\Omega, I)$, *then* $\mathcal{G}(\mathbf{x}, t) \in L_2(\Omega, I)$ *and*

$$(5.2)\qquad \| \mathcal{G}(t) \|_{L_2(\Omega)} \leqslant CB \int_0^t \| \mathbf{u}(\tau) \|_{\mathring{W}_2^2(\Omega)} \, d\tau,$$

for any $t \in I$, where c is a constant.

Proof. From our hypotheses it results that the function $G(x, t)$ of components

$$G_i(\mathrm{x}, t - \tau) = [G_{ijkl}(\mathrm{x}, t - \tau)\, u_{k,l}(x, \tau)]_{,j} .$$

belongs to $L_2(\Omega)$ for any $t \in I$. Furthermore, we have

$$\| \mathbf{G}(t) \|^2_{\mathbf{L}_2(\Omega)} \leqslant 2 \max_{i,j,k,l} \{ \operatorname*{ess\,sup}_{\substack{x \in \Omega \\ t \in I}} (G^2_{ijkl}), \operatorname*{ess\,sup}_{\substack{x \in \Omega \\ t \in I}} (G^2_{ijkl,j}) \} \| \mathbf{u}(t) \|^2_{\mathring{W}_2^2(\Omega)}$$

and there by $\mathcal{G} \in \mathbf{L}_2(\Omega, I)$.

Here we made allowance for $\mathring{\mathbf{W}}_2^2(\Omega) \subset \mathring{\mathbf{W}}_2^1(\Omega)$ algebraically and topologically i.e. $\| \mathbf{u} \|_{\mathring{W}_2^1(\Omega)} \leqslant \text{const.} \| \mathbf{u} \|_{\mathring{W}_2^2(\Omega)}$ for any $\mathbf{u} \in \mathring{\mathbf{W}}_2^2(\Omega)$.

The inequality (5.2) results by a direct calculation. If $\varphi \in \mathbf{D}(\Omega)$ then for every $t \in I$ we shall have

$$\| \mathcal{G}(t) \|_{L_2(\Omega)} = \sup_{\|\varphi\|_{\mathring{W}_2^1(\Omega)} = 1} \int_\Omega \mathcal{G}_i(\mathrm{x}, t)\, \varphi_i(\mathrm{x})\, d\mathrm{x} \leqslant B \int_0^t \| \mathbf{u}(\tau) \|_{\mathring{W}_2^1(\Omega)} \, d\tau \leqslant$$

$$\leqslant \text{const.} \quad B \int_0^t \| \mathbf{u}(\tau) \|_{\mathring{W}_2^2(\Omega)} \, d\tau,$$

and the lemma is proved.

Till now, we have only assumed the system (2.3), (3.2), of the associated elastic equilibrium, is elliptic with respect to Dirichlet's problem [1], [6], i.e. it satisfies (2.4). Further we assume the system (2.3), (3.2) is strongly elliptic [6], [7], uniformly with respect to $t \in I$, i.e., it satisfies

$$(5.4)\qquad C_{ijkl}(\mathrm{x}, t)\, \xi_i\, \xi_k\, \eta_j\, \eta_l \geqslant a \left(\sum_{i=1}^{3} \xi_i^2 \right) \left(\sum_{i=1}^{3} \eta_i^2 \right),$$

for every $t \in I$ and any vectors $\xi \equiv (\xi_1, \xi_2, \xi_3)$, $\eta \equiv (\eta_1, \eta_2, \eta_3)$ where the constant a is positive.

Theorem. *If* a) *the body force* $\mathbf{F}(\mathrm{x}, t) \in \mathbf{L}_2(\Omega, I)$, b) *the elasticities* $C_{ijkl}(\mathrm{x}, t) \in C^1(\bar{\Omega})$ *for every* $t \in I$, c) *the relaxation moduli satisfy the conditions of the Lemma,* d) $\Omega \in \mathfrak{N}^{(1),1}$, *and* e) *the system* (2.3), (3.2) *satisfies* (2.4) *and* (5.4),

then the weak solution $\mathbf{u}_d^{(r)}$ *of the displacement boundary value problem of the quasi-static linear viscoelasticity with zero displacement data is a.e. continuous on* Ω *for every* $t \in I$.

