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PULSATILE FLOW OF A MICROPOLAR FLUID IN A RIGID TUBE WITH LONGITUDINAL VIBRATION

BY

VALERIU A. SAVA

1. Introduction. The analysis of pulsating flows has been of interest for a long time. Such a motion would explain the blood flow in vascular beds.

Recently, Turk and al. [1] have used the linear micropolar theory of fluids as a model for blood. They have considered the pulsatile flow of a micropolar fluid through a rigid circular tube due to a sinusoidally varying pressure gradient.

In the present work the additional complexity of longitudinal wall vibration has been considered. The interest in the effects of such wall motion has resulted from an investigation of cardiovascular behavior in the presence of low-frequency vibration which is currently in progress. Such vibration effects are of interest for jackhammer operators, astronauts riding boosters, and other workers who perform in vibration environments. The solutions of governing partial differential equations are obtained through application of consecutive transformations: a finite Hankel transform is made on the space variable followed by a Laplace transform on time variable.

2. Formulation of the problem. For axisymmetric, incompressible, parallel flow the field equations governing the motion of the micropolar fluids [2] reduces to

$$
(1)\qquad
\begin{aligned}
&(\mu + k)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + k\frac{1}{r}\frac{\partial}{\partial r}(rw) - \frac{\partial p}{\partial z} = \rho\frac{\partial v}{\partial t},\\
&\gamma\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,w)\right] - k\frac{\partial v}{\partial r} - 2kw = \rho j\frac{\partial w}{\partial t},
\end{aligned}
$$

in which the cylindrical coordinate system (r, θ, z) has been used and where v is the velocity component in z-direction, w is the microrotation component in θ-direction, v and w being functions of (r, t) only; p is an unknown pressure to be specified or determined; ρ is the density and j the microinertia; μ, k, γ are the material constants.

Let us assume a sinusoidal pressure gradient of the form

$$-\frac{\partial p}{\partial z} = P_m + P_0 \sin \omega t, \tag{2}$$

where P_m is the mean pressure gradient, P_0 is the amplitude of sinusoidal pressure gradient and ω is the circular frequency of oscillation.

The boundary conditions are

$$v(R_0, t) = v_0 \sin(\Omega t + \varphi), \tag{3}$$

$$\frac{\partial}{\partial r}(rw) = 0, \quad \text{on } r = R_0, \tag{4}$$

where v_0 is the oscillation amplitude, Ω is frequency of vibration, and φ an initial phase angle. R_0 is the tube radius. In addition one wants

$$v(r, 0) = 0, \; w(r, 0) = 0, \; \left.\frac{\partial w}{\partial t}\right|_{t=0} = 0.$$

The boundary condition (4) represents a physically reasonable boundary condition for the microrotation at the tube wall when applying micropolar fluid theory to the flow of blood [1].

If we make the following transformation

$$u(r, t) = \frac{1}{r}\frac{\partial}{\partial r}(rw) \tag{5}$$

and substitute this expression and (2) in (1), we obtain the following transformed pair of coupled partial differential equations

$$\begin{aligned} &(\mu + k)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + ku + P_m(1 + \varepsilon \sin \omega t) = \rho\frac{\partial v}{\partial t}, \\ &\gamma\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - 2ku = \rho j\frac{\partial u}{\partial t}, \end{aligned} \tag{6}$$

where $\varepsilon = P_0/P_m$.

Now, taking the finite Hankel transformation [3] of equations (6) with respect to space variable r, and taking into account (3), (4) yields

$$\begin{aligned} &-(\mu + k)\,\xi_i^2\bar{v}_J + k\bar{u}_J + (\mu + k)\,R_0 v_0 \xi_i J_1(R_0\xi_i)\sin(\Omega t + \varphi) + \\ &+ \frac{P_m R_0(1 + \varepsilon \sin \omega t)}{\xi_i} J_1(R_0\xi_i) = \rho\frac{d\bar{v}_J}{dt}, \end{aligned} \tag{7}$$

$$\begin{aligned} &-\gamma\xi_i^2\,\bar{u}_J + k\xi_i^2\,\bar{v}_J - kR_0 v_0 \xi_i J_1(R_0\xi_i)\sin(\Omega t + \varphi) - \\ &- 2k\bar{u}_J = \rho j\frac{d\bar{u}_J}{dt}, \end{aligned} \tag{8}$$

where $\bar{v}_J = H_0\{v(r, t)\}$, $\bar{u}_J = H_0\{u(r, t)\}$, and

$$H_0\{f(r)\} = \int_0^{R_0} r f(r) J_0(\xi_i r)\, dr, \tag{9}$$

is the definition of finite Hankel transformation. The parameters ξ_i ($i = 1, 2, \ldots$) are choosen to be the positive roots of the transcendental equation

$$J_0(R_0\xi_i) = 0 \tag{10}$$

where J_0 is the Bessel function of order zero.

