are bounded below by 1 but they are not bounded above. Therefore, the statement of the Corollary 1 cannot be improved. We owe this remark to professor Allen Ziebur.

Let us consider now a particular case. Namely assume that $G(t)$ and $H(t)$ are diagonal matrices and $g_{ii}(t) = g(t) > 0$, $h_{ii}(t) = h(t) > 0$, $1, \ldots, n$. Then $G_{-1}(t) = G^{-1}(t)$, $H_{-1}(t) = H^{-1}(t)$ and G^{-1}, H^{-1} are also diagonal matrices. One has $G_{-1}(t)\, x(t) = x(t)/g(t)$ and $H_{-1}(t)\, x(t) = x(t)/h(t)$. In this way, one obtains the space C_g considered in (2).

Corollary 2. *A necessary and sufficient condition that*

$$C_g(I : E^n) = C_h(I : E^n),$$

with $g(t)$ and $h(t)$ continuous positive functions on I, is that

$$(10) \qquad 0 < \inf\{g(t)/h(t) : t \in I\} \leqslant \sup\{g(t)/h(t) : t \in I\} < +\infty .$$

Proof. The proof is immediate if we note that $H_{-1}(t)\, G(t)$ is a diagonal matrix whose elements on the main diagonal are equal to $g(t)/h(t)$.

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A CHARACTERIZATION OF THE PROBABILISTIC TOPOLOGICAL SPACES

BY

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1. The concept of a probabilistic topological space, introduced by M. J. F r a n k [3], is a natural generalization of the notion of a probabilistic metric space, as it appears in M e n g e r' s papers: just as a metric determines a topology, a probabilistic metric determines a probabilistic topology, (theorem 4.1 [3]). Frank shows that the θ-closure θ' by this theorem is additive, and therefore we use for the notion of a θ-closure the following definition:

A θ-closure on a set X is a mapping $\theta : \mathfrak{P}(X) \times I \to \mathfrak{P}(X)$ satisfying
(C1) $\theta(A, 0) = X$. (C2) $\theta(\varnothing, \lambda) = \varnothing$ if $\lambda \neq 0$.
(C3) $A \subset \theta(A, \lambda)$, (C4) $\theta(A \cup B, \lambda) = \theta(A, \lambda) \cup \theta(B, \lambda)$, (C5) $\theta(A, \lambda) \supseteq \theta(A, \mu)$ if $\lambda \geqslant \mu$: for every $A, B \subset X$, $\lambda, \mu \in I$, where $I = [0,1]$.

A θ-closure is left-continuous if: (LC) $\theta(A, \lambda) = \cap\{\theta(A, \mu) : \mu < \lambda\}$ for $\lambda \neq 0$. If θ is a left-continuous θ-closure on X and T is a t-function ($T(\lambda_1, \mu_1) \leqslant T(\lambda_2, \mu_2)$ whenever $\lambda_1 \leqslant \lambda_2$, $\mu_1 \leqslant \mu_2$) then the ordered triple (X, θ, T) is a probabilistic topological space if: (C6) $\theta(\theta(A, \mu), \lambda) \subseteq \theta(A, T(\lambda, \mu))$ if $A \neq \varnothing$.

We remark from (C2), (C3), (C4), that $\{\theta(., \lambda) : \lambda \in (0, 1]\}$ is a set of closure operators on X (see for instance [1], p. 237).

The purpose of this paper is to give a characterization of the probabilistic topological spaces by means of topogenous orders.

A. C s á s z á r has shown (see [2], (7.20)) that there is an equivalence between topological spaces and Császár topological spaces in the sense that, given one of this structures, it is possible to determine the other.

We proved that, between closure spaces and the spaces with a perfect topogenous order, there is the same equivalence (see [4]).

Using this result, we introduce the notion of a *probabilistic topogenous structure*, with the help of which we give the characterization stated for the probabilistic topological structures.

We recall that a topogenous order on X is a relation $< \subset \mathfrak{P}(X) \times \mathfrak{P}(X)$ such that:
(01) $\varnothing < \varnothing$, $X < X$, (02), if $A < B$ then $A \subset B$, (03) if $A \subset A' < B' \subset B$ then $A < B$, (Q) if $A < B$, $A' < B'$ then $A \cup A' < B \cup B'$, $A \cap A' < B \cap B'$, for every $A, A', B, B' \subset X$.