Proof. From the hypotheses a), b) and d) it results that $\mathbf{u}^{(1)} = \mathbf{u}_d^{(e)}$, the solution of the system (2.3), (3.4) belongs to $\mathring{\mathbf{W}}_2^2(\Omega)$ for any $t \in I$, [1], [6], and it satisfies

$$(5.5)\qquad \| \mathbf{u}_d^{(e)}(t) \|_{W_2^2(\Omega)} \leqslant a \| \mathbf{F}(t) \|_{L_2(\Omega)}, \quad t \in I.$$

Reasoning as in [1] it results $\mathbf{u}_d^{(e)} \in \mathring{\mathbf{W}}_2^2(\Omega, I)$ and

$$(5.6)\qquad \| \mathbf{u}_d^{(e)} \|_{\mathring{W}_2^2(\Omega, I)} \leqslant a \| \mathbf{F} \|_{L_2(\Omega, I)} .$$

Using the hypothesis c) and Lemma it follows that the system (3.3) for $n = 2$ has the weak solution $\mathbf{u}^{(2)} \in \mathring{\mathbf{W}}_2^2(\Omega, I)$. By an inductive argument it follows that the sequence (3.1) belongs to $\mathring{\mathbf{W}}_2^2(\Omega, I)$ and therefore (4.1) belongs to $\mathring{\mathbf{W}}_2^2(\Omega, I)$.

For the solution $\mathbf{v}^{(n)}$ of the system (4.2) we have the estimation

$$(5.7)\qquad \| \mathbf{v}_{(t)}^{(n)} \|_{\mathring{W}_2^2(\Omega)} \leqslant acB \int_0^t \| \mathbf{v}^{(n-1)}(\tau) \|_{\mathring{W}_2^2(\Omega)} \, d\tau,$$

for any $n > 2$ and $t \in I$. In deriving (5.7), we used Lemma, (5.2), the hypothesis c), and the limitation of the form (5.5) for the solution of an elliptic system satisfying these conditions [1], [6], [7].

From (5.6) and (5.7) we get

$$(5.8)\qquad \| \mathbf{v}_{(t)}^{(n)} \|_{\mathring{W}_2^2(\Omega)} < a \frac{(acBt)^{n-1}}{(n-1)!} \| \mathbf{F} \|_{L_2(\Omega, I)} .$$

for every $t \in I$ and $n \geqslant 2$. Since the sequence (3.1) belongs to $\mathring{\mathbf{W}}_2^2(\Omega, I)$ by (5.8), it follows

$$(5.9)\qquad \| \mathbf{v}^{(n)} \|_{\mathring{W}_2^2(\Omega, I)} \leqslant a \frac{(acBT)^{n-1}}{(n-1)!} \| \mathbf{F} \|_{L_2(\Omega, I)}$$

for any $n \geqslant 1$. The inequalities (5.9) imply the convergence in $\mathring{\mathbf{W}}_2^2(\Omega)$ of the series $\sum_{i=1}^{\infty} \mathbf{v}^{(i)}$ and this entails the convergence in $\mathring{\mathbf{W}}_2^2(\Omega)$ of the sequence (3.1) which has the limit $\mathbf{u}_d^{(r)}$ belonging to $\mathring{\mathbf{W}}_2^2(\Omega, I)$. By the imbedding theorem of Sobolev, it results that $\mathbf{u}_d^{(r)}$ is continuous a.e. and

$$(5.10)\qquad \| \mathbf{u}_d^{(r)}(t) \|_{C(\Omega)} \leqslant \text{const} \| \mathbf{u}_d^{(r)}(t) \|_{\mathring{W}_2^2(\Omega)}$$

for all $t \in I$, and

$$(5.11)\qquad \| \mathbf{u}_d^{(r)} \|_{C(\Omega, I)} \leqslant \text{const} \| \mathbf{u}_d^{(r)} \|_{\mathring{W}_2^2(\Omega)}$$

where $\| \cdot \|_{C(\Omega, I)}$ is the uniform norm.

