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ON EQUILIBRIUM SITES FOR INSTANTANEOUSLY ELASTIC MATERIALS

BY

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Introduction. Considering the class of thermomechanical simple materials introduced by M. E. Gurtin [1] and called *Instantaneously Elastic Materials* by C. Truesdell [2], we shall be concerned here with (a) the consequences of minimal property of free energetic functional at an equilibrium site and (b) the necessary conditions for the constant history and the jump continuation of constant history to be equilibrium histories. As consequences of minimal property of free energetic at an equilibrium site we find that the *equilibrium acoustic tensor* is symmetric and positive semidefinite and that the *instantaneous heat capacity* is less than or equal to *equilibrium heat capacity*. Defining the *equilibrium history* as one which ends at an equilibrium site, we obtain necessary conditions for the constant history and the jump continuation of the constant history to be equilibrium histories. Consequently these imply that the *equilibrium conductivity tensor* and the *instantaneous conductivity tensor* are symmetric and positive semidefinite. Here the equilibrium site is understood in a general sense, as defined in [1], [2], including the particular case when the temperature gradient vanishes. The obtained results generalize some of those contained in [3], [12].

After presenting some well known results in the theory of instantaneously elastic materials [1], [2], [4] in Section 1, the results pertaining to (a) shall be examined in Section 2, and those involving (b) in Section 3.

1. Preliminaries. Let B be a deformable body consisting of material points X. If $\mathbf{F}$ is the deformation gradient $\det \mathbf{F} > 0$, $\theta > 0$ is the absolute temperature, and g is the material gradient of temperature at a point X of B, then the set of all triples $\Lambda = (\mathbf{F}, \theta, \mathbf{g})$ forms a Euclidean space U of dimension 13 in the following definitions [4]:

$$\alpha\Lambda_1 + \beta\Lambda_2 = \alpha(\mathbf{F}_1, \theta_1, \mathbf{g}_1) + \beta(\mathbf{F}_2, \theta_2, \mathbf{g}_2) = (\alpha\mathbf{F}_1 + \beta\mathbf{F}_2, \alpha\theta_1 + \beta\theta_2, \alpha\mathbf{g}_1 + \beta\mathbf{g}_2)$$

$$\Lambda_1 \cdot \Lambda_2 = \mathbf{F}_1 \cdot \mathbf{F}_2 + \theta_1\theta_2 + \mathbf{g}_1 \cdot \mathbf{g}_2 = \operatorname{tr} \mathbf{F}_1 \mathbf{F}_2^T + \theta_1\theta_2 + \mathbf{g}_1 \cdot \mathbf{g}_2$$

A *site* is an element $\mathbf{A}$ of $\mathbf{U}$. A *process* is a function $\mathbf{A}(.) : (-\infty, \infty) \to \mathbf{U}$. The process $\mathbf{A}(.)$ occupies the site $(\mathbf{F}, \theta, \mathbf{g})$ at the time t if $\mathbf{A}(t) = (\mathbf{F}, \theta, \mathbf{g})$, [2]. The function $\mathbf{A}^t(.) : [0, \infty) \to \mathbf{U}$, associated with the process $\mathbf{A}(.)$ and defined by $\mathbf{A}^t(s) = \mathbf{A}(t - s)$ is the *total history* of this process up to time t. The *present value* of history $\mathbf{A}^t(.)$ is given by $\mathbf{A}^t(0) = \mathbf{A}(t)$; and the restriction of $\mathbf{A}^t(.)$ to $(0, \infty)$ is denoted by $\mathbf{A}^t_r(.)$ and is said to be the *past history* corresponding to the total history $\mathbf{A}^t(.)$, [5]. It is useful to represent the total history $\mathbf{A}^t(.)$ by the pair $(\mathbf{A}^t(0), \mathbf{A}^t_r(.)) = (\mathbf{A}(t), \mathbf{A}^t_r(.))$. In the following we will write $\mathbf{A}^t$ and $\mathbf{A}^t_r$ instead of $\mathbf{A}^t(.)$ and $\mathbf{A}^t_r(.)$.

An Instantaneously Elastic Material is characterized by six explicit axioms, [1], [2], restricting the *response functionals* $\overline{\mathbf{S}}$, $\overline{\Psi}$, $\overline{\eta}$, and $\overline{\mathbf{q}}$ which yield at each point $X \in B$ the present values of the Piola-Kirchoff stress tensor $\mathbf{S}(t)$, the free energetic $\Psi(t)$, the caloric $\eta(t)$, and the heat flux vector $\mathbf{q}(t)$, whenever the total history of a process up to time t is specified:

$$\text{(1.2)}\qquad \begin{aligned} &\mathbf{S}(t) = \overline{\mathbf{S}}(\mathbf{A}^t) = \overline{\mathbf{S}}(\mathbf{A}(t) : \mathbf{A}^t_r), \quad \Psi(t) = \overline{\Psi}(\mathbf{A}^t) = \overline{\Psi}(\mathbf{A}(t) : \mathbf{A}^t_r),\\ &\eta(t) = \overline{\eta}(\mathbf{A}^t) = \overline{\eta}(\mathbf{A}(t) ; \mathbf{A}^t_r), \quad \overline{\mathbf{q}}(t) = \mathbf{q}(\mathbf{A}^t) = \overline{\mathbf{q}}(\mathbf{A}(t) ; \mathbf{A}^t_r). \end{aligned}$$

The six axioms which define the material refer to the common domain $\mathcal{D}$ of the functionals in (1.2) and to their smoothness properties. If an adequate topology on $\mathcal{D}$ is chosen and the smoothness properties are referred to this topology, the abstract general results described in [1] [2] have a concrete explicit form [1], [4], [6].

