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ON $C_G(I, E_n)$ SPACES OF CONTINUOUS FUNCTIONS*)

BY

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Let K be a linear mapping from the euclidean space E^n into itself. By $\| x \|$ we denote the Euclidean norm of $x \in E^n$, while $| K | = \sup \{ \| Kx \| : \| x \| = 1\}$ is the norm of the mapping K. We denote further by $R(K)$ and $N(K)$ the range and the kernel (null space) of K. Let $P(K)$ be the projector of E^n on $R(K)$. By $\tilde{K}$ we shall denote the restriction of K to $(N(K))^{\perp}$ (the orthogonal complement of $N(K)$). Let $\widetilde{K}^{-1}$ be the inverse of $\widetilde{K}$, defined on $R(\widetilde{K}) = R(K)$ and let us define K_{-1} by $K_{-1} = \widetilde{K}^{-1} P(K)$. If $I = \{t : t \in E, 0 \leqslant t < \omega \leqslant +\infty\}$ and $\{e_1, e_2, \ldots, e_n\}$ is a basis in E^n, let $t \to G(t)$ be an $n \times n$ matrix function, continuous on I (i.e. a mapping from I into the space of linear transformations from E^n into itself, the basis $\{e_1, \ldots, e_n\}$ being fixed).

We say that a continuous function $x(t)$ from I into E^n belongs to $C_G(I, E^n)$, if and only if $P(G(t))\, x(t) = x(t)$ for any $t \in I$ and $v(t) = G_{-1}(t)\, x(t)$ is bounded on I. The norm in C_G is defined by

$$\| x \|_G = \sup \{ | G_{-1}(t)\, x(t) | ;\ t \in I\}$$

and, as shown in [1] C_G is a Banach space.

The aim of this paper is to give a necessary and sufficient condition for two Banach spaces C_G and C_H to contain the same elements. In this case it follows also that they are isomorphic.

We first prove the following result:

Lemma 1. *Let $G(t)$ and $H(t)$ be two $n \times n$ matrices, such that $R(G(t)) = R(H(t))$, $t \in I$. Then the following formulas hold:*

$$(H_{-1} G)_{-1} = G_{-1} H, \tag{1}$$

$$| G | \geqslant | G_{-1} |^{-1}, \qquad | H | \geqslant | H_{-1} |^{-1}. \tag{2}$$

Proof. Let us remark that

$$(H_{-1} G)_{-1} = (\widetilde{H_{-1} G})^{-1} P(H_{-1} G) = (\widetilde{H}^{-1} \widetilde{G})^{-1} P(H_{-1} G). \tag{3}$$

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Indeed we have

$$(4)\qquad \overgroup{H_{-1}G} = \overgroup{\widetilde{H}^{-1}P(H)G} = \overgroup{\widetilde{H}^{-1}G}.$$

if we observe that $R(H) = R\,(G)$. For the same reason, $N(G) = N(\widetilde{H}^{-1}G)$, from which we get $\overgroup{\widetilde{H}^{-1}G} = \widetilde{H}^{-1}\widetilde{G}$. Since $\widetilde{H}$ and $\widetilde{G}$ are invertible on their domains, we can write $(\widetilde{H}^{-1}\widetilde{G})^{-1} = \widetilde{G}^{-1}\widetilde{H}$. Therefore, taking into account (3) and (4) and the above formulas, we obtain

$$(H_{-1}G)_{-1} = (\widetilde{H}^{-1}\widetilde{G})^{-1}P\,(H_{-1}G) = (\widetilde{G}^{-1}\widetilde{H})\,P\,(H_{-1}G) =$$

$$(5)\qquad = (\widetilde{G}_{-1}\,P\,(G)\,\widetilde{H})\,P\,(H^{-1}G) = (G_{-1}\widetilde{H})\,P\,(H_{-1}G) = (\overgroup{G_{-1}H})\,P\,(H_{-1}G) =$$

$$= G_{-1}H,$$

because $R\,(H_{-1}G) = (N\,(H))^{\perp}$ and, therefore, for any $x \in E^n$ we can urite

$$(G_{-1}H)\,x = (\overgroup{G_{-1}H})\,P\,(H_{-1}G)\,x.$$

In order to prove relations (2), we remark that

$$|G| = \sup\{|Gx| : |x| = 1\} = \sup\{|Gx| : |x| = 1,\ x \in (N(G))^{\perp}\} =$$

$$= |\widetilde{G}| \geqslant |\widetilde{G}^{-1}|^{-1} = (\sup\{|\widetilde{G}^{-1}x| ; |x| = 1,\ x \in R\,(G)\})^{-1} =$$

$$= (\sup\{|\widetilde{G}^{-1}P\,(G)\,x| ; |x| = 1\})^{-1} = (\sup\{|G_{-1}x| : |x| = 1\})^{-1} = |G_{-1}|^{-1}$$

The lemma is now completely proved.