The Lemma and the Theorem are valid in less restrictive conditions; it is sufficient to assume that $C_{ijkl}(\mathrm{x})$ and $G_{ijkl}(\mathrm{x}, t)$ as functions of $\mathrm{x} \in \Omega$ satisfy the Lipschitz condition, uniformly with respect to $t \in I$.

6. We give now some preliminaries concerning the stress boundary value problem of the quasi-static linear viscoelasticity. We suppose $\Omega \in \mathfrak{N}^{(0),1}$. For such domains the unit normal vector $\mathbf{n}$ exists a.e. on Γ. We maintain the conditions a) and b) in Sec. 2 upon $C_{ijkl}(\mathrm{x})$ and $G_{ijkl}(\mathrm{x}, t)$. Let $\mathbf{Q}(\Omega)$ be the subset of functions $\varphi \in \mathbf{D}(\Omega)$ satisfying

$$\int_{\Omega} \varphi(\mathrm{x})\, d\mathrm{x} = 0, \int_{\Omega} \operatorname{rot} \varphi(\mathrm{x})\, d\mathrm{x} = 0. \tag{6.1}$$

Since on $\mathbf{Q}(\Omega)$ Korn's inequality is satisfied [1], [5], we have

$$\int_{\Omega} C_{ijkl}(\mathrm{x})\, \varphi_{i,j}(\mathrm{x})\, \varphi_{k,l}(\mathrm{x})\, d\mathrm{x} \geqslant a \int_{\Omega} \sum_{i,j=1}^{3} \varphi_{i,j}^{2}(\mathrm{x})\, d\mathrm{x}. \tag{6.2}$$

We denote by $\mathbf{H}(\Omega)$, the closure of $\mathbf{Q}(\Omega)$ in the norm induced by the scalar product [1]

$$(\mathbf{u}, \mathbf{v})_{\mathbf{H}(\Omega)} = \int_{\Omega} C_{ijkl}(\mathrm{x}, t)\, u_{i,j}(\mathrm{x}) u_{k,l}(\mathrm{x})\, d\mathrm{x}, \quad \mathbf{u}, \mathbf{v} \in \mathbf{Q}(\Omega), \tag{6.3}$$

and by $\mathbf{H}(\Omega, I)$ the linear space of measurable mappings of I onto $\mathbf{H}(\Omega)$ satisfying the condition

$$\operatorname*{ess\,sup}_{t \in I} \| \mathbf{u}(t) \|_{\mathbf{H}(\Omega)} < \infty.$$

$\mathbf{H}(\Omega, I)$ is a Banach space with the norm

$$\| \mathbf{u} \|_{\mathbf{H}(\Omega, I)} = \operatorname*{ess\,sup}_{t \in I} \| \mathbf{u}(t) \|_{\mathbf{H}(\Omega)}.$$

Obviously, any function from $H(\Omega, I)$ satisfies (6.1), (6.2), uniformly with respect to $t \in I$. From Poincaré's inequality [1], [5] and $(6.1)_1$, (6.2) it results $\mathbf{H}(\Omega, I) \subset \mathbf{W}_2^1(\Omega, I)$ algebraically and topologically, i.e. we have

$$\begin{aligned} &\| \mathbf{u}(t) \|_{\mathbf{W}_2^1(\Omega)} \leqslant \text{const.}\, \| \mathbf{u}(t) \|_{\mathbf{H}(\Omega)}, t \in I, \\ &\| \mathbf{u} \|_{\mathbf{W}_2^1(\Omega, I)} \leqslant \text{const.}\, \| \mathbf{u} \|_{\mathbf{H}(\Omega, I)}. \end{aligned} \tag{6.3}$$

Any element of $\mathbf{W}_2^1(\Omega, I)$ satisfying (6.1) belongs to $\mathbf{H}(\Omega, I)$ [1].