Now let us describe such a topology [1], [5], [7], [8]. Let h be a fixed influence function, i.e. a positive, continuous, monotone decreasing real valued function on $[0, \infty)$, square-integrable on $[0, \infty)$. The function h induces an inner product on all histories $\mathbf{A}^t$ defined by

$$\text{(1.3)}\qquad < \mathbf{A}^t_1(.), \mathbf{A}^t_2(.) > = \mathbf{A}^t_1(0) \cdot \mathbf{A}^t_2(0) + \int_0^\infty h^2(s)\, \mathbf{A}^t_1(s) \cdot \mathbf{A}^t_2(s)\, ds,$$

where $\mathbf{A}^t_1(s) \cdot \mathbf{A}^t_2(s)$ is defined by (1.1). The completion of all continuous $\mathbf{A}^t(.)$ which satisfy

$$\text{(1.4)}\qquad \| \mathbf{A}^t(.) \|^2 = < \mathbf{A}^t(.), \mathbf{A}^t(.) > < \infty$$

forms a Hilbert space $\mathcal{H}$ with (1.3) as its natural inner product. The set of functions $\mathbf{A}^t_r(.)$ with

$$\| \mathbf{A}^t_r(.) \|^2 = \int_0^\infty h^2(s)\, | \mathbf{A}^t_r(s) |^2\, ds$$

is a Hilbert subspace $\mathcal{H}_r$ of $\mathcal{H}$ [5].

We let the common domain $\mathcal{D}$ of functionals (1.2) be the set of all absolutely continuous $\mathbf{A}^t(.)$ from $\mathcal{H}$ for which $\mathbf{A}^t(0^+)$ exists and $\dot{\mathbf{A}}^t(s) = -d/ds(\mathbf{A}^t(s)) = d/dt\, \mathbf{A}^t(s)$ are in $\mathcal{H}$ [10].

The assumption that the material obeys the principle of fading memory in Coleman and Noll's sense [7], implies the existence and continuity of differential operators $\mathbf{D}_A = (D_F, D_\theta, D_g)$ and $\delta_A = (\delta_F, \delta_\theta, \delta_g)$ (first Fréchet derivative) defined by

$$\text{(1.5)}\qquad \begin{aligned} &D_F\overline{\Psi}(\mathbf{A}^t) = d/d\alpha[\overline{\Psi}(\mathbf{F} + \alpha\mathbf{A}, \theta, \mathbf{g} : \mathbf{A}^t_r)]_{\alpha=0} \quad D_\theta\overline{\Psi}(\mathbf{A}^t) = \partial/\partial\theta(\overline{\Psi}(\mathbf{F}, \theta, \mathbf{g} : \mathbf{A}^t_r)).\\ &D_g\overline{\Psi}(\mathbf{A}^t) \cdot \mathbf{V} = d/d\alpha(\overline{\Psi}(\mathbf{F}, \theta, \mathbf{g} + \alpha\mathbf{V} ; \mathbf{A}^t_r))_{\alpha=0} \end{aligned}$$

and

$$\text{(1.6)}\qquad \begin{aligned} &\delta_F\overline{\Psi}(\mathbf{A}^t \mid \mathbf{A}) = d/d\alpha(\overline{\Psi}(\mathbf{A}(t) : \mathbf{F}^t_r + \alpha\mathbf{A}, \theta^t_r, \mathbf{g}^t_r))|_{\alpha=0},\\ &\delta_\theta\overline{\Psi}(\mathbf{A}^t \mid a) = d/d\alpha(\overline{\Psi}(\mathbf{A}(t) : \mathbf{F}^t_r, \theta^t_r + \alpha a, g^t_r))|_{\alpha=0},\\ &\delta_g\overline{\Psi}(\mathbf{A}^t \mid \mathbf{a}) = d/d\alpha(\overline{\Psi}(\mathbf{A}(t) ; \mathbf{F}^t_r, \theta^t_r, g^t_r + \alpha\mathbf{a}))|_{\alpha=0}, \end{aligned}$$

for every $\mathbf{A}^t \in \mathcal{D}$ and every triple $(\mathbf{A}, a, \mathbf{a})$, with $\mathbf{A}$ a second order tensor, a a real number, and $\mathbf{a}$ an element of the Euclidean three dimensional vector space $\mathbf{E}_3$. In (1.5), $D_F\overline{\Psi}(\mathbf{A}^t) \cdot \mathbf{A} = \operatorname{tr} D_F\overline{\Psi}(\mathbf{A}^t)\mathbf{A}^T$, and $D_g\overline{\Psi}(\mathbf{A}^t) \cdot \mathbf{V}$ is the inner product in $\mathbf{E}_3$. In (1.6) the functionals $\delta_F\overline{\Psi}(\mathbf{A}^t \mid \mathbf{A})$, $\delta_\theta\overline{\Psi}(\mathbf{A}^t \mid a)$, and $\delta_g\overline{\Psi}(\mathbf{A}^t \mid \mathbf{a})$ are linear in $\mathbf{A}$, a, $\mathbf{a}$ and continuous in $\mathbf{A}^t$ [1], [4], [7], [8].