Now, if we consider the Laplace transform of equations (7) and (8) with respect to time, we get:

$$\begin{aligned} &-(\mu + k)\,\xi_i^2\bar{v}_{J^*} + k\bar{u}_{J^*} + \frac{P_m R_0}{\xi_i}\left(\frac{1}{s} + \frac{\varepsilon\omega}{s^2 + \omega^2}\right)J_1(R_0\xi_i) + \\ &+ (\mu + k)\,R_0 v_0 \xi_i\left(\frac{\Omega}{s^2 + \Omega^2}\cos\varphi + \frac{s}{s^2 + \Omega^2}\sin\varphi\right)J_1(R_0\xi_i) = \rho s\bar{v}_{J^*}. \end{aligned} \tag{11}$$

$$\begin{aligned} &-\gamma\xi_i^2\,\bar{u}_{J^*} + k\,\xi_i^2\bar{v}_{J^*} - kR_0 v_0 \xi_i\left(\frac{\Omega}{s^2 + \Omega^2}\cos\varphi + \frac{s}{s^2 + \Omega^2}\sin\varphi\right)J_1(R_0\xi_i) - \\ &- 2k\bar{u}_{J^*} = \rho j s\bar{u}_{J^*}. \end{aligned} \tag{12}$$

where $\bar{v}_{J^*} = L\{\bar{v}_J(\xi_i, t)\}$, $\bar{u}_{J^*} = L\{\bar{u}_J(\xi_i, t)\}$.

We have to solve the simultaneous linear algebraic equation (11) and (12) for $\bar{v}_{J^*}$ and $\bar{u}_{J^*}$. The results are

$$\begin{aligned} &\bar{v}_{J^*} = \frac{\gamma\xi_i^2 + 2k + \rho j s}{k\xi_i^2}\bar{u}_{J^*} + \frac{R_0 v_0 J_1(R_0\xi_i)}{\xi_i}\left(\frac{\Omega\cos\varphi + s\sin\varphi}{s^2 + \Omega^2}\right). \\ &\bar{u}_{J^*} = \frac{kP_m R_0\xi_i J_1(R_0\xi_i)}{\rho^2 j}\left[\frac{s^2 + \varepsilon\omega s + \omega^2}{s(s^2 + \omega^2)Q^*(s)}\right] - \\ &\quad - \frac{kv_0 R_0\xi_i J_1(R_0\xi_i)}{\rho j}\left[\frac{s^2\sin\varphi + s\Omega\cos\varphi}{(s^2 + \Omega^2)\,Q^*(s)}\right], \end{aligned} \tag{13}$$

where

$$Q^*(s) = s^2 + \frac{\xi_i^2}{\rho j}\left(j(\mu + k) + \gamma + \frac{2k}{\xi_i^2}s + \frac{\xi_i^2\gamma}{\rho^2 j}(\mu + k)(\xi_i^2 + \lambda^2)\right), \tag{14}$$

and $\lambda^2 = (2\mu + k)\,k/[\gamma(\mu + k)]$.

Now, it is possible to find the inverse Laplace transforms of (13) i.e. $\bar{v}_J(\xi_i, t) = L^{-1}\{\bar{v}_{J^*}(\xi_i, s)\}$ and $\bar{u}_J(\xi_i, t) = L^{-1}\{\bar{u}_{J^*}(\xi_i, s)\}$, through application of the residue theorem. The results are

$$\bar{v}_J(\xi_i, t) = \frac{P_m R_0 J_1(R_0\xi_i)}{\xi_i \rho^2 j r_1 r_2}\Big[(2k + \gamma\xi_i^2) + $$