Suppose $\mathbf{F}(x, t) \in \mathbf{L}_2(\Omega, I)$ and $\mathbf{p}(x, t) \in \mathbf{L}_2(\Omega, I)$, the body force and the stress on the boundary, satisfy necessary equilibrium conditions

$$\int_{\Omega} \mathbf{F}(\mathrm{x}, t)\, d\mathrm{x} + \int_{\Gamma} \mathbf{p}(\mathrm{x}, t)\, d\mathrm{x} = 0, \int_{\Omega} \mathbf{r} \times \mathbf{F}(\mathrm{x}, t)\, d\mathrm{x} + \int_{\Gamma} \mathbf{F} \times \mathbf{p}(\mathrm{x}, t)\, d\mathrm{x} = 0 \tag{6.4}$$

for every $t \in I$. Here $\mathbf{r}$ is the position vector of a point x.

Definition. *The function* $\mathbf{u} \in \mathbf{W}_2^1(\Omega, I)$ *is the weak solution of the stress boundary value problem of the quasi-static linear viscoelasticity with the stress* $\mathbf{p}(x, t)$ *on the boundary, if for every* $t \in I$ *and* $\varphi \in \mathbf{Q}(\Omega)$ *we have* [1], [6], [7],

$$\int_{\Omega} \sigma_{ij}(\mathrm{x}, t)\, \varphi_{i,j}(\mathrm{x})\, d\mathrm{x} = \mathcal{F}(\varphi) \tag{6.5}$$

where $\sigma_{ij}(\mathrm{x}, t)$ *is given by* (2.1) *and*

$$\mathcal{F}(\varphi) = \int_{\Omega} F_i(\mathrm{x}, t)\, \varphi_i(\mathrm{x})\, d\mathrm{x} + \int_{\Gamma} \mathrm{p}(\mathrm{x}, t)\, \varphi_i(\mathrm{x})\, d\mathrm{x}. \tag{6.6}$$

We shall denote by $\mathbf{u}_s^{(r)}$ *this solution.*

Lemma. *For every* $\mathbf{u}^0 \in \mathbf{W}_2^1(\Omega, I)$ *satisfying* (6.1), *the system*

$$-\left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}(\mathrm{x}, t) \right]_{,j} = \left[\int_0^t G_{ijkl}(\mathrm{x}, t - \tau)\, u_{k,l}^0(\mathrm{x}, \tau)\, d\tau \right]_{,j} \tag{6.7}$$

has a unique weak solution $\mathbf{u} \in \mathbf{W}_2^1(\Omega, I)$ *satisfying* (6.1) *for every* $t \in I$ *and the condition*

$$\left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t - \tau)\, u_{k,l}^0(\mathrm{x}, \tau)\, d\tau \right] n_j = 0, \tag{6.8}$$

on Ω — *i.e. we have*

$$\int_{\Omega} \left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t - \tau)\, u_{k,l}^0(\mathrm{x}, \tau)\, d\tau \right] \varphi_{i,j}(\mathrm{x})\, d\mathrm{x} = 0 \tag{6.9}$$

for any $\varphi \in \mathbf{Q}(\Omega)$ *and* $t \in I$. *Furthermore, the weak solution satisfies the limitation*

$$\| \mathbf{u}(t) \|_{\mathbf{W}_2^1(\Omega)} \leqslant CM \int_0^t \| \mathbf{u}^0(\tau) \|_{\mathbf{W}_2^1(\Omega)}\, d\tau, \quad t \in I, \tag{6.10}$$

which entails

$$\| \mathbf{u} \|_{\mathbf{W}_2^1(\Omega, I)} \leqslant CMT \| \mathbf{u}^0 \|_{\mathbf{W}_2^1(\Omega, I)}, \tag{6.11}$$

where $M = B/a$.

