similar to (3.10), where $D = (BT + a)/a$. The inequality has the significance and the interpretation similar to (3.10)

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THEORY OF LOADED ELASTIC CYLINDERS

BY

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1. Introduction. Since the appearence of Saint-Venant's celebrated memoirs [14], [15] an immeasurable amount of effort by many investigators has been devoted to the problem which is known nowadays as Saint-Venant's problem. Aspects of the aforementioned problem have been studied extensively.

Most of the papers concerned with Saint-Venant's problem are restricted to homogeneous or piecewise homogeneous cylinders. Nowinski and Turski [11] have considered the case of an isotropic and inhomogeneous cylinder where the elastic coefficients are independent of the axial coordinate, they being prescribed functions of the remaining coordinates. This problem was later studied by Radu [12], [13] and it was entirely solved when the Poisson's ratio is constant. A method to solve Saint-Venant's problem which avoids restrictions of this type was given by Ieşan [7], [8]. In [9] was solved Saint-Venant's problem when the cross-section is occupied by different inhomogeneous and anisotropic elastic materials. The case of piecewise homogeneous cylinders was treated by Borş [2].

In the present paper we consider a more general problem assuming that the cylinder is subject to body forces and to surface tractions on the lateral surface and to appropriate stress resultants over its ends.

The theory of loaded cylinders was initiated by Almansi [1] and Michell [10] and it was developed in various papers (see e.g. Borş [2], [3], Ieşan [6], Chiriţă [4]).

2. Statement of the Problem. Throughout this paper $\mathcal{R}$ denotes the interior of a right cylinder of length l with the generic cross-section Σ and the lateral surface B. We use a rectangular Cartesian coordinate system Ox_k $(k = 1, 2, 3)$. This system is chosen such that the x_3-axis is parallel to the generators of $\mathcal{R}$ and x_1Ox_2-plane contains one of terminal sections. We call $\Sigma^{(0)}$ the cross-section located at $x_3 = 0$ and $\Sigma^{(l)}$ the cross-section which lies in the plane $x_3 = l$. We denote by L the boundary of the generic cross-section Σ.

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to run over the integers (1, 2), whereas Latin subscripts — unless otherwise specified — to run over (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

In this paper we consider the linear theory of classical elasticity. Let u_i denote the components of the displacement vector field. The components of the infinitesimal strain field are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

The stress-strain relations in the case of an anisotropic elastic medium are

$$t_{ij} = C_{ijkl}\, e_{kl}, \tag{2.2}$$

where t_{ij} are the components of the stress tensor and C_{ijkl} are the components of the elasticity tensor which obey the symmetry relations

$$C_{ijkl} = C_{jikl} = C_{klij}. \tag{2.3}$$

As a consequence of the relations (2.1), (2.3) the stress-strain relations take the form

$$t_{ij} = C_{ijkl}\, u_{k,l}. \tag{2.4}$$

We assume that Σ is a C^1-smooth domain (Fichera [5]). Let L_1 and L_2 be two disjoint subsets of L, such that $L = L_1 \cup L_2$. Let Γ be a curve contained in Σ satisfying the condition that $L_\rho \cup \Gamma$ is the boundary of a domain Σ_ρ contained in Σ, such that $\Sigma_1 \cap \Sigma_2 = \varnothing$. Suppose that Σ_1 and Σ_2 are occupied by two elastic materials. Let $C_{ijkl}^{(\rho)}$ be the elastic coefficients relative to Σ_ρ.

We denote by $\mathcal{R}_\rho$, the domain occupied by the material with the elastic coefficients $C_{ijkl}^{(\rho)}$.

Throughout this paper we assume that

$$C_{ijkl}^{(\rho)} = C_{ijkl}^{(\rho)}(x_1, x_2) \text{ in } \mathcal{R}_\rho. \tag{2.5}$$

The functions $C_{ijkl}^{(\rho)}$ are supposed to belong to C^∞, and the elastic potential corresponding to the body which occupies $\mathcal{R}_\rho$ is assumed to be a positive definite quadratic form.

We can consider Σ as being occupied by an elastic medium, which has the elastic coefficients, in general, discontinuous along Γ.

The equations of equilibrium are

$$t_{ji,j} + f_i^{(\rho)} = 0, \text{ in } \mathcal{R}_\rho, \tag{2.6}$$

where $f_i^{(\rho)}$ are the components of body force vector.

The displacement vector and the stress vector must be continuous in passing from one medium to another so that we have the conditions

$$[u_i]_1 = [u_i]_2, \quad [t_{i\beta}]_1 \nu_\beta = [t_{i\beta}]_2 \nu_\beta \text{ on } \Gamma, \tag{2.7}$$

where we have indicated that the expressions between paranthesis are calculated for the domains Σ_1 and Σ_2, respectively, and ν_β are the direction cosines of the normal vector to Γ and outward to Σ_1.

On the lateral surface of the cylinder we consider the following conditions

$$[t_{i\alpha}\, n_\alpha]_\rho = \tilde{t}_i^{(\rho)}, \text{ on } B_\rho, \tag{2.8}$$

where B_ρ are subsets of B corresponding to the two materials and n_α are the direction cosines of the exterior normal to B.

As in [1], we assume that the body forces and the tractions applied on the lateral surface are polynomials of r-degree in the axial coordinate x_3, namely

$$f_i^{(\rho)} = \sum_{k=0}^{r} F_{ik}^{(\rho)}(x_1, x_2)\, x_3^k, \text{ in } \mathcal{R}_\rho, \tag{2.9}$$

$$\tilde{t}_i^{(\rho)} = \sum_{k=0}^{r} p_{ik}^{(\rho)}(x_1, x_2)\, x_3^k, \text{ on } B_\rho, \tag{2.10}$$

where $F_{ik}^{(\rho)}$ and $p_{ik}^{(\rho)}$ are prescribed functions which are supposed to belong to C^∞.