The necessary and sufficient conditions for the Clausius-Duhem inequality to hold are [1], [4], [6], [8]:

$$\text{(1.7)}\qquad \begin{aligned} &\mathbf{S}(t) = \overline{\mathbf{S}}(\mathbf{A}^t) = D_F\overline{\Psi}(\mathbf{A}^t) \qquad && D_g\overline{\Psi}(\mathbf{A}^t) = 0,\\ &\eta(t) = \overline{\eta}(\mathbf{A}^t) = -D_\theta\overline{\Psi}(\mathbf{A}^t), && \overline{\mathbf{q}}(\mathbf{A}^t) \cdot g \leqslant \theta^2\sigma(t), \end{aligned}$$

where

$$\text{(1.8)}\qquad \sigma(t) = -\frac{1}{\theta}\delta_A\overline{\Psi}(\mathbf{A}^t \mid \dot{\mathbf{A}}^t_r) = -\frac{1}{\theta}[\delta_F\overline{\Psi}(\mathbf{A}^t \mid \dot{F}^t_r) + \delta_\theta\overline{\Psi}(\mathbf{A}^t \mid \dot{\theta}^t_r) + \delta_g\overline{\Psi}(\mathbf{A}^t \mid \dot{\mathbf{g}}^t_r)]$$

is the *internal dissipation*.

From $(1.7)_3$ it results that $\overline{\Psi}$, and therefore $\overline{\mathbf{S}}$ and $\overline{\eta}$, must be independent of the present value of $\mathbf{g}$, although they may be affected by the past history $\mathbf{g}^t_r$ of $\mathbf{g}$.

We can now write (1.2) in the form

$$\text{(1.9)}\qquad \begin{aligned} &\Psi(t) = \overline{\Psi}(\mathbf{F}, \theta ; \mathbf{A}^t_r), \qquad && \eta(t) = -D_\theta\overline{\Psi}(\mathbf{F}, \theta ; \mathbf{A}^t_r),\\ &\mathbf{S}(t) = D_F\overline{\Psi}(\mathbf{F}, \theta ; \mathbf{A}^t_r), && \mathbf{q}(t) = \overline{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g} ; \mathbf{A}^t_r). \end{aligned}$$

If $\mathbf{A} = (\mathbf{F}, \theta, \mathbf{g})$ is a site and $\mathbf{A}(t) = \mathbf{A}$, $t \in (-\infty, \infty)$ is the constant process, we denote by $\mathbf{A}^+(s) = \mathbf{A}$, $s \in [0, \infty)$, the constant history with value $\mathbf{A}$, and with $\mathbf{A}^+_r(s) = \mathbf{A}$, $s \in (0, \infty)$, the past history corresponding to $\mathbf{A}^+$. We say that $\mathbf{A}^+$ is the constant history *corresponding to the site* $\mathbf{A}$.

If $\mathcal{F}$ is one of the functionals in (1.2), it is denoted by $\widehat{\mathcal{F}}$ the *equilibrium response function* corresponding to $\overline{\mathcal{F}}$.

$$\widehat{\mathcal{F}}(\mathbf{A}) \equiv \overline{\mathcal{F}}(\mathbf{A}^+) = \overline{\mathcal{F}}(\mathbf{A}, \mathbf{A}_r^+).$$

This definition makes sense because $\mathbf{A}^+ \in \mathcal{D}$ as $|\mathbf{A}| < \infty$. Taking into account $(1.9)_1$, we write explicitly the above notation for the functional Ψ:

$$\widehat{\Psi}(\mathbf{A}) = \widehat{\Psi}(\mathbf{F}, \theta, \mathbf{g}) \equiv \overline{\Psi}(\mathbf{F}, \theta; \mathbf{F}_r^+, \theta_r^+, \mathbf{g}_r^+) = \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+) \tag{1.10}$$

Definition 1. [1], [2]. *The site* $(\mathbf{F}, \theta, \mathbf{g})$ *is an equilibrium site if for all processes* $\mathbf{A}(.) = (\mathbf{F}(.), \theta(.), \mathbf{g}(.))$ *occupying the site* $(\mathbf{F}, \theta, \mathbf{g})$ *at time* t *we have*

$$\mathbf{q}(\mathbf{F}, \theta, \mathbf{g}; \mathbf{A}_r^t) \cdot \mathbf{g} = 0. \tag{1.11}$$

Definition 2. *Let* $\mathbf{A} = (\mathbf{F}, \theta, \mathbf{g})$ *be an equilibrium site. A history* $\mathbf{A}^t$ *with* $\mathbf{A}^t(0) = (\mathbf{F}, \theta, \mathbf{g})$, *which satisfy* (1.11) *is called the equilibrium history corresponding to the site* $\mathbf{A}$.

From Definition 1, it results that every site $(\mathbf{F}, \theta, \mathbf{0})$ is an equilibrium site if there exists a process $\mathbf{A}(.)$ occuping this site a some time t. For this reason and in order to make sense to speak of derivatives of the equilibrium functions, it is necessary to consider the connected open subset $\mathcal{E}$ of all equilibrium sites including at least one of the equilibrium sites $(\mathbf{A}, \theta, \mathbf{0})$ [2].