Let $<$ be a topogenous order on X and let $<^c$ be the order defined by $A <^c B$ if $X - B < X - A$ for every $A, B \subset X$. Using Császár's terminology, we call $<^c$ the complementary order of $<$. We say that a topogenous order $<$ is perfect if:

(*P*) If $A_\gamma < B$ for every $\gamma \in \Gamma$ then $\cup \{A_\gamma : \gamma \in \Gamma\} < B$, for every $A_\gamma, B \subset X, \gamma \in \Gamma$.

A syntopogenous structure on X is a family of topogenous orders, $\mathfrak{S} = \{<_a\}_{a \in \lambda}$, on X satisfying:

(S1) For every $<_a, <_b \in \mathfrak{S}$ there is $<_c \in \mathfrak{S}$ such that $<_a, <_b \subset <_c$.

(S2) For every $<_a \in \mathfrak{S}$ there is $<_b \in \mathfrak{S}$ such that for every $A, B \subset X$ with $A <_a B$ there exists $C \subset X$ such that $A <_b C <_b B$.

We remark that, using topogenous structure terminology, we can introduce in a natural way some structures placed between the probabilistic metrics structures and probabilistic topological structures; we call these structures, taking into account, their construction, *probabilistic uniform structures*. Their study is the topic of a future note.

Topogenous orders and closure spaces. Proposition 1. *Let X be an abstract set and $<$ be a topogenous order on X; the mapping* $(-) : \mathfrak{P}(X) \to \mathfrak{P}(X)$ *defined by* $\bar{A} = \cap \{B : A <^c B\}$ *for every* $A \subset X$ *is a closure operator on X.*

Proof. 1). $\bar{\varnothing} = \cap \{B : \varnothing <^c B\} = \varnothing$. 2). $X - \bar{A} = \cup \{X - B : A <^c B\} = \cup \{X - B : X - B < X - A\} \subset X - A$. 3). $\bar{A}_1 \cup \bar{A}_2 = (\cap \{B' : A_1 <^c B'\}) \cup (\cap \{B'' : A_2 <^c B''\}) \subset \cap \{B : A_1 \cup A_2 <^c B\} = \overline{A_1 \cup A_2}$ ($A_1 \cup A_2 <^c B$ implies $A_1 <^c B$, $A_2 <^c B$). Now, let $x \in \overline{A_1 \cup A_2}$ and $x \overline{\in} \bar{A}_1 \cup \bar{A}_2$; then $x \overline{\in} \bar{A}_1$ and $x \overline{\in} \bar{A}_2$, hence there exist B_1, B_2 such that $A_1 <^c B_1$, $A_2 <^c B_2$ and $x \overline{\in} B_1$, $x \overline{\in} B_2$; from (*Q*) and $X - B_1 < X - A_1$, $X - B_2 < X - A_2$, we have $(X - B_1) \cap (X - B_2) < (X - A_1) \cap (X - A_2)$ or $X - B_1 \cup B_2 < X - A_1 \cup A_2$, hence

$A_1 \cup A_2 <^c B_1 \cup B_2$; but $x \overline{\in} B_1 \cup B_2$, hence $x \overline{\in} \overline{A_1 \cup A_2}$. This is a contradiction.

Definition 1. *The mapping* $(^-)$, *defined as above, is called the closure induced by the order* $<$.

Theorem 1. *Let* $(^-)$ *be a closure operator on X. Then there exists a unique perfect topogenous order on X which induces* $(^-)$.

Proof. We define $< \subset \mathfrak{P}(X) \times \mathfrak{P}(X)$ by $A < B$ iff $A \cap \overline{X - B} = \varnothing$. Then:

(01) $\varnothing \cap \overline{X - \varnothing} = \varnothing$ hence $\varnothing < \varnothing$, $X \cap \overline{X - X} = \varnothing$, hence $X < X$; (02) $A < B$ implies $A \cap \overline{X - B} = \varnothing$, hence $A \subset X - \overline{X - B} \subset B$; (03). $A_1 \subset A < B \subset B_1$ implies $A_1 \subset X - \overline{X - B} \subset X - \overline{X - B_1}$, hence $A_1 < B_1$; (Q) if $A_1 < B_1$, $A_2 < B_2$ then $A_1 \cap \overline{X - B_1} = \varnothing = A_2 \cap \overline{X - B_2}$. But $(A_1 \cap A_2) \cap \overline{X - B_1 \cap B_2} = (A_1 \cap A_2) \cap (\overline{X - B_1}) \cup (\overline{X - B_2}) = (A_1 \cap A_2 \cap \overline{X - B_1}) \cup (A_1 \cap A_2 \cap \overline{X - B_2}) = \varnothing$; hence $A_1 \cap A_2 < B_1 \cap B_2$. Now, $(A_1 \cup A_2) \cap \overline{X - B_1 \cup B_2} = (A_1 \cup A_2) \cap (\overline{X - B_1}) \cap \overline{(X - B_2)} = [A_1 \cap (\overline{X - B_1}) \cap \overline{(X - B_2)}] \cup [A_2 \cap (\overline{X - B_1}) \cap (\overline{X - B_2})] = \varnothing$ hence $A_1 \cup A_2 < B_1 \cup B_2$; P is verified; indeed: if $A_\gamma < B$ $\gamma \in \Gamma$ then $A_\gamma \subset X - \overline{X - B}$ for every $\gamma \in \Gamma$, hence $\cup \{A_\gamma : \gamma \in \Gamma\} \subset X - \overline{X - B}$ or $\cup \{A_\gamma : \gamma \in \Gamma\} < B$. Therefore $<$ is a perfect topogenous order on X.

Let $(^{-1})$ be the closure operator induced by $<$. For every $A \subset X$ $\bar{A}^1 = \cap \{B : A <^c B\} = \cap \{B : X - B < X - A\} = \cap \{B : (X - B) \cap \bar{A} = \varnothing\} = \cap \{B : \bar{A} \subset B\} = \bar{A}$. Hence $(^{-1}) = (^-)$. If $<_1$ is another perfect topogenous order which induces $(^-)$, then we have: $A < B$ if and only if $X - B <^c X - A$ iff $\overline{X - B} \subset X - A$ iff $\cap \{B' : X - B <^c_1 B'\} \subset X - A$ iff $\cap \{B' : X - B' <_1 B\} \subset X - A$. Let $\mathfrak{A} = \{B' : X - B' <_1 B\}$. Since $<_1$ is perfect, $\cup \{X - B' : B' \in \mathfrak{A}\} <_1 B$ iff $X - B <^c_1 (\cap \{B' : B' \in \mathfrak{A}\}) \subset X - A$ iff $A <_1 B$; hence $< = <_1$.

Consequence: if $\mathfrak{S}_p$ is the family of all perfect topogenous orders on X and $\mathfrak{J}$ is the family of closure operators on X, then the mapping $f : \mathfrak{S}_p \to \mathfrak{J}$, $f(<) = (^-)$, where $(^-)$ is the closure induced by $<$, is a bijection.

3. Probabilistic topogenous structures and probabilistic topological spaces.

Definition 2. Let T be a t-function and $\mathfrak{Q} = \{<_\lambda\}_{\lambda \in I}$ be a family of orders on X, where $A <_0 B$ iff $A = \varnothing$ and $<_\lambda$ is a perfect topogenous order on X for every $\lambda \in (0,1]$. Then we call the ordered pair $(\mathfrak{Q}, T)$ a *probabilistic topogenous structure on X if*:

(PT1). If $\lambda \leqslant \mu$ then $<_\lambda \subseteq <_\mu$. (PT2) if $\{x\} <_\lambda A$ there is $\mu < \lambda$ such that $\{x\} <_\mu A$, for $\lambda \neq 0$, $A \neq \varnothing$. (PT3) if $A <^c_{T(\lambda,\mu)} B$ then there exists C such that $A <^c_\mu C <^c_\lambda B$; for every $\lambda, \mu \in I$, $A, B \subset X$.

If $(\mathfrak{Q}, T)$ is a probabilistic topogenous structure on X we say that the ordered triple $(X, \mathfrak{Q}, T)$ is a *probabilistic topogenous space*.

Remarks. 1) The relation $<_0$ verifies all axioms of a perfect topogenous order except one: $X <_0 X$. 2). If $(\mathfrak{Q}, T)$ is a probabilistic topogenous structure on X, then $\mathfrak{Q}$ is a syntopogenous structure if and only if $<_1$ is dense.