$$+ \frac{(\rho j r_1 + \gamma\xi_i^2 + 2k)(r_1^2 + \varepsilon\omega r_1 + \omega^2)}{r_1^2 + \omega^2}\left(\frac{r_2}{r_1 - r_2}\right)e^{r_1 t} +$$

$$+ \frac{(\rho j r_2 + \gamma\xi_i^2 + 2k)(r_2^2 + \varepsilon\omega r_2 + \omega^2)}{r_2^2 + \omega^2}\left(\frac{r_1}{r_2 - r_1}\right)e^{r_2 t} +$$

$$+ \frac{\varepsilon r_1 r_2}{(r_1^2 + \omega^2)(r_2^2 + \omega^2)}(\alpha_1 \cos\omega t + \beta_1 \sin\omega t)\Big] - \tag{15}$$

$$- \frac{v_0 R_0 J_1(R_0\xi_i)}{\xi_i \rho j}\Big[\frac{(\rho j r_1 + 2k + \gamma\xi_i^2)(r_1 \sin\varphi + \Omega\cos\varphi)}{r_1^2 + \Omega^2}\left(\frac{r_1}{r_1 - r_2}\right)e^{r_1 t} +$$

$$+ \frac{(\rho j r_2 + 2k + \gamma\xi_i^2)(r_2 \sin\varphi + \Omega\cos\varphi)}{r_2^2 + \Omega^2}\left(\frac{r_2}{r_2 - r_1}\right)e^{r_2 t} +$$

$$+ \frac{\Omega}{(r_1^2 + \Omega^2)(r_2^2 + \Omega^2)}(\alpha_4 \cos\Omega t + \beta_4 \sin\Omega t)\Big],$$

$$\bar{u}_J(\xi_i, t) = \frac{k P_m R_0 \xi_i J_1(R_0\xi_i)}{\rho^2 j r_1 r_2}\Big[1 + \frac{r_1^2 + \varepsilon\omega r_1 + \omega^2}{r_1^2 + \omega}\left(\frac{r_2}{r_1 - r_2}\right)e^{r_1 t} +$$

$$+ \frac{r_2^2 + \varepsilon\omega r_2 + \omega^2}{r_2^2 + \omega^2}\left(\frac{r_1}{r_2 - r_1}\right)e^{r_2 t} +$$

$$+ \frac{\varepsilon r_1 r_2}{(r_1^2 + \omega^2)(r_2^2 + \omega^2)}(\alpha_2 \cos\omega t + \beta_2 \sin\omega t)\Big] - \tag{16}$$

$$- \frac{k v_0 R_0 \xi_i J_1(R_0\xi_i)}{\rho j}\Big[\frac{r_1 \sin\varphi + \Omega\cos\varphi}{r_1^2 + \Omega^2}\left(\frac{r_1}{r_1 - r_2}\right)e^{r_1 t} +$$

$$+ \frac{r_2 \sin\varphi + \Omega\cos\varphi}{r_2^2 + \Omega^2}\left(\frac{r_2}{r_2 - r_1}\right)e^{r_2 t} +$$

$$+ \frac{\Omega}{(r_1^2 + \Omega^2)(r_2^2 + \Omega^2)}(\alpha_3 \cos\Omega t + \beta_3 \sin\Omega t)\Big],$$

where

$$\alpha_1 = \rho j\omega(r_1 r_2 - \omega^2) + \omega(r_1 + r_2)(2k + \gamma\xi_i^2), \quad \alpha_2 = \omega(r_1 + r_2),$$

$$\alpha_3 = \Omega(r_1 + r_2)\sin\varphi - r_1 r_2 \cos\varphi + \Omega^2\cos\varphi,$$

$$\alpha_4 = (\gamma\xi_i^2 + 2k)[\Omega(r_1 + r_2)\sin\varphi - r_1 r_3 \cos\varphi + \Omega^2\cos\varphi] + \rho j\Omega[\Omega^2 \sin\varphi -$$

$$- r_1 r_2 \sin\varphi - \Omega(r_1 + r_2)\cos\varphi],$$

$$\beta_1 = (r_1 r_2 - \omega^2)(2k + \gamma\xi_i^2) - \rho j\omega^2(r_1 + r_2), \quad \beta_2 = r_1 r_2 - \omega^2,$$

$$\beta_3 = \Omega(r_1 + r_2)\cos\varphi - r_1 r_2 \sin\varphi + \Omega^2\sin\varphi,$$

$$\beta_4 = (2k + \gamma\xi_i^2)[\Omega(r_1 + r_2)\cos\varphi - r_1 r_2 \sin\varphi + \Omega^2\sin\varphi] +$$

$$+ \rho j\Omega[\Omega^2\cos\varphi - r_1 r_2 \cos\varphi + \Omega(r_1 + r_2)\sin\varphi],$$

and r_1, r_2 are the roots of the quadratic expression $Q^*(s)$ given by (14).