In connection with equilibrium sites we need to remind the *Theorem of Consistency with Thermostatics* [1].

If the equilibrium free energy function $\widehat{\Psi}$ is differentiable in $(\mathbf{A}, \theta, \mathbf{a}) \in \mathcal{E}$, then:

i) $\widehat{\Psi}(.)$ in $(\mathbf{A}, \theta, \mathbf{a})$ is a function of $(\mathbf{A}, \theta)$ only,

$$\widehat{\Psi}(\mathbf{A}, \theta, \mathbf{a}) = \widehat{\Psi}(\mathbf{A}, \theta) \tag{1.12}$$

ii) $\widehat{\Psi}(\mathbf{A}, \theta)$ is a potential for the stress and caloric equilibrium functions

$$\widehat{\mathbf{S}}(\mathbf{A}, \theta) = \frac{\partial}{\partial \mathbf{A}} \widehat{\Psi}(\mathbf{A}, \theta), \quad \widehat{\eta}(\mathbf{A}, \theta) = -\frac{\partial}{\partial \theta} \widehat{\Psi}(\mathbf{A}, \theta). \tag{1.13}$$

iii) among all histories ending at a given equilibrium site, that which has remained fixed at that site gives the least possible value of the free energy functional at that site; i.e. for all processes $\mathbf{A}(.)$ with $\mathbf{A}(t) = (\mathbf{F}, \theta, \mathbf{g}) \in \mathcal{E}$, we have

$$\widehat{\Psi}(\mathbf{F}, \theta) \leqslant \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^t), \tag{1.14}$$

for every $\mathbf{A}_r^t \in \mathcal{H}_r$.

The derivative in $(1.13)_1$ is defined by

$$\mathbf{B} \cdot \frac{\partial}{\partial \mathbf{A}} \widehat{\Psi}(\mathbf{A}, \theta) = \frac{d}{d\alpha} \widehat{\Psi}(\mathbf{A} + \alpha \mathbf{B}, \theta)_{\alpha=0}, \tag{1.15}$$

B being any second order tensor.

2. Consequences of the minimal property of the free energetic at an equilibrium site. In this section we assume Ψ is twice continuously Fréchet differentiable on its domain of definition $\mathcal{D}$. From (1.14) and (1.10) we have

$$\overline{\Psi}(\mathbf{F}, \theta; \mathbf{F}_r^+, \theta_r^+, \mathbf{g}_r^+) \leqslant \overline{\Psi}(\mathbf{F}, \theta; \mathbf{F}_r^t, \theta_r^t, \mathbf{g}_r^t) \tag{2.1}$$

for every $(\mathbf{F}_r^t, \theta_r^t, \mathbf{g}_r^t) \in \mathcal{H}_r$ and every $(\mathbf{F}, \theta)$ such that $(\mathbf{F}, \theta, \mathbf{g}) \in \mathcal{E}$, where $\mathbf{A}_r^+ = (\mathbf{F}_r^+, \theta_r^+, \mathbf{g}_r^+)$ is the past history corresponding to the constant history $\mathbf{A}^+(s) = (\mathbf{F}, \theta, \mathbf{g})$, $s \in [0, \infty)$.

The assumption that $\overline{\Psi}$ is twice continuously differentiable implies the existence and continuity of second Fréchet derivatives

$$\delta_F^2 \overline{\Psi}, \delta_\theta^2 \overline{\Psi}, \delta_g^2 \overline{\Psi}, \delta_F \delta_\theta \overline{\Psi} = \delta_\theta \delta_F \overline{\Psi}, \delta_F \delta_g \overline{\Psi} = \delta_g \delta_F \overline{\Psi}, \delta_\theta \delta_g \overline{\Psi} = \delta_g \delta_\theta \overline{\Psi},$$

$$D_F^2 \overline{\Psi}, D_F \delta_F \overline{\Psi} = \delta_F D_F \overline{\Psi}, D_\theta^2 \overline{\Psi}, D_\theta \delta_\theta \overline{\Psi} = \delta_\theta D_\theta \overline{\Psi}$$

and they are defined as in [11], [12].

The minimal property (2.1) implies.

$$\delta_F \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}) + \delta_\theta \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | a) + \delta_g \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{a}) = 0.$$

$$\begin{aligned} &\delta_F^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}, \mathbf{A}) + \delta_\theta^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | a, a) + \delta_g^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{a}, \mathbf{a}) + \\ &+ 2[\delta_F \delta_\theta \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}, a) + \delta_F \delta_g \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}, \mathbf{a}) + \\ &+ \delta_\theta \delta_g \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{a}, a)] \geqslant 0 \end{aligned} \tag{2.2}$$

for every $(\mathbf{A}, a, \mathbf{a}) \in \mathbf{U}$. In (2.2), $\mathbf{A}_r^+$ is the past history of the constant equilibrium history corresponding to the equilibrium site $\mathbf{A} = (\mathbf{F}, \theta, \mathbf{g})$.

