whence $\{x\} <_\lambda X - A$. From (PT2), there is $\mu < \lambda$ such that $\{x\} <_\mu X - A$ hence $\{x\} \cap \theta(A, \mu) = \varnothing$. It follows that $x \overline{\in} \cap \{\theta(A, \mu) : \mu < \lambda\}$. Now, from (C5), $\theta(A, \lambda) \subset \cap \{\theta(A, \mu) : \mu < \lambda\}$. Hence θ is a left-continuous θ-closure on X. (C6): Let $A \subset X$, $A \neq \varnothing$. If $\mu = 0$ $\theta(\theta(A, \mu), \lambda) = X$ for every $\lambda \in I$; $\theta(A, T(\lambda, 0)) = \cap \{B : A <^c_{T(\lambda, 0)} B\}$; from (PT3) there exists C such that $A <^c_0 C <^c_\lambda B$ hence $C = X$ and $B = X$. Hence $\theta(A, T(\lambda, 0)) = X$ and $\theta(\theta(A, 0), \lambda) \subset \theta(A, T(\lambda, 0))$. If $\lambda = 0$, $\mu \neq 0$ $\theta(\theta(A, \mu), 0) = X$; $\theta(A, T(0, \mu)) = \cap \{B : A <^c_{T(0,\mu)} B\}$. From (PT3) it follows that there exists C such that $A <^c_\mu C <^c_0 B$ hence $B = X$ and $\theta(A, T(0, \mu)) = X$. If $\lambda \neq 0$, $\mu \neq 0$, $\theta(A, T(\lambda, \mu)) = \cap \{B : A <^c_{T(\lambda,\mu)} B\}$.

Let $\mathfrak{A} = \{B : A <^c_{T(\lambda,\mu)} B\}$: for every $B \in \mathfrak{A}$ from (PT3) there exists C_B such that $A <^c_\mu C <^c_\lambda B$ hence $C_B \supset \theta(A, \mu)$, $\theta(C_B, \lambda)$
$\supset \theta(\theta(A, \mu), \lambda)$.

But $B \supset \theta(C_B, \lambda)$ hence $B \supset \theta(\theta(A, \mu), \lambda)$ and $\theta(A, T(\lambda, \mu)) \supset \supset \theta(\theta(A, \mu), \lambda)$.

Definition 3. *We say that the structure of a probabilistic topological space (X, θ, T) is induced by the probabilistic topogenous structure $(\mathfrak{L}, T)$ and we call the operator θ, the operator induced by this structure.*

Theorem 2. *Let (X, θ, T) be a probabilistic topological space. Then there exists exactly one probabilistic topogenous structure $(\mathfrak{L}, T)$ on X, which induces the probabilistic topological structure of the space.*

Proof. Let $\mathfrak{L} = \{<_\lambda\}_{\lambda \in I}$ be such that: for $\lambda = 0$ $A <_0 B$ iff $A = \varnothing$; for $\lambda \in (0, 1]$, $A <_\lambda B$, iff $<_\lambda$ is the unique perfect topogenous order which induces $\theta(., \lambda)$ (Theorem 1). (PT1): for every $\lambda, \mu \in I$ with $\lambda \leqslant \mu$ and $A <_\lambda B$ it follows $A \cap \theta(X - B, \lambda) = \varnothing$. Now, from (C5), $\theta(X - B, \lambda) \supset \theta(X - B, \mu)$. Hence $A \cup \theta(X - B, \mu) = \varnothing$ implies $A <_\mu$ $<_\lambda B$. It follows $<_\lambda \subseteq <_\mu$. (PT2): Let $A \subset X$, $A \neq \varnothing$ and $\lambda > 0$. $\{x\} <_\lambda$ $<_\lambda A$ implies $\{x\} \cap \theta(X - A, \lambda) = \varnothing$. But $\theta(X - A, \lambda) = \cap \{\theta(X - - A, \mu) : \mu < \lambda\}$. Hence there is $\mu < \lambda$ such that $\{x\} \cap \theta(X - A, \mu) = = \varnothing$ or $\{x\} <_\mu A$. (PT3): For every $\lambda, \mu \in I$ with $A <^c_{T(\lambda, \mu)} B$, let $C = = \theta(A, \mu)$. Then $A <^c_\mu C$ (because $(X - C) \cap \theta(A, \mu) = \varnothing$). Then $(X - - B) \cap \theta(A, T(\lambda, \mu)) = \varnothing$, hence.

$\theta(A, T(\lambda, \mu)) \subset B$. From (C6), $\theta(\theta(A, \mu), \lambda) \subset B$ hence $\theta(C, \lambda) \subset B$ therefore $C <^c_\lambda B$. Hence $(\mathfrak{L}, T)$ is a probabilistic topogenous structure on X. Every $<_\lambda$ induces the operator $\theta(., \lambda)$, hence the θ-closure θ is induced by $(\mathfrak{L}, T)$, therefore the probabilistic topological structure is induced by the probabilistic topogenous structure. The uniqueness of this structure is a consequence of the uniqueness of $<_\lambda$ (Theorem 1).

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NEAR-MODULES OVER SPECIAL NEAR-RINGS

BY

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Dedicated to our Professor Gheorghe Gheorghiev, on the occasion of his seventieth birthday

The nonunitary modules with noncommutative addition over a ring with identity has been studied in [1] by the first author. In [2], we found that there exists a class of distributive near-rings with a finite number of central idempotents and we studied there some properties of these near-rings.