Let the loading applied on $\Sigma^{(0)}$ be statically equivalent to a force $R\,(R_i)$ and a moment $M\,(M_i)$. On $\Sigma^{(l)}$ there are applied tractions in a way which fulfills the equilibrium conditions of a rigid body. Thus, for $x_3 = 0$, we have the following conditions

$$\int_\Sigma t_{\alpha 3}\, d\sigma = -R_\alpha, \tag{2.11}$$

$$\int_\Sigma t_{33}\, d\sigma = -R_3, \tag{2.12}$$

$$\int_\Sigma x_\alpha t_{33}\, d\sigma = \varepsilon_{\alpha\beta 3} M_\beta, \tag{2.13}$$

$$\int_\Sigma \varepsilon_{\alpha\beta 3}\, x_\alpha t_{\beta 3}\, d\sigma = -M_3, \tag{2.14}$$

where ε_{ijk} is the alternating symbol.

The problem consists in the determination of a solution of the equations (2.4), (2.6) which satisfies the conditions (2.7), (2.8), (2.11)–(2.14) where $C_{ijkl}^{(\rho)}$, $f_i^{(\rho)}$ and $\tilde{t}_i^{(\rho)}$ are given by (2.5), (2.9), (2.10), respectively.

Let us denote by (A) the problem of determination of a solution of equations (2.4), (2.6) with the body force

$$f_i^{(\rho)} = F_{in}^{(\rho)}(x_1, x_2)\, x_3^n, \tag{2.15}$$

which satisfies the conditions (2.7), (2.11)–(2.14) and

$$[t_{i\alpha}\, n_\alpha]_\rho = p_{in}^{(\rho)}(x_1, x_2)\, x_3^n, \text{ on } B_\rho. \tag{2.16}$$

In (2.15) and (2.16) n is a positive integer or zero and the functions $F_{in}^{(\rho)}$, $p_{in}^{(\rho)}$ are prescribed.

If we know the solution of the problem (A) for any n then, according to the linearity of the equations, we can determine the solution of the initial problem.

We denote by $B^{(0)}$ the problem (A) for $n=0$ and by $B^{(s)}$ the problem (A) when $n=s$ $(s=1,\dots,r)$ and $R_i = M_i = 0$. If the components of the displacement vector from the problem $B^{(k)}$ $(k=0,1,\dots,r)$ are u_{ik} then the components of the displacement vector of the initial problem are given by

$$u_i = \sum_{k=0}^{r} u_{ik}. \tag{2.17}$$

Thus, to solve the initial problem, we use the method of induction. In Section 4 we solve the problem $B^{(0)}$. In Section 5 we shall establish the solution of the problem $B^{(n+1)}$ when the solution of the problem $B^{(n)}$ (with $R_i = M_i = 0$) is known.

3. Auxiliary Plane Strain Problems. In what follows we will have occasion to use four special problems $P^{(s)}$ $(s=1,2,3,4)$ of generalized plane strain. We denote by $v_i^{(s)}$, $\sigma_{ij}^{(s)}$ the components of the displacement vector and the components of the stress tensor from the problem $P^{(s)}$. The problems $P^{(s)}$ are characterized by the equations

$$\sigma_{i\alpha}^{(s)} = C^{(\rho)}_{i\alpha k\beta}\, v^{(s)}_{k,\beta}, \tag{3.1}$$

$$\begin{aligned} &\sigma^{(\beta)}_{i\alpha,\alpha} + (C^{(\rho)}_{i\alpha 33} x_\beta)_{,\alpha} = 0, \qquad (\beta = 1, 2),\\ &\sigma^{(3)}_{i\alpha,\alpha} + C^{(\rho)}_{i\alpha 33,\alpha} = 0, \qquad \sigma^{(4)}_{i\alpha,\alpha} - \varepsilon_{\gamma\beta 3}(C^{(\rho)}_{i\alpha\gamma 3} x_\beta)_{,\alpha} = 0, \ \text{in } \Sigma_\rho, \end{aligned} \tag{3.2}$$

and the following conditions

$$\begin{aligned} &\sigma^{(\beta)}_{i\alpha} n_\alpha = - C^{(\rho)}_{i\alpha 33} x_\beta n_\alpha, \qquad (\beta = 1, 2),\\ &\sigma^{(3)}_{i\alpha} n_\alpha = - C^{(\rho)}_{i\alpha 33} n_\alpha, \qquad \sigma^{(4)}_{i\alpha} n_\alpha = \varepsilon_{\gamma\beta 3} C^{(\rho)}_{i\alpha\gamma 3} x_\beta n_\alpha, \ \text{on } L_\rho, \end{aligned} \tag{3.3}$$

$$[v^{(s)}_i]_1 = [v^{(s)}_i]_2, \quad [\sigma^{(s)}_{i\alpha}]_1 \nu_\alpha = [\sigma^{(s)}_{i\alpha}]_2 \nu_\alpha + g^{(s)}_i, \ \text{on } \Gamma, \tag{3.4}$$

where

$$\begin{aligned} &g^{(\beta)}_i = [C^{(2)}_{i\alpha 33} - C^{(1)}_{i\alpha 33}]\, x_\beta \nu_\alpha, \qquad g^{(4)}_i = \varepsilon_{\gamma\beta 3}[C^{(1)}_{i\alpha\gamma 3} - C^{(2)}_{i\alpha\gamma 3}]\, x_\beta \nu_\alpha.\\ &g^{(3)}_i = [C^{(2)}_{i\alpha 33} - C^{(1)}_{i\alpha 33}]\, \nu_\alpha, \end{aligned} \tag{3.5}$$

It is known Ieşan [9] that the necessary and sufficient conditions for the existence of the solution are satisfied for each boundary value problem $P^{(s)}$. In what follows we assume that the functions $v_i^{(s)}$, $\sigma_{ij}^{(s)}$ are known.