Theorem 1. *Let $G\,(t)$ and $H\,(t)$ be two $n \times n$ matrices, continuous on I. A necessary and sufficient condition for C_G and C_H to have the same elements is that*

$$(6)\qquad R\,(G\,(t)) = R\,(H\,(t)), \qquad t \in I.$$

and

$$(7)\qquad \alpha = \sup\{|H_{-1}(t)\,G\,(t)| : t \in I\} < +\infty,\ \beta = \sup\{|G_{-1}(t)\,H\,(t)| ;\ t \in I\} < +\infty.$$

If (6) and (7) are verified, then $\|x\|_G$ and $\|x\|_H$ are equivalent norms.

Proof. Let $g_i\,(t)$ $(h_i\,(t))$ be the i-th column of the matrix $G(t)$ $(H\,(t))$ for $i = 1,\ldots, n$. The vectors $g_i\,(t)$ $(h_i\,(t))$ are continuous functions from I into E^n: $g_i\,(t) = G\,(t)\,e_i$ $(h_i\,(t) = H\,(t)\,e_i)$, for $1 \leqslant i \leqslant n$. We prove first that our condition is necessary. Let us remark that $g_i \in C_G$. Indeed $P\,(G\,(t))\,g_i\,(t) = g_i(t)$, $R\,(G\,(t))$ being spanned by the vectors $g_i\,(t)$, $1 \leqslant i \leqslant n$. We have further

$$|G_{-1}(t)\,g_i\,(t)| = |G_{-1}(t)\,G\,(t)\,e_i| \leqslant |G_{-1}(t)\,G\,(t)| = 1.$$

taking into account that $x \in (N\,(G))^{\perp}$ implies $G_{-1}(t)\,G\,(t)\,x = x$. If C_G and C_H have the same elements, then $g_i \in C_G$ implies $g_i \in C_H$ for $1 \leqslant i \leqslant n$. Therefore, $R\,(G\,(t)) \in R\,(H\,(t))$. If we interchange G and H we obtain the converse inclusions Hence $R\,(G\,(t)) = R\,(H\,(t))$.

Now we prove that conditions (7) are also necessary. We remark that for any $n \times n$ matrix A, with column vector $a_i 1 \leqslant i \leqslant n$, we have $|A| \leqslant$ $\leqslant |a_1|^2 + \ldots + |a_n|^2$. Hence, we shall have $|H_{-1}\ (t)\ G\ (t)| \leqslant$ $\sum_1^n{}_i\ |H_{-1}\ (t)\ g_i\ (t)|$. Therefore, if $\alpha = +\infty$, then $\sup\{|H_{-1}\ (t)$ $g_i\ (t)| :\ t \in I\} = +\infty$ for at least one index i, $1 \leqslant i \leqslant n$. This means that $g_i\,(t) \notin C_H$ for such index i, and consequently $C_G \neq C_H$. By a similar argument one obtains that the second condition in (7) is also necessary.

Let us now prove that this conditions suffice. If $x\,(t) \in C_G\ (I, E^n)$, we have $P\,(H\,(t))\,x\,(t) = x\,(t)$ according to (6). Furthermore,

$$(8)\qquad \begin{aligned} |H_{-1}(t)\,x\,(t)| &= |H_{-1}(t)\,G\,(t)\,G_{-1}(t)\,x\,(t)| \leqslant \sup\{|H_{-1}(t)\,G\,(t)| ;\\ &\quad t \in I\}.\ \{\sup|G_{-1}(t)\,x\,(t)| ;\ t \in I\} = \alpha\,\|x\|_G, \end{aligned}$$

which shows that $x\,(t) \in C_H\,(I\,;E^n)$. Similarly we prove that $C_H \subset C_G$ using (6) and the second condition (7). Therefore, conditions (6) and (7) suffice to show that $C_G = G_H$.

The last assertion of the theorem follows easily from the above considerations. Indeed from (7) we derive $\|x\|_H \leqslant \|x\|_G$. Since $\|x_G\| \leqslant$ $\leqslant \beta\,\|x\|_H$, we can write

$$(9)\qquad \beta^{-1}\,\|x\|_G \leqslant \|x\|_H \leqslant \alpha\,\|x\|_G, \qquad x \in C_G = C_H,$$

which ends the proof.

Corollary 1. *Let $G\,(t)$ and $H\,(t)$ be two matrices which are continuous on I. A necessary condition that $C_G = C_H$ is that $R\,(G\,(t)) = R\,(H\,(t))$ for any $t \in I$ and*

$$0 < \inf\{|G_{-1}(t)\,H\,(t)| ;\quad t \in I\},$$

$$0 < \inf\{|H_{-1}(t)\,G\,(t)| ;\quad t \in I\}.$$

Proof. This follows easily using Lemma 1. From Lemma 1, $|H_{-1}(t)\,G\,(t)| \geqslant |G_{-1}(t)\,H\,(t)|^{-1}$. Consequently, the condition $\sup\{|H_{-1}(t)\,G(t)| ;$ $t \in I\} < +\infty$ implies $0 < \inf\{|G_{-1}(t)\,H\,(t)| ;\ t \in I\}$. A similar statement holds if we change the roles of G and H.