Proof. From (PT1) it follows that $\mathfrak{Q}$ satisfies (S1). If (S2) holds, then for $\lambda = 1$ there exists $\mu = 1$ (it is necessary that the order $<\mu$ be finer than $<_1$) such that if $A <_1 B$ there exists C with $A <_1 C <_1 B$, hence $<_1$ is dense. Conversely, if $<_1$ is dense, for any $\lambda \in I$ there is $\mu = 1$ such that $A <_\lambda B$ implies $A <_1 B$ and there exists C such that $A <_1 C <_1 B$.

Proposition 2. *Let $(X, \mathfrak{Q}, T)$ be a probabilistic topogenous space. Then the mapping* $\theta : \mathfrak{P}(X) \times I \to P(X)$, $\theta(A, \lambda) = \cap \{B : A <^c_\lambda B\}$ *for each A, λ, is a left-continuous θ-closure on X and $(X, \mathfrak{Q}, T)$ is a probabilistic topological space.*

Proof. (C1): If $A \subseteq X$ $\theta(A, 0) = \cap \{B : A <^c_0 B\} = \cap \{B : X - B = \varnothing\} = X$. Now, we observe that $\theta(., \lambda)$ is the closure operator induced by $<_\lambda$ for every $\lambda \in (0.1]$. It is seen that $A \subset \theta(A, 0) = X$ for every $A \subset X$ and $\theta(A \cup B, 0) = X = \theta(A, 0) \cup \theta(B, 0)$. It then follows that (*C*2), (*C*3), (*C*4) are satisfied. (*C*5): For every $\lambda, \mu \in I$ with $\lambda \leqslant \mu$ from (PT1) $<_\lambda \subseteq <_\mu$. Hence $A <^c_\lambda B$ implies $A <^c_\mu B$ and $\{B : A <^c_\lambda B\} \subset \{B : A <^c_\mu B\}$. Thus $\theta(A, \lambda) = \cap \{B : A <^c_\lambda B\} \supset \cap \{B : A <^c_\mu B\} = \theta(A, \mu)$. (*LC*): If $\bar{x} \in \theta(A, \lambda)$ then $\{x\} \cap \theta(A, \lambda) = \varnothing$, whence $\{x\} <_\lambda X - A$. From (PT2), there is $\mu < \lambda$ such that $\{x\} <_\mu X - A$ hence $\{x\} \cap \theta(A, \mu) = \varnothing$. It follows that $x \notin \cap \{\theta(A, \mu) : \mu < \lambda\}$. Now, from (C5), $\theta(A, \lambda) \subset \cap \{\theta(A, \mu) : \mu < \lambda\}$. Hence θ is a left-continuous θ-closure on X. (C6): Let $A \subset X$, $A \neq \varnothing$. If $\mu = 0$ $\theta(\theta(A, \mu), \lambda) = X$ for every $\lambda \in I$: $\theta(A, T(\lambda, 0)) = \cap \{B : A <^\xi_{T(\lambda, 0)} B\}$; from (PT3) there exists C such that $A <^\xi_0 C <^\xi_\lambda B$ hence $C = X$ and $B = X$. Hence $\theta(A, T(\lambda, 0)) = X$ and $\theta(\theta(A, 0), \lambda) \subset \theta(A, T(\lambda, 0))$. If $\lambda = 0$, $\mu \neq 0$ $\theta(\theta(A, \mu), 0) = X$; $\theta(A, T(0, \mu)) = \cap \{B : A <^\xi_{T(0,\mu)} B\}$. From (PT3) it follows that there exists C such that $A <^\xi_\mu C <^\xi_0 B$ hence $B = X$ and $\theta(A, T(0, \mu)) = X$. If $\lambda \neq 0$, $\mu \neq 0$, $\theta(A, T(\lambda, \mu)) = \cap \{B : A <^\xi_{T(\lambda,\mu)} B\}$.

Let $\mathfrak{A} = \{B : A <^\xi_{T(\lambda,\mu)} B\}$: for every $B \in \mathfrak{A}$ from (PT3) there exists C_B such that $A <^\xi_\mu C <^\xi_\lambda B$ hence $C_B \supset \theta(A, \mu)$, $\theta(C_B, \lambda) \supset \theta(\theta(A, \mu), \lambda)$.

But $B \supset \theta(C_B, \lambda)$ hence $B \supset \theta(\theta(A, \mu), \lambda)$ and $\theta(A, T(\lambda, \mu)) \supset \theta(\theta(A, \mu), \lambda)$.