4. Theory of Uniformly Loaded Cylinders. We assume that the body forces have the form

$$f_i^{(\rho)} = G_i^{(\rho)}(x_1, x_2), \ \text{in } \mathcal{R}_\rho, \tag{4.1}$$

and that the conditions on the lateral surface are

$$[t_{i\alpha} n_\alpha]_\rho = p_i^{(\rho)}(x_1, x_2), \ \text{on } B_\rho. \tag{4.2}$$

In this section we establish a solution of the equations (2.4), (2.6) which satisfies the conditions (2.7), (4.2), (2.11) – (2.14), when the body forces are given by (4.1).

We look for a solution in the form

$$\begin{aligned} u_\alpha = &-\frac{1}{2} a_\alpha x_3^2 - \frac{1}{6} b_\alpha x_3^3 - \frac{1}{24} c_\alpha x_3^4 - \varepsilon_{3\alpha\beta} x_\beta \left(a_4 x_3 + \frac{1}{2} b_4 x_3^2 + \frac{1}{6} c_4 x_3^3\right) +\\ &+ \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2} c_s x_3^2\right) v_\alpha^{(s)} + w_\alpha(x_1, x_2) + x_3 v_\alpha(x_1, x_2),\\ u_3 = &(a_1 x_1 + a_2 x_2 + a_3) x_3 + \frac{1}{2}(b_1 x_1 + b_2 x_2 + b_3) x_3^2 + \frac{1}{6}(c_1 x_1 +\\ &c_2 x_2 + c_3) x_3^3 + \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2} c_s x_3^2\right) v_3^{(s)} + w_3(x_1, x_2) + x_3 v_3(x_1, x_2), \end{aligned} \tag{4.3}$$

where $v_i^{(s)}$ are the components of the displacement vectors from the auxiliary plane strain problems considered in Section 3, w_i, v_i are unknown functions and a_s, b_s, c_s $(s = 1, 2, 3, 4)$ are unknown constants.

Using (2.4), from (4.3) we obtain

$$\begin{aligned} t_{ij} = C^{(\rho)}_{ij33}\Big[&(a_1 x_1 + a_2 x_2 + a_3) + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \frac{1}{2}(c_1 x_1 +\\ &c_2 x_2 + c_3) x_3^2\Big] - C^{(\rho)}_{ij\alpha 3}\, \varepsilon_{\alpha\beta 3} x_\beta \left(a_4 + b_4 x_3 + \frac{1}{2} c_4 x_3^2\right) + \sum_{s=1}^{4}\Big(a_s + b_s x_3 +\\ &+ \frac{1}{2} c_s x_3^2\Big)\sigma^{(s)}_{ij} + k^{(0)}_{ij} + x_3 k^{(1)}_{ij} + \tau_{ij} + x_3 s_{ij}, \end{aligned} \tag{4.4}$$

where

$$\tau_{ij} = C^{(\rho)}_{ijk\alpha} w_{k,\alpha}, \tag{4.5}$$

$$s_{ij} = C^{(\rho)}_{ijk\alpha} v_{k,\alpha}, \tag{4.6}$$

$$k^{(0)}_{ij} = C^{(\rho)}_{ijk3} v_k + \sum_{s=1}^{4} b_s C^{(\rho)}_{ijk3} v^{(s)}_k, \quad k^{(1)}_{ij} = \sum_{s=1}^{4} c_s C^{(\rho)}_{ijk3} v^{(s)}_k. \tag{4.7}$$

The relations (4.4) can be written in the form

$$t_{ij} = \sum_{s=1}^{4}\left(a_s + b_s x_3 + \frac{1}{2} c_s x_3^2\right) t^{(s)}_{ij} + k^{(0)}_{ij} + x_3 k^{(1)}_{ij} + \tau_{ij} + x_3 s_{ij}, \tag{4.8}$$

where

$$t^{(\alpha)}_{ij} = C^{(\rho)}_{ij33} x_\alpha + \sigma^{(\alpha)}_{ij}, \quad t^{(3)}_{ij} = C^{(\rho)}_{ij33} + \sigma^{(3)}_{ij}, \quad t^{(4)}_{ij} = -C^{(\rho)}_{ij\alpha 3}\, \varepsilon_{\alpha\beta 3} x_\beta + \sigma^{(4)}_{ij}. \tag{4.9}$$

In view of the relations (3.2) the equilibrium equations are reduced to

$$\tau_{i\alpha,\alpha} + k^{(0)}_{i\alpha,\alpha} + \sum_{s=1}^{4} b_s t^{(s)}_{i3} + s_{i3} + k^{(1)}_{i3} + G_i^{(\rho)} = 0, \tag{4.10}$$

$$s_{i\alpha,\alpha} + k^{(1)}_{i\alpha,\alpha} + \sum_{s=1}^{4} c_s t^{(s)}_{i3} = 0, \text{ in } \Sigma_\rho . \tag{4.11}$$

The conditions (2.7) are satisfied if

$$[w_i]_1 = [w_i]_2 , \quad [\tau_{i\alpha}]_1 \nu_\alpha = [\tau_{i\alpha}]_2 \nu_\alpha + \chi^{(0)}_i , \tag{4.12}$$

$$[v_i]_1 = [v_i]_2 , \quad [s_{i\alpha}]_1 \nu_\alpha = [s_{i\alpha}]_2 \nu_\alpha + \chi^{(1)}_i , \text{ on } \Gamma, \tag{4.13}$$

where

$$\chi^{(\zeta)}_i = \{[k^{(\zeta)}_{i\alpha}]_2 - [k^{(\zeta)}_{i\alpha}]_1\} \nu_\alpha , \quad (\zeta = 0, 1). \tag{4.14}$$