From (1.10), in our hypothesis on $\overline{\Psi}$, we have

$$\begin{aligned} \mathbf{A} \cdot \frac{\partial^2 \widehat{\Psi}(\mathbf{A})}{\partial \mathbf{F}^2} \mathbf{B} &= \mathbf{A} \cdot D_F^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+) \mathbf{B} + \mathbf{A} \cdot D_F \delta_F \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{B}) + \\ &+ \mathbf{B} \cdot D_F \delta_F \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}) + \delta_F^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}, \mathbf{B}). \end{aligned} \tag{2.3}$$

for any second order tensors $\mathbf{A}$, $\mathbf{B}$, and

$$\frac{\partial^2 \widehat{\Psi}(\mathbf{A})}{\partial \theta^2} = D_\theta^2 \overline{\Psi}(F, \theta; \mathbf{A}_r^+) + 2 D_\theta \delta_\theta \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | 1) + \delta_\theta^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | 1, 1), \tag{2.4}$$

for any $\mathbf{A} = (\mathbf{F}, \theta, \mathbf{g}) \in \mathcal{D}$, $\mathbf{A}_r^+$ having the same significance as in **(1.10)**.

If we take $\mathbf{a} = 0$, $a = 0$ in (2.2) we obtain

$$\delta_F \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}) = 0, \quad \delta_F^2 \overline{\Psi}(\mathbf{F}, \theta; \mathbf{A}_r^+ | \mathbf{A}, \mathbf{A}) \geqslant 0. \tag{2.5}$$

Since $(2.5)_1$ holds on the set $\mathcal{E}$ of equilibrium sites, it follows that

$$(2.6) \qquad \frac{\partial}{\partial \mathbf{F}} \delta_{\mathbf{F}} \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^t \mid \mathbf{A}) = 0$$

on $\mathcal{E}$. Here the derivative $\partial / \partial \mathbf{F}$ is defined by (1.15).

From (2.6) we obtain

$$(2.7) \qquad \mathbf{B} \,.\, D_{\mathbf{F}} \delta_{\mathbf{F}} \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid \mathbf{A}) + \delta_{\mathbf{F}}^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid \mathbf{A}, \mathbf{B}) = 0$$

and the similar relation by changing $\mathbf{A}$ with $\mathbf{B}$.

Now assuming in (2.3) that $\mathbf{A}$ is an equilibrium site and A_r is as in (2.2), from (1.12), $(1.13)_1$ and (2.7) we get

$$(2.8) \qquad \mathbf{A} \,.\, \mathbf{Q} (\mathbf{A}) \mathbf{B} = \delta_{\mathbf{F}}^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid \mathbf{A}, \mathbf{B}), \qquad \mathbf{A} . \mathbf{Q} (\mathbf{A}) \mathbf{A} \geqslant 0.$$

In (2.8), the fourth order tensor $\mathbf{Q} (\mathbf{A})$ is the *instantaneous acoustic tensor* [12] and is given by

$$(2.9) \qquad \mathbf{Q} (\mathbf{A}) = D_{\mathbf{F}}^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+) - \frac{\partial^2 \hat{\Psi}}{\partial \mathbf{F}^2} (\mathbf{F}, \theta) = D_{\mathbf{F}} \mathbf{S} (\mathbf{F}, \theta ; \mathbf{A}_r^+) - \frac{\partial \hat{\mathbf{S}}}{\partial \mathbf{F}} (\mathbf{F}, \theta).$$

From (2.8), we have

Theorem 1. *The instantaneous acoustic tensor is symmetric and positive semidefinite.*

Since from $(1.13)_1$ it follows that $\partial \hat{\mathbf{S}} (\mathbf{F}, \theta) / \partial \mathbf{F}$ is symmetric, $(2.8)_1$ implies that $D_{\mathbf{F}} \mathbf{S} (\mathbf{F}, \theta ; \mathbf{A}_r^+)$ is symmetric. This result generalizes to the case of instantaneous elastic materials, the result of C o l e m a n in § 8 of [9] and that of C o l e m a n and G u r t i n obtained in 4.32 of [12]. For linear viscoelastic materials the result (2.8) is derived in [4] from other considerations.

The quantities $\hat{C} (F, \theta) = - (\partial^2 \hat{\Psi} (\mathbf{F}, \theta) / \partial \theta^2) \theta$ and $c (\mathbf{F}, \theta ; \mathbf{A}_r^+) = - D_\theta^2 (\Psi (\mathbf{F}, \theta ; \mathbf{A}_r^+)) \theta$, with $\mathbf{A}$ equilibrium site are called *equilibrium heat capacity* and *instantaneous heat capacity*, respectively [12].

Theorem 2. *The equilibrium heat capacity is greater than or equal to instantaneous heat capacity.*

$$(2.10) \qquad \hat{c} (\mathbf{F}, \theta) \geqslant c (\mathbf{F}, \theta ; \mathbf{A}_r^+).$$

Proof. If in (2.2) we take $\mathbf{A} = 0$, $\mathbf{a} = 0$, it results

$$(2.11) \qquad \delta_\theta \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1) = 0, \qquad \delta_\theta^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1 \,.\, 1) \geqslant 0.$$

From $(2.11)_1$ we have

$$\frac{d}{d \theta} \delta_\theta \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1) = 0,$$

on $\mathcal{E}$ and therefore

$$(2.12) \qquad D_\theta \delta_\theta \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1) + \delta_\theta^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1, 1) = 0.$$

Assuming in (2.4) that $\mathbf{A}$ is an equilibrium site and $\mathbf{A}_r$ the past history of the constant equilibrium history corresponding to $\mathbf{A}$, by (1.12), (2.4), and (2.12) we have

$$\hat{c} (\mathbf{F}, \theta) = c (\mathbf{F}, \theta ; \mathbf{A}_r^+) + \theta \delta_\theta^2 \overline{\Psi} (\mathbf{F}, \theta ; \mathbf{A}_r^+ \mid 1, 1).$$

The theorem results from the above formula and $(2.11)_2$. The theorem generalizes, to the case of instantaneous elastic materials, the result (4.31) of [12] and (3.14) of [3]. (In the last paper the rigid heat conductor bodies are considered and the equilibrium sites are those with vanishing temperature gradient).