Definition 3. *We say that the structure of a probabilistic topological space (X, θ, T) is induced by the probabilistic topogenous structure $(\mathfrak{S}, T)$ and we call the operator θ, the operator induced by this structure.*

Theorem 2. *Let (X, θ, T) be a probabilistic topological space. Then there exists exactly one probabilistic topogenous structure $(\mathfrak{S}, T)$ on X, which induces the probabilistic topological structure of the space.*

Proof. Let $\mathfrak{S} = \{<_\lambda\}_{\lambda \ni I}$ be such that: for $\lambda = 0$ $A <_0 B$ iff $A = \varnothing$; for $\lambda \in (0, 1]$, $A <_\lambda B$, iff $<_\lambda$ is the unique perfect topogenous order which induces $\theta(., \lambda)$ (Theorem 1). (PT1): for every $\lambda, \mu \in I$ with $\lambda \leqslant \mu$ and $A <_\lambda B$ it follows $A \cap \theta(X - B, \lambda) = \varnothing$. Now, from (C5), $\theta(X - B, \lambda) \supset \theta(X - B, \mu)$. Hence $A \cup \theta(X - B, \mu) = \varnothing$ implies $A <_\mu$ $<_\lambda B$. It follows $<_\lambda \subseteq <_\mu$. (PT2): Let $A \subset X$, $A \neq \varnothing$ and $\lambda > 0$, $\{x\} <_\lambda$ $<_\lambda A$ implies $\{x\} \cap \theta(X - A, \lambda) = \varnothing$. But $\theta(X - A, \lambda) = \cap \{\theta(X - A, \mu) : \mu < \lambda\}$. Hence there is $\mu < \lambda$ such that $\{x\} \cap \theta(X - A, \mu) = \varnothing$ or $\{x\} <_\mu A$. (PT3): For every $\lambda, \mu \in I$ with $A <^\xi_{T(\lambda, \mu)} B$, let $C = \theta(A, \mu)$. Then $A <^\xi_\mu C$ (because $(X - C) \cap \theta(A, \mu) = \varnothing$). Then $(X - B) \cap \theta(A, T(\lambda, \mu)) = \varnothing$, hence.

$\theta(A, T(\lambda, \mu)) \subset B$. From (C6), $\theta(\theta(A, \mu), \lambda) \subset B$ hence $\theta(C, \lambda) \subset B$ therefore $C <^\xi_\lambda B$. Hence $(\mathfrak{S}, T)$ is a probabilistic topogenous structure on X. Every $<_\lambda$ induces the operator $\theta(., \lambda)$, hence the θ-closure θ is induced by $(\mathfrak{S}, T)$, therefore the probabilistic topological structure is induced by the probabilistic topogenous structure. The uniqueness of this structure is a consequence of the uniqueness of $<_\lambda$ (Theorem 1).

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NEAR-MODULES OVER SPECIAL NEAR-RINGS

BY

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Dedicated to our Professor Gheorghe Gheorghiev, on the occasion of his seventieth birthday

The nonunitary modules with noncommutative addition over a ring with identity has been studied in [1] by the first author. In [2], we found that there exists a class of distributive near-rings with a finite number of central idempotents and we studied there some properties of these near-rings.

If the ring A has only a finite number of central idempotents, then an A-module M with noncommutative addition has a decomposition as a semidirect sum of a submodule with commutative addition and a normal submodule. We proved it in [3].

In this paper, we consider the case of a N-module with noncommutative addition, where N is a distributive near-ring with a finite number of central idempotents, none of them of order 2. For such a near-module, we obtain a decomposition theorem as a semidirect sum of a submodule with commutative addition and a normal submodule.

1. Let N be a distributive near-ring with a finite number of central idempotents, none of them of order 2. We proved that, in this case, N has 2^n central idempotents, where n is the number of its primitive central idempotents, where n is the number of its primitive central idempotents (i. e. those central idempotents which are not decomposable as a sum of two other central idempotents (Theorem 3.3, [2]). For such a near-ring, we have the following:

Theorem 1. (Theorem 3.6, [2]). *If N is a distributive near-ring with a finite number of central idempotents, none of them having the order 2, then N is the semidirect sum:*

$$N = \left(\bigcap_{i=1}^{n} K_{e_i}\right) \dotplus \left(\bigoplus_{i=1}^{n} Ne_i\right),$$

where $e_1, \ldots, e_n$ are its primitive central idempotents, $e = \sum_{i=1}^{n} e_i$ is the identity