From the conditions on the lateral surface (4.2) we obtain the following boundary conditions:

$$[\tau_{i\alpha} n_\alpha]_\rho = P^{(\rho)}_i , \tag{4.15}$$

$$[s_{i\alpha} n_\alpha]_\rho = Q^{(\rho)}_i , \text{ on } L_\rho , \tag{4.16}$$

where

$$P^{(\rho)}_i = p^{(\rho)}_i - k^{(0)}_{i\alpha} n_\alpha , \quad Q^{(\rho)}_i = - k^{(1)}_{i\alpha} n_\alpha . \tag{4.17}$$

Let us consider the boundary value problem (4.5), (4.10), (4.12), (4.15). The necessary and sufficient conditions for the existence of the solution of this problem reduce to

$$\begin{gathered} \sum_{\rho=1}^{2} \int_{L_\rho} p^{(\rho)}_i \, ds + \sum_{\rho=1}^{2} \int_{\Sigma_\rho} G^{(\rho)}_i \, d\sigma + \int_{\Sigma} (t_{i3,3})_{x_3=0} \, d\sigma = 0, \\ \sum_{\rho=1}^{2} \int_{L_\rho} \varepsilon_{3\alpha\beta} x_\alpha p^{(\rho)}_\beta \, ds + \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta} x_\alpha G^{(\rho)}_\beta \, d\sigma + \int_{\Sigma} \varepsilon_{3\alpha\beta} x_\alpha (t_{\beta3,3})_{x_3=0} \, d\sigma = 0. \end{gathered} \tag{4.18}$$

The similar conditions for the existence of the solution of the generalized plane strain problem (4.6), (4.11), (4.13), (4.16) are

$$\int_{\Sigma} \sum_{s=1}^{4} c_s t^{(s)}_{i3} \, d\sigma = 0, \quad \int_{\Sigma} \sum_{s=1}^{4} c_s \varepsilon_{\alpha\beta3} x_\alpha t^{(s)}_{\beta3} \, d\sigma = 0. \tag{4.19}$$

Taking into account the relations (3.2) — (3.5) and (4.9) it follows

$$t^{(s)}_{i\alpha,\alpha} = 0, \text{ in } \Sigma_\rho , \tag{4.20}$$

$$[t^{(s)}_{i\alpha}]_1 \nu_\alpha = [t^{(s)}_{i\alpha}]_2 \nu_\alpha , \text{ on } \Gamma, \tag{4.21}$$

$$t^{(s)}_{i\alpha} n_\alpha = 0, \text{ on } L_\rho , \ (s = 1, 2, 3, 4). \tag{4.22}$$

Using (4.20) — (4.22) we can write

$$\int_{\Sigma} t^{(s)}_{\alpha3} \, d\sigma = \int_{\Sigma} [t^{(s)}_{\alpha3} + x_\alpha t^{(s)}_{\beta3,\beta}] \, d\sigma = \int_{\Sigma} (x_\alpha t^{(s)}_{\beta3})_{,\beta} \, d\sigma = 0. \tag{4.23}$$

Thus the first two conditions (4.19) are satisfied. The remaining conditions lead to

$$\sum_{s=1}^{4} L_{ms} c_s = 0, \quad (m = 3, 4), \tag{4.24}$$

where

$$\begin{aligned} L_{3\alpha} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [C^{(\rho)}_{3333} x_\alpha + \sigma^{(\alpha)}_{33}] \, d\sigma, & L_{4\alpha} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\gamma\beta} x_\gamma [C^{(\rho)}_{\beta333} x_\alpha + \sigma^{(\alpha)}_{\beta3}] \, d\sigma, \\ L_{33} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [C^{(\rho)}_{3333} + \sigma^{(3)}_{33}] \, d\sigma, & L_{43} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\gamma\beta} x_\gamma [C^{(\rho)}_{\beta333} + \sigma^{(3)}_{\beta3}] \, d\sigma, \\ L_{34} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [C^{(\rho)}_{33\alpha3} \varepsilon_{\beta\alpha3} x_\beta + \sigma^{(4)}_{33}] \, d\sigma, & L_{44} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\gamma\beta} x_\gamma [C^{(\rho)}_{\beta3\nu3} \varepsilon_{\alpha\nu3} x_\alpha + \sigma^{(4)}_{\beta3}] \, d\sigma. \end{aligned} \tag{4.25}$$

On the basis of the equilibrium equations we can write

$$\begin{aligned} t_{\alpha3,3} &= t_{\alpha3,3} + x_\alpha (t_{i3,i} + G^{(\rho)}_3)_{,3} = (t_{\alpha3} + x_\alpha t_{i3,i})_{,3} = [(x_\alpha t_{\beta3})_{,\beta} + x_\alpha t_{33,3}]_{,3} = \\ &= (x_\alpha t_{\beta3,3})_{,\beta} + x_\alpha t_{33,33} = \Big\{ x_\alpha \Big[\sum_{s=1}^{4} (b_s + c_s x_3) t^{(s)}_{\beta3} + s_{\beta3} + k^{(1)}_{\beta3} \Big] \Big\}_{,\beta} + x_\alpha t_{33,33} , \end{aligned} \tag{4.26}$$

such that, in view of (4.20) — (4.22), (4.13), (4.16) we have

$$\begin{aligned} \int_{\Sigma} t_{\alpha3,3} \, d\sigma &= \int_{L} x_\alpha \Big[\sum_{s=1}^{4} (b_s + c_s x_3) t^{(s)}_{\beta3} + s_{\beta3} + k^{(1)}_{\beta3} \Big] n_\beta \, ds + \int_{\Sigma} x_\alpha t_{33,33} \, d\sigma = \\ &= \int_{\Sigma} x_\alpha t_{33,33} \, d\sigma = \int_{\Sigma} x_\alpha \sum_{s=1}^{4} c_s t^{(s)}_{33} \, d\sigma . \end{aligned} \tag{4.27}$$