Proof. Because the right hand side of (6.7) belongs to the dual space of $\mathbf{H}(\Omega, I)$, from Riesz's theorem it results the existence and the uniqueness of the solution. The measurability of the solution is obtained by reasoning as in [1]. Let us derive (6.10) and (6.11). As every function in $\mathbf{H}(\Omega, I)$ is the limit of a sequence of measurable and essentially bounded functions $\varphi : I \to \mathbf{Q}(\Omega)$ in the norm (6.3) and (6.9) holds for any such function, it results that the weak solution of the system (6.7) satisfies

$$\left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t - \tau)\, u_{k,l}(\tau)\, d\tau \right] u_{i,j}(\mathrm{x}, t)\, d\mathrm{x} = 0 \tag{6.12}$$

for every $t \in I$.

Now, taking into account (6.2), (6.3), and Cauchy's inequalities, we obtain (6.10), (6.11). We make mention of the following

Theorem [1], [5], [6]. *If* $\mathbf{F}(\mathrm{x}, t) \in \mathbf{L}_2(\Omega, I)$ *and* $\mathbf{p}(x, t) \in \mathbf{L}_2(\Omega, I)$ *satisfy* (6.4) *then the associated elastic stress boundary value problem*

$$(6.13) \qquad -\sigma_{ij,j}^{(e)}(\mathrm{x}, t) = F_i(\mathrm{x}, t), \qquad \sigma_{ij}^{(e)}(\mathrm{x}, t) = p_i(\mathrm{x}, t), \text{ on } \Gamma,$$

where $\sigma_{ij}^{(e)}(\mathrm{x}, t)$ *is given by* (2.3), *has a unique weak solution* $\mathbf{u}_s^{(e)}(x, t) \in W_2^1(\Omega, I)$ *satisfying* (6.4) *for every* $t \in I$ — *i.e.*

$$(6.14) \qquad \int_{\Omega} C_{ijkl}(\mathrm{x}, t)\, u_{k,l}(\mathrm{x}, t)\, \varphi_{i,j}(\mathrm{x})\, dx = \mathcal{F}(\varphi)$$

for any $\varphi \in \mathbf{Q}(\Omega)$ *and* $t \in I$. *The solution* $\mathbf{u}_s^{(e)}$ *satisfies*

$$(6.15) \qquad \| \mathbf{u}_s^{(e)}(t) \|_{\mathbf{W}_2^1(\Omega)} \leqslant c\, [\, \| \mathbf{F} \|_{\mathbf{L}_2(\Omega, I)} + \| \mathbf{p} \|_{\mathbf{L}_2(\Gamma, I)}],$$

c being a constant.

The solution $\mathbf{u}_s^{(e)}$ is called the weak elastic solution associated to the weak solution of the stress boundary value problem of the quasi-static linear viscoelasticity $\mathbf{u}_s^{(v)}$ [4].

7. In this section we shall derive results similar to (3.10) and (4.6) for the weak solution $\mathbf{u}_s^{(v)}$.

Let

$$(7.1) \qquad \mathbf{u}_s^{(e)} = \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(n)}, \ldots$$

be the sequence of functions from $\mathbf{W}_2^1(\Omega, I)$ satisfying (6.1), $\mathbf{u}^{(n)}$, $n \geqslant 2$, being the weak solution of the system