To determine the properties of r_1 and r_2 it is necessary to specify a value for j. We shall consider the limiting case in which

$$\nu = 1/2\, \nabla \times V$$

or in component form

$$w(r, t) = -\frac{1}{2}\frac{\partial v(r,t)}{\partial r}. \tag{17}$$

Substituting (17) in equations (1) yields, after simplification

$$j = 2\gamma/(2\mu + k) \tag{18}$$

Using the value for j given by (18) in (14) we see that $Q^*(s)$ has two real distinct negative roots r_1 and r_2.

To complete the solution, it remains to obtain the inverse finite Hankel transforms of $\bar{v}_J(\xi_i, t)$ and $\bar{u}_J(\xi_i, t)$ by use of

$$f(r) = H_0^{-1}(\bar{f}_J) = \frac{2}{R_0^2}\sum_i \bar{f}_J(\xi_i)\frac{J_0(r\xi)}{\{J_1(R_0\xi_i)\}^2}.$$

Hence

$$v(r, t) = \frac{2}{R_0^2}\sum_i \bar{v}_J(\xi_i, t)\frac{J_0(r\xi_i)}{\{J_1(R_0\xi_i)\}^2}, \tag{19}$$

$$u(r, t) = \frac{2}{R_0^2}\sum_i \bar{u}_J(\xi_i, t)\frac{J_0(r\xi_i)}{\{J_1(R_0\xi_i)\}^2}. \tag{20}$$

From the equation (5) with the help of (20) it results

$$w(r, t) = \frac{2}{R_0^2}\sum_i \frac{\bar{u}_J(\xi_i, t)}{\xi_i}\frac{J_0(r\xi_i)}{\{J_1(R_0\xi_i)\}^2}. \tag{21}$$

Combining equations (15) and (16) with equations (19) and (21), respectively, the complete transient velocity and microrotation profiles for the flow of micropolar fluid through a circular tube with longitudinal vibration are obtained.

If we allow time to approach infinity it follows

$$v(r, t) = \frac{2P_m}{\rho^2 j R_0} \sum_i \frac{J_0(r\xi_i)}{\xi_i r_1 r_2 J_1(R_0\xi_i)} \Big[(2k + \gamma \xi_i^2) + \tag{22}$$
$$+ \frac{\varepsilon r_1 r_2}{(r_1^2 + \omega^2)(r_2^2 + \omega^2)} (\alpha_1 \cos \omega t + \beta_1 \sin \omega t) \Big] -$$
$$- \frac{2v_0\Omega}{\rho j R_0} \sum_i \frac{J_0(r\xi_i)}{\xi_i r_1 r_2 J_1(R_0\xi_i)} \cdot \frac{\alpha_1 \cos \Omega t + \beta_1 \sin \Omega t}{(r_1^2 + \Omega^2)(r_2^2 + \Omega^2)};$$

$$w(r, t) = \frac{2kP_m}{\rho^2 j R_0} \sum_i \frac{J_1(r\xi_i)}{r_1 r_2 J_1(R_0\xi_i)} \Big[1 + \frac{\varepsilon r_1 r_2}{(r_1^2+\omega^2)(r_2^2+\omega^2)} (\alpha_2 \cos \omega t + \tag{23}$$
$$+ \beta_2 \sin \omega t) \Big] - \frac{2kv_0\Omega}{\rho j R_0} \sum_i \frac{J_1(r\xi_i)}{r_1 r_2 J_1(R_0\xi_i)} \cdot \frac{\alpha_3 \cos \Omega t + \beta_3 \sin \Omega t}{(r_1^2 + \Omega^2)(r_2^2 + \Omega^2)}.$$

In comparison with the results of [1], we see, that wall oscillations can significantly alter the pulsating flow in a rigid tube.