The first two conditions (4.18) reduce to

$$\sum_{s=1}^{4} L_{\alpha s} c_s = - \sum_{\rho=1}^{2} \int_{L_\rho} p^{(\rho)}_\alpha \, ds - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} G^{(\rho)}_\alpha \, d\sigma, \tag{4.28}$$

where

$$\begin{aligned} L_{\alpha\beta} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [C^{(\rho)}_{3333} x_\beta + \sigma^{(\beta)}_{33}] \, d\sigma, & L_{\alpha4} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [C^{(\rho)}_{33\nu3} \varepsilon_{\beta\nu3} x_\beta + \sigma^{(4)}_{33}] \, d\sigma. \\ L_{\alpha3} &= \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha [C^{(\rho)}_{3333} + \sigma^{(3)}_{33}] \, d\sigma, \end{aligned} \tag{4.29}$$

In [9] it is proven that det $(L_{rs}) \neq 0$, so that the system (4.28), (4.24) determines uniquely the constants c_s. Thus, the conditions (4.19) are satisfied and in what follows we assume that the functions v_i and s_{ij} are known.

The remaining conditions from (4.18) lead to

$$\sum_{s=1}^{4} L_{3s} b_s = -\sum_{\rho=1}^{2} \int_{L_\rho} p_3^{(\rho)} ds - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} G_3^{(\rho)} d\sigma - \int_{\Sigma} [s_{33} + h_{33}^{(1)}] d\sigma,$$

$$(4.30)\quad \sum_{s=1}^{4} L_{4s} b_s = -\sum_{\rho=1}^{2} \int_{L_\rho} \varepsilon_{\alpha\beta 3} x_\alpha p_\beta^{(\rho)} ds - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{\alpha\beta 3} x_\alpha G_\beta^{(\rho)} d\sigma - \int_{\Sigma} \varepsilon_{\alpha\beta 3} x_\alpha [s_{\beta 3} + h_{\beta 3}^{(1)}] d\sigma.$$

Using the relations (2.6) — (2.8) we can write

$$(4.31)\quad \int_{\Sigma} t_{\alpha 3} d\sigma = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [t_{\alpha 3} + x_\alpha (t_{3i,i} + G_3^{(\rho)})] d\sigma - \sum_{\rho=1}^{2} \int_{L_\rho} x_\alpha p_3^{(\rho)} ds + \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha (t_{33,3} + G_3^{(\rho)}) d\sigma.$$

Taking into account (4.8) and (4.31), from the conditions (2.11) we obtain

$$(4.32)\quad \sum_{s=1}^{4} L_{\alpha s} b_s = -R_\alpha - \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha G_3^{(\rho)} d\sigma - \sum_{\rho=1}^{2} \int_{L_\rho} x_\alpha p_3^{(\rho)} ds - \int_{\Sigma} x_\alpha (s_{33} + h_{33}^{(1)}) d\sigma.$$

The equations (4.32), (4.30) determine the constants b_s. Thus, the conditions (4.18) are satisfied and in the following we consider the functions w_i, τ_{ij} to be known.

From the conditions (2.12) — (2.14), in view of (4.8) we obtain the following equations for the unknown constants a_s

$$(4.33)\quad \sum_{s=1}^{4} L_{ms} a_s = A_m, \quad (m = 1, 2, 3, 4),$$

where

$$A_\alpha = \varepsilon_{\alpha\beta 3} M_\beta - \int_{\Sigma} x_\alpha [h_{33}^{(0)} + \tau_{33}] d\sigma,$$

$$(4.34)$$

$$A_3 = -R_3 - \int_{\Sigma} [h_{33}^{(0)} + \tau_{33}] d\sigma, \quad A_4 = -M_3 - \int_{\Sigma} \varepsilon_{\alpha\beta 3} x_\alpha [h_{\beta 3}^{(0)} + \tau_{\beta 3}] d\sigma.$$

The constants a_s are determined by the system (4.33). Thus the considered problem is solved.

5. Recurrence Process. Let us establish the solution of the problem $B^{(n+1)}$ assuming that the solution of the problem $B^{(n)}$ (in which $R_i = M_i = 0$) is known.

As the solution of the problem $B^{(n)}$ is known for any $F_{in}^{(\rho)}$ and $p_{in}^{(\rho)}$ we can know the solution of the problem $B^{(n)}$ for $f_i^{(\rho)} = F_{i(n+1)}^{(\rho)}(x_1, x_2)\, x_3^n$ and $\tilde{p}_i^{(\rho)} = p_{i(n+1)}^{(\rho)}(x_1, x_2) x_3^n$.

We denote by u_i^*, t_{ij}^*, respectively, the components of the displacement vector and the components of the stress tensor of this problem.

Let u_i, t_{ij} be the analogous functions for the problem $B^{(n+1)}$.

We look for a solution of the problem $B^{(n+1)}$ in the form

$$(5.1)\quad u_i = (n + 1)\left[\int_0^{x_3} u_3^* dx_3 + v_i\right],$$

where $v_i(x_1, x_2, x_3)$ are unknown functions.