3. Equilibrium and instantaneous conductivity tensors. From the definition of the equilibrium site it follows, by derivating (1.11) with respect to $\mathbf{g}$, and taking $\mathbf{g} = 0$, that [6]

$$(3.1) \qquad \mathbf{q} (\mathbf{F}, \theta, 0 ; \mathbf{A}_r^t) = 0,$$

or every history $\mathbf{A}^t$ ending in the equilibrium site $(\mathbf{F}, \theta, \mathbf{0}) \in \mathcal{E}$, regardless of the values of $\mathbf{F}$ and θ.

If $(\mathbf{F}, \theta, \mathbf{g})$, $(\mathbf{F}, \theta, \mathbf{0}) \in \mathcal{E}$, and these two equilibrium sites are sufficiently close in the topology of $\mathcal{E}$ defined by (1.1), we have

$$(3.2) \qquad \mathbf{q} \mathbf{F} (, \theta, \mathbf{g} ; \mathbf{A}_r^t) = K (\mathbf{F}, \theta, \mathbf{A}_r^t) \mathbf{g} + O (1) \mathbf{g}.$$

where

$$(3.3) \qquad K (\mathbf{F}, \theta ; \mathbf{A}_r^t) = D_{\mathbf{g}} \mathbf{q} (\mathbf{F}, \theta ; \mathbf{0} ; \mathbf{A}_r^t)$$

and $0 (1) \to 0$ as $\mid \mathbf{g} \mid \to 0$.

From relation (3.2) it follows that at every equilibrium site $(\mathbf{F}, \theta, \mathbf{g})$, sufficiently close to the equilibrium site $(\mathbf{F}, \theta, \mathbf{0})$, the heat flux vector is linear in $\mathbf{g}$; that is, the Fourier's law holds to within an error of order one in $\mid \mathbf{g} \mid$.

We have to note that the concept of equilibrium site is defined for a material point, that is, it is a local concept. Also, we note that there is at least a process such that $\mathbf{F}$, θ, $\mathbf{g}$ can be independently given in a material point X of the body [8].

If $\mathbf{A}^+ (s) = (\mathbf{F}, \theta, \mathbf{g})$ is the constant history corresponding to this process, from (1.8) we have $\sigma (t) = 0$, and therefore by $(1.7)_4$ it follows

$$(3.4) \qquad \overline{\mathbf{q}} (\mathbf{F}, \theta, \mathbf{g} ; \mathbf{A}_r^+) \,.\, \mathbf{g} \geqslant 0,$$

$\mathbf{A}_r^+$ being defined by $\mathbf{A}_r^+ (s) = (\mathbf{F}, \theta, \mathbf{g})$, $s \in (0, \infty)$.

Therefore by (3.4) and Definition 2, any constant history ending in the site $(\mathbf{F}, \theta, \mathbf{g})$ is a possible equilibrium history corresponding to this site.

Theorem 3. *If the site* $\mathbf{A} = (\mathbf{F}, \theta, \mathbf{g})$ *is an equilibrium site and* $\mathbf{A}^+ (s) = (\mathbf{F}, \theta, \mathbf{g})$, $s \in [0, \infty)$ *is its equilibrium history, then we have*

$$(3.5) \qquad \mathbf{g} \,.\, [D_{\mathbf{g}} \, \mathbf{q} (\mathbf{A}^+) \mathbf{u} + \delta_{\mathbf{g}} \, \mathbf{q} (\mathbf{A}^+ \mid \mathbf{u})] + \mathbf{q} (\mathbf{A}^+) \,.\, \mathbf{u} = 0.$$

$$\mathbf{g} \cdot [D_{\mathbf{g}}^2 \bar{\mathbf{q}}(\Lambda^+)\mathbf{u} \otimes \mathbf{u} + 2D_{\mathbf{g}}\,\delta_{\mathbf{g}}\,\mathbf{q}(\Lambda^+ \mid \mathbf{u})\,\mathbf{u} + \delta_{\mathbf{g}}^2\,\bar{\mathbf{q}}(\Lambda^+ \mid \mathbf{u}, \mathbf{u})] + \tag{3.6}$$
$$+ \mathbf{u} \cdot [D_{\mathbf{g}}\,\mathbf{q}(\Lambda^+) + \delta_{\mathbf{g}}\,\mathbf{q}(\Lambda^+ \mid .)]\,\mathbf{u} \leqslant 0,$$

for every $\mathbf{u} \in E_3$. *In* (3.4) $\mathbf{u} \otimes \mathbf{u}$ *is the tensor product.*

Proof. In proving this theorem, we need to suppose the heat flux functional $\mathbf{q}$ is twice continuously differentiable.