These results suggest, also, that important changes in peripheral vascular resistance and the shear stress to which blood vessels are exposed may occur during whole-body vibration.

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RECENZII

D. T. WHITESIDE (Editor): *The Mathematical Papers of Isaac Newton*. Volume VII. 1961—1965. Cambridge University Press, 1976. LXVIII + 706 p.

Volumul cuprinde cercetările matematice ale lui Newton din perioada ultimilor 5 ani petrecuţi la Cambridge. Acestea au trei centre principale de interes: metodele fluxiunilor şi seriilor, geometria clasică pură şi geometira analitică carteziană. Prima parte a volumului reproduce rezultatele obţinute în extinderea tehnicilor de calcul combinat al fluxiunilor şi fluentelor şi de exprimare prin serii infinite. A doua parte, care este cea mai extinsă, prezintă pentru prima dată sub formă tipărită tratatul detaliat al lui Newton asupra metodelor de analiză geometrică utilizate de geometrii antici şi urmaşii lor. A treia parte prezintă textul final al lucrării „Enumeratio Linearum Tertii Ordinis" (Enumerarea curbelor de ordinul al III-lea), redactată de Newton în 1695 pe baza versiunilor anterioare; aceasta tratează clasificarea algebrică a curbelor cubice.

Jack Weinstein

HANS REICHARDT: *Gauss und die nicht-euklidische Geometrie*. Teubner Verlagsgesellschaft, Leipzig, 1976, 116 p.

Aniversarea bicentenarului naşterii lui Gauss (30 aprilie 1777) a dat autorului prilejul de a evoca contribuţia lui Gauss la apariţia şi dezvoltarea geometriei neeuclidiene. Materialul este grupat în trei capitole. În primul (14 p.)se prezintă antecedentele geometriei neeuclidiene, diversele încercări de demonstrare directă sau indirectă, a axiomei paralelelor, necontradicţia geometriei euclidiene. Capitolul II (66 p.) conţine rezultatele lui Gauss în geometria neeuclidiană, aşa cum le găsim în notiţele sale şi în corespondenţa sa cu sau despre Taurinus, Gerling, Schumacher, Bessel, Wolfgang şi Johann Bolyai, Lobacevski, Riemann; se dau extrase esenţiale din însemnări şi din scrisori. Ultimul capitol (23 p.) subliniază influenţa moştenirii lui Gauss — recunoaşterea progresivă a geometriei neeuclidiene, realizarea unor modele, apariţia unor direcţii noi în geometrie (Felix Klein, Sophus Lie, David Hilbert). În încheiere se dă o bibliografie de 34 titluri şi un indice de nume şi noţiuni.

Deşi densă, expunerea este de o claritate desăvîrşită, care permite parcurgerea cărţii nu numai fără dificultăţi, dar chiar cu plăcere. De altfel, autorul, geometru reputat, membru al Academiei de Ştiinţe a Republicii Democrate Germane, este un profund cunoscător al operei lui Gauss, el fiind şi coordonatorul volumului omagial publicat în 1955 la centenarul morţii lui Gauss.

D. Rimer

D. S. KOREIMANN: *Lexicon der angewandten Datenverarbeitung*, Walter de Gruyter, Berlin, New York, 1977, 71 fig., 340 p.

Lexiconul referitor la prelucrarea datelor cu caracter aplicativ conţine un număr mare de articole care se referă la cei mai vehiculaţi termeni dintr-o serie de domenii ale informaticii cu caracter aplicativ: analiza sistemelor informaţionale şi proiectarea acestora, conducerea cu mijloace de calcul a proceselor de producţie, software de bază, procese de calcul în timp real, etc. Lucrarea este utilă tuturor informaticienilor precum şi celor care doresc să-şi clarifice sensul unor noţiuni de informatică de largă circulaţie.

Dan Simovici

HEINRICH ROTH: *Der Lehrer und seine Wissenschaft, Errinertes und Aktuelles, Ein Interview von Dagmar Friedrich zu seinem 70. Geburtstag*, Hermann Schroedel Verlag, Hannover Dortmund, Darmstadt, Berlin (1976), 168 p.

La a 70-a aniversare a naşterii sale, Heinrich Roth, profesor de pedagogie la Universitatea din Göttingen, autorul a numeroase tratate de bază în probleme psiho-pedagogice îşi