The components of the stress tensor are given by

$$(5.2)\quad t_{ij} = (n + 1)\left[\int_0^{x_3} t_{ij}^* dx_3 + \tau_{ij} + h_{ij}^{(\rho)}\right].$$

where

$$(5.3)\quad \tau_{ij} = C_{ijkl}^{(\rho)} v_{k,l},$$

$$(5.4)\quad h_{ij}^{(\rho)} = C_{ijk3}^{(\rho)} u_k^*(x_1, x_2, 0).$$

By using the relations (5.2), the equilibrium equations reduce to

$$(5.5)\quad \tau_{ji,j} + H_i^{(\rho)} = 0, \text{ in } \mathcal{R}_\rho.$$

where

$$(5.6)\quad H_i^{(\rho)} = h_{i\alpha,\alpha}^{(\rho)} + t_{i3}^*(x_1, x_2, 0).$$

The conditions (2.7), (2.8) lead to the following conditions for the functions v_i and τ_{ij}

$$(5.7)\quad [v_i]_1 = [v_i]_2, \ [\tau_{i\alpha}]_1 \nu_\alpha = [\tau_{i\alpha}]_2 \nu_\alpha + \chi_i, \text{ on } \Gamma,$$

$$(5.8)\quad [\tau_{i\alpha} n_\alpha]_\rho = \tau_i^{(\rho)}, \text{ on } B_\rho,$$

where

$$(5.9)\quad \chi_i = \chi_i(x_1, x_2) = (h_{i\alpha}^{(2)} - h_{i\alpha}^{(1)})\nu_\alpha, \ \tau_i^{(\rho)} = \tau_i^{(\rho)}(x_1, x_2) = -h_{i\alpha}^{(\rho)} n_\alpha.$$

From the conditions on the end $\Sigma^{(0)}$ we obtain

$$(5.10)\quad \int_{\Sigma} \tau_{i3} d\sigma = -P_i, \quad \int_{\Sigma} \varepsilon_{ijk} x_j \tau_{k3} d\sigma = -N_i,$$

where

$$(5.11)\quad P_i = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} h_{i3}^{(\rho)} d\sigma, \quad N_i = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{ijk} x_j h_{k3}^{(\rho)} d\sigma.$$

Thus, the functions v_i are the components of the displacement vector in the problem characterized by the equations (5.3), (5.5), (5.7), (5.8). In this problem the load is independent of x_3.

If χ_i would vanish then this problem reduces to that solved in Section 4. However, it is easy to see that also for $\chi_i \neq 0$ the solution is (4.3). Moreover, by using (5.6), (5.9), (4.24), (4.28), (4.30), (4.32) and the divergence theorem we obtain $c_s = b_s = 0$ $(s = 1, 2, 3, 4)$, $v_i = 0$, so that the solution (5.1) has the form

$$u = (n+1)\left[\int_0^{x_3} u^* \, dx_3 + \hat{u}\,[a_s] + w\right]. \tag{5.12}$$

In (5.12) we denoted by $\hat{u}\,[a_s]$ the displacement vector whose components are obtained from (4.3) taking $c_s = b_s = 0$, $v_i = w_i = 0$. It is known that $\hat{u}\,[a_s]$ represents the displacement vector from the problem of extension, bending and torsion (Ieşan [9]).

6. An Alternative Form of the Solution. In this section we give a treatment of the problem using the procedure established by Bors [3].

Taking into account the recurrence process and the solution (4.3) it is easy to see that we can seek the solution of the problem in the form

$$\begin{aligned} u_\alpha = -\sum_{k=0}^{r+2} \frac{1}{(k+1)(k+2)}\, a_\alpha^{(k)} x_3^2 + \frac{1}{k+1}\, \varepsilon_{3\alpha\beta}\, a_4^{(k)} x_\beta x_3 \,- \\ - \sum_{s=1}^{4} a_s^{(k)} v_\alpha^{(s)} \Big] x_3^k + \sum_{k=0}^{r+1} w_\alpha^{(k)} x_3^k, \end{aligned} \tag{6.1}$$

$$u_3 = \sum_{k=0}^{r+2}\left[\frac{1}{k+1}\left(a_1^{(k)} x_1 + a_2^{(k)} x_2 + a_3^{(k)}\right) x_3 + \sum_{s=1}^{4} a_s^{(k)} v_3^{(s)}\right] x_3^k + \sum_{k=0}^{r+1} w_3^{(k)} x_3^k$$

where $v_i^{(s)}$ are the solutions of the auxiliary plane problems considered in Section 3, $w_i^{(k)}(x_1, x_2)$, $(k = 0, 1, \ldots, r+1)$ are unknown functions and $a_s^{(k)}$ $(s = 1, 2, 3, 4;\ k = 0, 1, \ldots, r+2)$ are unknown constants.

From (6.1), (2.4), (4.9) we obtain

$$t_{ij} = \sum_{k=0}^{r+2} \sum_{s=1}^{4} a_s^{(k)}\, t_{ij}^{(s)}\, x_3^k + \sum_{k=0}^{r+1} \pi_{ij}^{(k)} x_3^k + \sum_{k=0}^{r+2} k \mu_{ij}^{(k)} x_3^{k-1}, \tag{6.2}$$

where

$$\begin{gathered} \pi_{ij}^{(k)} = C_{ijm\alpha}^{(\rho)}\, w_{m.\alpha}^{(k)}, \quad \mu_{ij}^{(k)} = C_{ijm3}^{(\rho)}\left(\sum_{s=1}^{4} a_s^{(k)} v_m^{(s)} + w_m^{(k)}\right), \quad k = 0, 1, \ldots, r+2) \\ w_m^{(r+1+p)} \equiv 0, \quad (p = 1, 2, \ldots). \end{gathered} \tag{6.3}$$

On the basis of the equations (4.20) the equilibrium equations (2.6) reduce to

$$\pi_{i\alpha,\alpha}^{(k)} + \mathcal{F}_{ik}^{(\rho)} = 0, \text{ in } \Sigma_\rho, \quad (k = 0, 1, \ldots, r+1), \tag{6.4}$$

where

$$\begin{aligned} \mathcal{F}_{ik}^{(\rho)} = F_{ik}^{(\rho)} + (k+1) \sum_{s=1}^{4} a_s^{(k+1)} t_{i3}^{(s)} + (k+1)\left[\mu_{i3.3}^{(k+1)} + \pi_{i3}^{(k+1)} + \right. \\ \left. + (k+2)\, \mu_{i3}^{(k+2)}\right], \quad F_{i(r+p)}^{(\rho)} \equiv 0, \ (p = 1, 2, \ldots). \end{aligned} \tag{6.5}$$