Consider the function

$$f(\xi) = \mathbf{q}(F, 0, g + \xi u; F, 0, g + \xi u) \cdot (g + \xi u). \tag{3.7}$$

If $\Lambda^+(s) = (\mathbf{F}, 0, \mathbf{g})$, $s \in [0, \infty)$ is an equilibrium history, then it must follow that $f'(\xi)|_{\xi=0} = 0$ and $f''(\xi)_{\xi=0} \leqslant 0$, for every vector $\mathbf{u}$. Computing $f'(\xi)$ and $f''(\xi)$, and taking $\xi = 0$, we obtain the stated results (3.5) and (3.6).

Remark 1. What is usually known as *equilibrium conductivity* tensor $K(\infty)$ is now given by the tensor associated with the bilinear form $K(\infty)$ (...) and defined by

$$-K(\infty)(\mathbf{u}, \mathbf{v}) = \mathbf{g} \cdot [D_{\mathbf{g}}^2\,\mathbf{q}(\Lambda^+)(\mathbf{u}, \mathbf{v}) + 2D_{\mathbf{g}}\,\delta_{\mathbf{g}}\,\mathbf{q}(\Lambda^+ \mid \mathbf{u})\,\mathbf{v} + \tag{3.8}$$
$$+ \delta_{\mathbf{g}}^2\,\mathbf{q}(\Lambda^+ \mid \mathbf{u}, \mathbf{v})] + \mathbf{u} \cdot [D_{\mathbf{g}}\,\mathbf{q}(\Lambda^+) + \delta_{\mathbf{g}}\,\mathbf{q}(\Lambda^+ \mid .)]\,\mathbf{v}$$

The inequality (3.6) shows that the equilibrium conductivity tensor is pozitive semidefinite.

Remark 2. In the particular case of the equilibrium history $\Lambda_0^+(s) = (\mathbf{F}, 0, \mathbf{0})$, $s \in [0, \infty)$, we have from (3.5)

$$\bar{\mathbf{q}}(\Lambda_0^+) \cdot \mathbf{u} = 0 \tag{3.9}$$

for any $\mathbf{u}$, and this leads to the already obtained result (3.1). In this caser the equilibrium conductivity tensor is given by

$$-K(\infty) = D_{\mathbf{g}}\,\bar{\mathbf{q}}(\Lambda_0^+) + \delta_{\mathbf{g}}\,\mathbf{q}(\Lambda_0^+ \mid .), \tag{3.10}$$

and (3.6) becomes

$$\mathbf{u} \cdot K(\infty)\,\mathbf{u} \geqslant 0, \tag{3.11}$$

that is, $K(\infty)$ is positive semidefinite [3].

Let $\Lambda_0 = (\mathbf{F}_0, \theta_0, \mathbf{g}_0)$ be a constant process and $I(\Lambda_0)$ the subset of all histories in $\mathcal{D}$ satisfying the condition $\Lambda^t(s) = \Lambda_0$ on (s_0, ∞), $s_0 > 0$. If $\alpha > 1$, the history $\Lambda_\alpha^t(.) \in I(\Lambda_0)$ defined by $\Lambda_\alpha^t(s) = \Lambda^t(\alpha s)$, $s \in (0, \infty)$ and $\Lambda_\alpha^t(0) = \Lambda = (\mathbf{F}, 0, \mathbf{g})$ is called the accelerated history corresponding to $\Lambda^t(.) \in I(\Lambda_0)$ [3]. It is known that

$$\lim_{\alpha \to \infty} \| \Lambda_\alpha^t(.) - \bar{\Lambda}(.) \| = 0, \tag{3.12}$$

where $\bar{\Lambda}(.)$ is defined by $\bar{\Lambda}(s) = \Lambda_0$ on $(0, \infty)$ and $\bar{\Lambda}(0) = \Lambda$, [3], [9]. The history $\bar{\Lambda}(.)$ is the *jump continuation* of the constant history $\Lambda^+(s) = \Lambda_0$, $s \in [0, \infty)$, with the *jump* $\Lambda - \Lambda_0$ [9].

From the continuity of $\delta_\Lambda \bar{\Psi}(\Lambda_\alpha^t \mid \dot{\Lambda}_\alpha^t)$ in both arguments and (3.12) we have [3],

$$\lim_{\alpha \to 0} \delta_\Lambda \bar{\Psi}(\Lambda_\alpha^t \mid \dot{\Lambda}_\alpha^t) = 0. \tag{3.13}$$

The continuity of $\mathbf{q}$, (3.12), (3.13), (1.8) and (1.7) lead to

$$\bar{\mathbf{q}}(\bar{\Lambda}) \cdot \mathbf{g} \leqslant 0, \tag{3.14}$$

or

$$\mathbf{q}(\Lambda; \Lambda_0) \cdot \mathbf{g} = \mathbf{q}(\mathbf{F}, 0, \mathbf{g}; \Lambda_0) \cdot \mathbf{g} \leqslant 0, \tag{3.15}$$

because $\Lambda_r(s) = \Lambda_0$ on $(0, \infty)$.

By Definition 2, it follows that the history $\bar{\Lambda}(.)$ ending in the site $(\mathbf{F}, 0, \mathbf{g})$ is a possible equilibrium history corresponding to this site.