$$(7.2) \qquad \begin{gathered} -[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}^{(n)}(\mathrm{x}, t)]_{,j} = F_i(\mathrm{x}, t) + \left[\int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, u_{k,l}^{(n-1)}(\mathrm{x}, \tau)\, d\tau \right]_{,j}, \\ \left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}^{(n)}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, u_{k,l}^{(n-1)}(x, \tau)\, d\tau \right] n_j = p_i(\mathrm{x}, t) \text{ on } \Gamma. \end{gathered}$$

i.e. we have

$$(7.3) \qquad \int_{\Omega} \left[C_{ijkl}(\mathrm{x}, t)\, u_{k,l}^{(n)}(x, t) + \int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, u_{k,l}^{(n-1)}(\mathrm{x}, \tau)\, d\tau \right] \varphi_{i,j}(\mathrm{x})\, d\mathrm{x} = \mathcal{F}(\varphi),$$

for any $\varphi \in \mathbf{Q}(\Omega)$. In [3] we proved that (7.1) converges in the norm of $\mathbf{W}_2^1(\Omega, I)$ and its limit is $\mathbf{u}_s^{(v)}$ — i.e.

$$(7.4) \qquad \lim_{n \to \infty} \mathbf{u}^{(n)} = \mathbf{u}_s^{(v)}.$$

If we denote

$$(7.5) \qquad \mathbf{u}_s^{(e)} = \mathbf{v}^{(1)}, \quad \mathbf{v}^{(n)} = \mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}, \quad n \geqslant 2,$$

then, from (7.2) we get that $\mathbf{v}^{(n)}$, $n \geqslant 2$ is the weak solution of the system

$$(7.6) \qquad \begin{gathered} -[C_{ijkl}(\mathrm{x}, t)\, v_{k,l}^{(n)}(\mathrm{x}, t)]_{,j} = \left[\int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, v_{k,l}^{(n-1)}(\mathrm{x}, \tau)\, d\tau \right]_{,j}, \\ \left[C_{ijkl}(\mathrm{x}, t)\, v_{k,l}^{(n)}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, v_{k,l}^{(n-1)}(\mathrm{x}, \tau)\, d\tau \right] n_j = 0 \text{ on } \Gamma. \end{gathered}$$

i.e. for any $\varphi \in \mathbf{Q}(\Omega)$, the relation :

$$(7.7) \qquad \int_{\Omega} \left[C_{ijkl}(\mathrm{x}, t)\, v_{k,l}^{(n)}(\mathrm{x}, t) + \int_0^t G_{ijkl}(\mathrm{x}, t-\tau)\, v_{k,l}^{(n)}(\mathrm{x}, \tau)\, d\tau \right] \varphi_{i,j}(\mathrm{x})\, d\mathrm{x} = 0$$

holds. The relation (6.1) is satisfied by any $\mathbf{v}^{(n)}$ of (7.5). By using the Lemma in Sec. 6 and the inequalities (6.10), (6.15) we have

$$(7.8) \qquad \| \mathbf{v}^{(n)}(t) \|_{\mathbf{W}_2^1(\Omega)} \leqslant cM \int_0^t \| \mathbf{v}_{(\tau)}^{(n-1)} \|_{\mathbf{W}_2^1(\Omega)}\, d\tau, \quad n \geqslant 2,$$

and therefore

$$(7.9) \qquad \begin{gathered} \| \mathbf{v}^{(n)}(t) \|_{\mathbf{W}_2^1(\Omega)} \leqslant c\, \frac{(CMt)^{n-1}}{(n-1)!} [\, \| \mathbf{F} \|_{\mathbf{L}_2(\Omega, I)} + \| \mathbf{p} \|_{\mathbf{L}_2(\Gamma, I)}], \\ \| \mathbf{v}^{(n)} \|_{\mathbf{W}_2^1(\Omega, I)} \leqslant c\, \frac{(CMT)^{n}}{(n-1)!} [\, \| \mathbf{F} \|_{\mathbf{L}_2(\Omega, I)} + \| \mathbf{p} \|_{\mathbf{L}_2(\Gamma, I)}, \end{gathered}$$

where $M = B/a$.