Taking into account (3.4), (4.21), (4.22), (6.1) and (6.2), the conditions (2.7) and (2.8) lead to

$$[w_i^{(k)}]_1 = [w_i^{(k)}]_2, \quad [\pi_{i\alpha}^{(k)}]_1 \nu_\alpha = [\pi_{i\alpha}^{(k)}]_2 \nu_\alpha + \gamma_i^{(k)}, \text{ on } \Gamma, \tag{6.6}$$

$$[\pi_{i\alpha}^{(k)} n_\alpha]_\rho = q_{ik}^{(\rho)}, \text{ on } L_\rho, \quad (k = 0, 1, \ldots, r+1), \tag{6.7}$$

where

$$\begin{gathered} \gamma_i^{(k)} = (k+1)\left([\mu_{i\alpha}^{(k+1)}]_2 - [\mu_{i\alpha}^{(k+1)}]_1\right) \nu_\alpha, \\ q_{ik}^{(\rho)} = p_{ik}^{(\rho)} - (k+1)\, \mu_{i\alpha}^{(k+1)} n_\alpha, \quad p_{i(r+p)}^{(\rho)} \equiv 0, \quad (p = 1, 2, \ldots). \end{gathered} \tag{6.8}$$

Thus, the functions $w_i^{(k)}$ are the components of the displacement vector in the generalized plane strain problem (6.3), (6.4), (6.6), (6.7). The necessary and sufficient conditions for the existence of the solution of this problem are

$$\begin{gathered} \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \mathcal{F}_{ik}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} q_{ik}^{(\rho)}\, ds + \int_\Gamma \gamma_i^{(k)}\, ds = 0, \\ \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha\, \mathcal{F}_{\beta k}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha\, q_{\beta k}^{(\rho)}\, ds + \int_\Gamma \varepsilon_{3\alpha\beta}\, x_\alpha\, \gamma_\beta^{(k)}\, ds = 0, \\ (k = 0, 1, \ldots, r+1). \end{gathered} \tag{6.9}$$

Taking into account (6.5), (6.8) and divergence theorem, from (6.9) we obtain

$$\begin{gathered} \sum_{\rho=1}^{2} \int_{\Sigma_\rho} F_{ik}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} p_{ik}^{(\rho)}\, ds + (k+1) \sum_{s=1}^{4} a_s^{(k+1)} \int_\Sigma t_{i3}^{(s)}\, d\sigma + \\ + (k+1) \int_\Sigma \left[\pi_{i3}^{(k+1)} + (k+2)\, \mu_{i3}^{(k+2)}\right] d\sigma = 0, \\ \sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha\, F_{\beta k}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha\, p_{\beta k}^{(\rho)}\, ds + (k+1) \sum_{s=1}^{4} a_s^{(k+1)}\, \cdot \\ \cdot \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha\, t_{\beta 3}^{(s)}\, d\sigma + (k+1) \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha \left[\pi_{\beta 3}^{(k+1)} + (k+2)\, \mu_{\beta 3}^{(k+2)}\right] d\sigma = 0, \\ (k = 0, 1, \ldots, r+1). \end{gathered} \tag{6.10}$$

In view of (4.23) and using the identity

$$(6.11)\qquad \int_{\Sigma} \tau_{\alpha 3}^{(k+1)}\, d\sigma = \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha \mathcal{F}_{3(k+1)}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} x_\alpha q_{3(k+1)}^{(\rho)}\, ds + \int_{\Gamma} x_\alpha \tau_{i3}^{(k+1)}\, ds ,$$

the first two conditions (6.10) reduce to

$$(6.12)\qquad \sum_{s=1}^{4} L_{\alpha s}\, a_s^{(k+2)} = N_\alpha^{(k)} ,$$

where

$$N_\alpha^{(k)} = -\frac{1}{(k+1)(k+2)} \left\{ \sum_{\rho=1}^{2} \int_{\Sigma_\rho} [F_{\alpha k}^{(\rho)} + (k+1)\, x_\alpha F_{3(k+1)}^{(\rho)}]\, d\sigma + \right.$$

$$(6.13)\qquad \left. + \sum_{\rho=1}^{2} \int_{L_\rho} [p_{\alpha k}^{(\rho)} + (k+1)\, x_\alpha p_{3(k+1)}^{(\rho)}]\, ds \right\} - \int_{\Sigma} x_\alpha [\tau_{33}^{(k+2)} + (k+3)\, \mu_{33}^{(k+3)}]\, d\sigma ,$$

$$(k = 0, 1, \ldots, r).$$

The remaining conditions (6.10) lead to

$$(6.14)\qquad \sum_{s=1}^{4} L_{ms}\, a_s^{(k+1)} = N_m^{(k)} , \qquad (m = 3, 4\,;\ k = 0, 1, \ldots, r+1),$$

where

$$N_3^{(k)} = -\frac{1}{k+1} \left[\sum_{\rho=1}^{2} \int_{\Sigma_\rho} F_{3k}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} p_{3k}^{(\rho)}\, ds \right] - \int_{\Sigma} [\tau_{33}^{(k+1)} + (k+2)\, \mu_{33}^{(k+2)}]\, d\sigma ,$$

$$(6.15)\qquad N_4^{(k)} = -\frac{1}{k+1} \left[\sum_{\rho=1}^{2} \int_{\Sigma_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha F_{\beta k}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} \varepsilon_{3\alpha\beta}\, x_\alpha p_{\beta k}^{(\rho)}\, ds \right| -$$

$$- \int_{\Sigma} \varepsilon_{3\alpha\beta}\, x_\alpha [\tau_{\beta 3}^{(k+1)} + (k+2)\, \mu_{\beta 3}^{(k+2)}]\, d\sigma .$$

Let us note that $a_s^{(r+2+p)} = w_i^{(r+1+p)} = \tau_{ij}^{(r+1+p)} = F_{i(r+p)}^{(\rho)} = p_{i(r+p)}^{(\rho)} = 0$, $(p = 1, 2, \ldots)$.