Remark. Let us consider the matrices

$$G\,(t) = \begin{pmatrix} 1 & g\,(t) \\ 0 & 1 \end{pmatrix}, \qquad G_{-1}(t) = G^{-1}(t) = \begin{pmatrix} 1 & -g\,(t) \\ 0 & 1 \end{pmatrix},$$

with $g\,(t)$ continuous and unbounded on I. Then

$$|G\,(t)| = |G_{-1}(t)| = \sqrt{2^{-1}}\,.\,\sqrt{2 + g^2\,(t)\,|g(t)|\sqrt{4 + g^2\,(t)}}$$

are bounded below by 1 but they are not bounded above. Therefore, the statement of the Corollary 1 cannot be improved. We owe this remark to professor Allen Ziebur.

Let us consider now a particular case. Namely assume that $G(t)$ and $H(t)$ are diagonal matrices and $g_{ii}(t) = g(t) > 0$, $h_{ii}(t) = h(t) > 0$, $1, \ldots, n$. Then $G_{-1}(t) = G^{-1}(t)$, $H_{-1}(t) = H^{-1}(t)$ and G^{-1}, H^{-1} are also diagonal matrices. One has $G_{-1}(t)\, x(t) = x(t)/g(t)$ and $H_{-1}(t)\, x(t) = x(t)/h(t)$. In this way, one obtains the space C_g considered in (2).

Corollary 2. *A necessary and sufficient condition that*

$$C_g(I : E^n) = C_h(I : E^n),$$

with $g(t)$ and $h(t)$ continuous positive functions on I, is that

$$(10) \qquad 0 < \inf \{ g(t)/h(t) : t \in I \} \leqslant \sup \{ g(t)/h(t) : t \in I \} < +\infty .$$

Proof. The proof is immediate if we note that $H_{-1}(t)\, G(t)$ is a diagonal matrix whose elements on the main diagonal are equal to $g(t)/h(t)$.

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A CHARACTERIZATION OF THE PROBABILISTIC TOPOLOGICAL SPACES

BY

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1. The concept of a probabilistic topological space, introduced by M. J. F r a n k [3], is a natural generalization of the notion of a probabilistic metric space, as it appears in M e n g e r' s papers: just as a metric determines a topology, a probabilistic metric determines a probabilistic topology, (theorem 4.1 [3]). Frank shows that the θ-closure θ' by this theorem is additive, and therefore we use for the notion of a θ-closure the following definition:

A θ-closure on a set X is a mapping $\theta : \mathfrak{P}(X) \times I \to \mathfrak{P}(X)$ satisfying
(C1) $\theta(A, 0) = X$, (C2) $\theta(\varnothing, \lambda) = \varnothing$ if $\lambda \neq 0$,
(C3) $A \subset \theta(A, \lambda)$, (C4) $\theta(A \cup B, \lambda) = \theta(A, \lambda) \cup \theta(B, \lambda)$, (C5) $\theta(A, \lambda) \supseteq \theta(A, \mu)$ if $\lambda \geqslant \mu$; for every $A, B \subset X$, $\lambda, \mu \in I$, where $I = [0,1]$.

A θ-closure is left-continuous if: (LC) $\theta(A, \lambda) = \cap \{ \theta(A, \mu) : \mu < \lambda \}$ for $\lambda \neq 0$. If θ is a left-continuous θ-closure on X and T is a t-function ($T(\lambda_1, \mu_1) \leqslant T(\lambda_2, \mu_2)$ whenever $\lambda_1 \leqslant \lambda_2$, $\mu_1 \leqslant \mu_2$) then the ordered triple (X, θ, T) is a probabilistic topological space if: (C6) $\theta(\theta(A, \mu), \lambda) \subseteq \theta(A, T(\lambda, \mu))$ if $A \neq \varnothing$.

We remark from (C2), (C3), (C4), that $\{ \theta(\cdot, \lambda) : \lambda \in (0, 1] \}$ is a set of closure operators on X (see for instance [1], p. 237).

The purpose of this paper is to give a characterization of the probabilistic topological spaces by means of topogenous orders.

A. C s á s z á r has shown (see [2], (7.20)) that there is an equivalence between topological spaces and Császár topological spaces in the sense that, given one of this structures, it is possible to determine the other.

We proved that, between closure spaces and the spaces with a perfect topogenous order, there is the same equivalence (see [4]).

Using this result, we introduce the notion of a *probabilistic topogenous structure*, with the help of which we give the characterization stated for the probabilistic topological structures.

We recall that a topogenous order on X is a relation $< \subset \mathfrak{P}(X) \times \mathfrak{P}(X)$ such that:

(01) $\varnothing < \varnothing$, $X < X$, (02), if $A < B$ then $A \subset B$, (03) if $A \subset A' < B' \subset B$ then $A < B$, (Q) if $A < B$, $A' < B'$ then $A \cup A' < B \cup B'$, $A \cap A' < B \cap B'$, for every $A, A', B, B' \subset X$.