If we proceed as for obtaining (4.6) we get here

$$(7.10) \qquad \| \mathbf{u}_s^{(v)} - \mathbf{u}_s^{(e)} \|_{\mathbf{W}_2^1(\Omega, I)} \leqslant c\, (e^{cMT} - 1)\, [\, \| \mathbf{F} \|_{\mathbf{L}_2(\Omega, I)} + \| p \|_{\mathbf{L}_2(\Gamma, I)}],$$

which gives an estimation for $\mathbf{u}_s^{(v)} - \mathbf{u}_s^{(e)}$ in the norm of $\mathbf{W}_2^1(\Omega, I)$.

The relation (7.10) has the following mechanical interpretations.

1°. If the „viscosity" B of the viscoelastic material is sufficiently small, the weak viscoelastic solution $\mathbf{u}_s^{(v)}$ can be sufficiently well approximated by the weak elastic solution $\mathbf{u}_s^{(e)}$ in the norm of $\mathbf{W}_2^1(\Omega, I)$.

2°. For a viscoelastic body with a given „viscosity" B, the weak elastic solution $\mathbf{u}_s^{(e)}$ approximates sufficiently well the weak viscoelastic solution $\mathbf{u}_s^{(v)}$ if the length T of the interval I is sufficiently small.

Without difficulties we can obtain the limitation

$$(7.11) \qquad \| \mathbf{u}_s^{(v)} \|_{\mathbf{W}_2^1(\Omega, I)} \leqslant \frac{c}{a} e^{D}\, [\, \| \mathbf{F} \|_{\mathbf{L}_2(\Omega, I)} + \| \mathbf{p} \|_{\mathbf{L}_2(\Gamma, I)}]$$

similar to (3.10), where $D = (BT + a)/a$. The inequality has the significance and the interpretation similar to (3.10)

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THEORY OF LOADED ELASTIC CYLINDERS

BY

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1. Introduction. Since the appearence of Saint-Venant's celebrated memoirs [14], [15] an immeasurable amount of effort by many investigators has been devoted to the problem which is known nowadays as Saint-Venant's problem. Aspects of the aforementioned problem have been studied extensively.

Most of the papers concerned with Saint-Venant's problem are restricted to homogeneous or piecewise homogeneous cylinders. Nowinski and Turski [11] have considered the case of an isotropic and inhomogeneous cylinder where the elastic coefficients are independent of the axial coordinate, they being prescribed functions of the remaining coordinates. This problem was later studied by Radu [12], [13] and it was entirely solved when the Poisson's ratio is constant. A method to solve Saint-Venant's problem which avoids restrictions of this type was given by Ieșan [7], [8]. In [9] was solved Saint-Venant's problem when the cross-section is occupied by different inhomogeneous and anisotropic elastic materials. The case of piecewise homogeneous cylinders was treated by Borș [2].

In the present paper we consider a more general problem assuming that the cylinder is subject to body forces and to surface tractions on the lateral surface and to appropriate stress resultants over its ends.

The theory of loaded cylinders was initiated by Almansi [1] and Michell [10] and it was developed in various papers (see e.g. Borș [2], [3], Ieșan [6], Chiriță [4]).

2. Statement of the Problem. Throughout this paper $\mathcal{R}$ denotes the interior of a right cylinder of length l with the generic cross-section Σ and the lateral surface B. We use a rectangular Cartesian coordinate system Ox_k $(k = 1, 2, 3)$. This system is chosen such that the x_3-axis is parallel to the generators of $\mathcal{R}$ and x_1Ox_2-plane contains one of terminal sections. We call $\Sigma^{(0)}$ the cross-section located at $x_3 = 0$ and $\Sigma^{(l)}$ the cross-section which lies in the plane $x_3 = l$. We denote by L the boundary of the generic cross-section Σ.