From the conditions (2.11), by using of (6.2), we obtain

$$(6.16)\qquad \sum_{s=1}^{4} L_{\alpha s}\, a_s^{(1)} = -\widetilde{R}_\alpha ,$$

where

$$(6.17)\qquad \widetilde{R}_\alpha = R_\alpha + \sum_{\rho=1}^{2} \int_{\Sigma_\rho} x_\alpha F_{30}^{(\rho)}\, d\sigma + \sum_{\rho=1}^{2} \int_{L_\rho} x_\alpha p_{30}^{(\rho)}\, ds + \int_{\Sigma} x_\alpha ([\tau_{33}^{(1)} + 2\mu_{33}^{(2)}]\, d\sigma .$$

On the basis of (6.2), the conditions (2.12)–(2.14) become

$$(6.18)\qquad \sum_{s=1}^{4} L_{js}\, a_s^{(0)} = \widetilde{M}_j , \qquad (j = 1, 2, 3, 4),$$

where

$$(6.19)\qquad \widetilde{M}_\alpha = \varepsilon_{\alpha\beta 3} M_\beta - \int_{\Sigma} x_\alpha [\tau_{33}^{(0)} + \mu_{33}^{(1)}]\, d\sigma ,$$

$$\widetilde{M}_3 = -R_3 - \int_{\Sigma} [\tau_{33}^{(0)} + \mu_{33}^{(1)}]\, d\sigma , \qquad \widetilde{M}_4 = -M_3 - \int_{\Sigma} \varepsilon_{\alpha\beta 3}\, x_\alpha [\tau_{\beta 3}^{(0)} + \mu_{\beta 3}^{(1)}] d\sigma .$$

The unknown functions and constants are determined in the following order. Firstly, the relations (6.12) for $k = r$ together the relations (6.14) for $k = r + 1$ determine uniquely the unknown $a_s^{(r+2)}$. As $a_s^{(r+2)}$ are known it follows that the functions $\mathcal{F}_{i(r+1)}^{(\rho)}$, $q_{i(r+1)}^{(\rho)}$, $\tau_{ii}^{(r+1)}$ are known. The conditions (6.9) for $k = r + 1$ are satisfied so that the plane strain problem characterized by the equations (6.3), (6.4), (6.6) and (6.7), for $k = r + 1$, determine the functions $w_i^{(r+1)}$. Then we consider the relations (6.12) for $k = r - 1$ and the relations (6.14) for $k = r$. These equations determine the constants $a_s^{(r+1)}$. Then we determine the functions $w_i^{(r)}$ and so on. From (6.14), for $k = 0$, and (6.16) we obtain the constants $a_s^{(1)}$. The plane strain problem characterized by the equations (6.3), (6.4), (6.6), (6.7) for $k = 0$ determine the functions $w_i^{(0)}$. Finally, from (6.19) we determine the constants $a_s^{(0)}$. Thus the initial problem is solved.

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ON EQUILIBRIUM SITES FOR INSTANTANEOUSLY ELASTIC MATERIALS

BY

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Introduction. Considering the class of thermomechanical simple materials introduced by M. E. Gurtin [1] and called *Instantaneously Elastic Materials* by C. Truesdell [2], we shall be concerned here with (a) the consequences of minimal property of free energetic functional at an equilibrium site and (b) the necessary conditions for the constant history and the jump continuation of constant history to be equilibrium histories. As consequences of minimal property of free energetic at an equilibrium site we find that the *equilibrium acoustic tensor* is symmetric and positive semidefinite and that the *instantaneous heat capacity* is less than or equal to *equilibrium heat capacity*. Defining the *equilibrium history* as one which ends at an equilibrium site, we obtain necessary conditions for the constant history and the jump continuation of the constant history to be equilibrium histories. Consequently these imply that the *equilibrium conductivity tensor* and the *instantaneous conductivity tensor* are symmetric and positive semidefinite. Here the equilibrium site is understood in a general sense, as defined in [1], [2], including the particular case when the temperature gradient vanishes. The obtained results generalize some of those contained in [3], [12].

After presenting some well known results in the theory of instantaneously elastic materials [1], [2], [4] in Section 1, the results pertaining to (a) shall be examined in Section 2, and those involving (b) in Section 3.

1. Preliminaries. Let B be a deformable body consisting of material points X. If $\mathbf{F}$ is the deformation gradient $\det \mathbf{F} > 0$, $\theta > 0$ is the absolute temperature, and g is the material gradient of temperature at a point X of B, then the set of all triples $\Lambda = (\mathbf{F}, \theta, \mathbf{g})$ forms a Euclidean space $\mathbf{U}$ of dimension 13 in the following definitions [4]:

$$\alpha\Lambda_1 + \beta\Lambda_2 = \alpha(\mathbf{F}_1, \theta_1, \mathbf{g}_1) + \beta(\mathbf{F}_2, \theta_2, \mathbf{g}_2) = (\alpha\mathbf{F}_1 + \beta\mathbf{F}_2, \alpha\theta_1 + \beta\theta_2, \alpha\mathbf{g}_1 + \beta\mathbf{g}_2)$$

$$\Lambda_1 . \Lambda_2 = \mathbf{F}_1 . \mathbf{F}_2 + \theta_1 \theta_2 + \mathbf{g}_1 \cdot \mathbf{g}_2 = \operatorname{tr} \mathbf{F}_1 \mathbf{F}_2^T + \theta_1 \theta_2 + \mathbf{g}_1 \cdot \mathbf{g}_2$$