We can now formulate the

Theorem 4. *If the site* $\Lambda = (\mathbf{F}, 0, \mathbf{g})$ *is an equilibrium site and the jump continuation history* $\bar{\Lambda}(.)$ *of the constant history* $\Lambda^+(s) = \Lambda_0$ *with the jump* $\Lambda - \Lambda_0$ *is one of its equilibrium histories then the conditions*

$$\mathbf{g} \cdot D_{\mathbf{g}}\,\mathbf{q}(\Lambda; \Lambda_0)\,\mathbf{u} + \mathbf{u} \cdot \bar{\mathbf{q}}(\Lambda; \Lambda_0) = 0, \tag{3.16}$$
$$\mathbf{g} \cdot D_{\mathbf{g}}^2 \bar{\mathbf{q}}(\Lambda; \Lambda_0)\mathbf{u} \otimes \mathbf{u} + 2\mathbf{u} \cdot D_{\mathbf{g}}\,\bar{\mathbf{q}}(\Lambda; \Lambda_0)\,\mathbf{u} \leqslant 0,$$

hold for every vector $\mathbf{u}$.

Proof. Consider the function

$$f(\xi) = \mathbf{q}(\mathbf{F}, 0, \mathbf{g} + \xi\mathbf{u}; \Lambda_0) \cdot (\mathbf{g} + \xi\mathbf{u}),$$

$\mathbf{u}$ being a non-zero arbitrarily chosen vector.

If $(\mathbf{F}, 0, \mathbf{g})$ is an equilibrium site and $\bar{\Lambda}(.)$ is its equilibrium history, then the function $\mathbf{q}(\mathbf{F}, 0, \mathbf{g}; \Lambda_0) \cdot \mathbf{g}$ as a function of $\mathbf{g}$, has a maximum at the site $(\mathbf{F}, 0, \mathbf{g})$, and therefore $f(\xi)$ has a maximum at $\xi = 0$ for every $\mathbf{u}$. It follows that $f'(\xi)_{\xi=0} = 0$ and $f''(\xi)_{\xi=0} = 0$, and from these we obtain (3.16).

Remark 3. The *instantaneous conductivity tensor* $K(0)$ is now given by the tensor associated with the bilinear form

$$-K(0)(\mathbf{u}, \mathbf{v}) = \mathbf{g} \cdot D_{\mathbf{g}}\,\bar{\mathbf{q}}(\Lambda; \Lambda_0)(\mathbf{u}, \mathbf{v}) + 2\mathbf{u} \cdot D_{\mathbf{g}}\,\bar{\mathbf{q}}(\Lambda; \Lambda_0)\,\mathbf{v}. \tag{3.17}$$

The inequality $(3.16)_2$ shows that the instantaneous conductivity tensor $K(0)$ is positive semidefinite.

Remark 4. In the case $\mathbf{g} = \mathbf{0}$, we obtain from $(3.16)_1$ $\mathbf{u} \cdot \mathbf{q}(\mathbf{F}, 0, \mathbf{0}; \Lambda_0) = 0$, for every u and this leads to

$$\mathbf{q}(\mathbf{F}, 0, \mathbf{0}; \Lambda_0) = \mathbf{0}.$$

For this case, the instantaneous conductivity tensor is given by [3], $K(0) = 2D_{\mathbf{g}}\,\bar{\mathbf{q}}(\mathbf{F}, 0, \mathbf{0}; \Lambda_0)$, and by $(3.16)_2$ we have $\mathbf{u} \cdot K(0)\,\mathbf{u} \geqslant 0$.

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PULSATILE FLOW OF A MICROPOLAR FLUID IN A RIGID TUBE WITH LONGITUDINAL VIBRATION

BY

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1. Introduction. The analysis of pulsating flows has been of interest for a long time. Such a motion would explain the blood flow in vascular beds.

Recently, T u r k and al. [1] have used the linear micropolar theory of fluids as a model for blood. They have considered the pulsatile flow of a micropolar fluid through a rigid circular tube due to a sinusoidally varying pressure gradient.

In the present work the additional complexity of longitudinal wall vibration has been considered. The interest in the effects of such wall motion has resulted from an investigation of cardiovascular behavior in the presence of low-frequency vibration which is currently in progress. Such vibration effects are of interest for jackhammer operators, astronauts riding boosters, and other workers who perform in vibration environments. The solutions of governing partial differential equations are obtained through application of consecutive transformations: a finite Hankel transform is made on the space variable followed by a Laplace transform on time variable.

2. Formulation of the problem. For axisymmetric, incompressible, parallel flow the field equations governing the motion of the micropolar fluids [2] reduces to

$$
(1)\qquad
\begin{aligned}
&(\mu + k)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + k\frac{1}{r}\frac{\partial}{\partial r}(rw) - \frac{\partial p}{\partial z} = \rho\frac{\partial v}{\partial t},\\
&\gamma\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,w)\right] - k\frac{\partial v}{\partial r} - 2kw = \rho j\frac{\partial w}{\partial t},
\end{aligned}
$$

in which the cylindrical coordinate system (r, θ, z) has been used and where v is the velocity component in z-direction, w is the microrotation component in θ-direction, v and w being functions of (r, t) only; p is an unknown pressure to be specified or determined; ρ is the density and j the microinertia; μ, k, γ are the material constants.

Let us assume a sinusoidal pressure gradient of the form