If the ring A has only a finite number of central idempotents, then an A-module M with noncommutative addition has a decomposition as a semidirect sum of a submodule with commutative addition and a normal submodule. We proved it in [3].

In this paper, we consider the case of a N-module with noncommutative addition, where N is a distributive near-ring with a finite number of central idempotents, none of them of order 2. For such a near-module, we obtain a decomposition theorem as a semidirect sum of a submodule with commutative addition and a normal submodule.

1. Let N be a distributive near-ring with a finite number of central idempotents, none of them of order 2. We proved that, in this case, N has 2^n central idempotents, where n is the number of its primitive central idempotents, where n is the number of its primitive central idempotents (i. e. those central idempotents which are not decomposable as a sum of two other central idempotents (Theorem 3.3, [2]). For such a near-ring, we have the following:

Theorem 1. (Theorem 3.6, [2]). *If N is a distributive near-ring with a finite number of central idempotents, none of them having the order 2, then N is the semidirect sum:*

$$N = \left(\bigcap_{i=1}^{n} K_{e_i}\right) \dotplus \left(\bigoplus_{i=1}^{n} N e_i\right),$$

where $e_1, \ldots, e_n$ are its primitive central idempotents, $e = \sum_{i=1}^{n} e_i$ is the identity *of the subring* $Ne = \bigoplus_{i=1}^{n} Ne_i$ *and* $K_e = \bigcap_{i=1}^{n} K_{e_i}$ *is a two-sided ideal of* N, *which does not contain other nonzero central idempotents. In addition,* $N/K_e \simeq Ne$.

Definition 1. *A left near-module over the distributive near-ring* N *is an additive group* $(M, +)$, *generally noncommutative, together with a multiplication.* $: N \times M \to M$, *such that the following conditions hold:*

$$n\,.\,(x + y) = n\,.\,x + n\,.\,y,\ \forall x,\ y \in M,\ \forall n \in N, \tag{1}$$

$$(n_1 + n_2)\,.\,x = n_1\,.\,x + n_2\,.\,x,\ \forall x \in M,\ \forall n_1,\ n_2 \in N, \tag{2}$$

$$n_1\,.\,(n_2\,.\,x) = (n_1 n_2)\,.\,x,\ \forall x \in M,\ n_1,\ n_2 \in N. \tag{3}$$

The notions of submodule and homomorphism of near-modules, as well as the kernel and image of a homomorphism of near-modules are those suggested by the same notions in the theory of usual unitary modules over rings with identity.

Definition 2. *We shall say that the near-module* M *is the semidirect sum of its subnear-modules* M_1 *and* M_2, *iff* $M = M_1 + M_2$, $M_1 \cap M_2 = \{0\}$ *and* M_1 *is a normal submodule of* M *(that is* $(M_1, +)$ *is a normal subgroup of* $(M, +)$*)*.

Note that for any $x,\ y \in M$, $n_1,\ n_2 \in N$, we have:

$$n_2\,.\,x + n_1\,.\,y = n_1\,.\,y + n_2\,.\,x, \tag{4}$$

as we can show by applying (1) and (2) to the product: $(n_1 + n_2)\,.\,(x + y)$.

2. Recall that a central idempotent $a \in N$, $a \neq 0$, is characterized by the conditions: $a^2 = a$ and $an = na$, $\forall n \in N$.

We have:

Theorem 2. *If* $a \in N$, $a \neq 0$ *is a central idempotent of the distributive near-ring* N *and* M *is a left near-module over* N, *then:*

1°. *The mapping* $\varphi_a : M \to M$, *given by* $\varphi_a(x) = a.x$, $\forall x \in M$, *is an idempotent endomorphism of* M;

2°. $M = \operatorname{Ker} \varphi_a + \operatorname{Im} \varphi_a$, *where the sum is a semidirect one, since* $\operatorname{Ker} \varphi_a$ *is a normal subnear-module of* M;

3°. $\operatorname{Im} \varphi_a$ *is a unitary module over the subring* Na *of* N ($\operatorname{Im} \varphi_a, +$) *is an abelian group) and* $\operatorname{Im} \varphi_a \simeq M/\operatorname{Ker} \varphi_a$. *Note that* $\operatorname{Im} \varphi_a = \{x \mid x \in M,\ ax = x\}$.

Proof: 1°. We have, indeed, $\varphi_a(x + y) = \varphi_a(x) + \varphi_a(y)$ and $\varphi_a(nx) = a\,.\,(n\,.\,x) = (an)\,.\,x = (na)\,.\,x = n\,.\,(a\,.\,x) = n\,.\,\varphi_a(x)$, $\forall x,\ y \in M$, $\forall n \in N$. In addition, $\varphi_a(\varphi_a(x)) = a\,.\,(a\,.\,x) = a^2\,.\,x = a\,.\,x = \varphi_a(x)$, $\forall x \in M$. Hence $\varphi_a \circ \varphi_a = \varphi_a$.

2°. For every $x \in M$, we have $x = (x - a\,.\,x) + a\,.\,x$ and, obviously $x - a\,.\,x \in \operatorname{Ker} \varphi_a$, $a\,.\,x \in \operatorname{Im} \varphi_a$. $\operatorname{Ker} \varphi_a$ is a normal submodule of M, as a kernel of a homomorphism of near-modules. $\operatorname{Im} \varphi_a = \{x \mid x \in M,\ a\,.\,x = x\}$, because $a^2 = a$, it is a submodule of M and $\operatorname{Ker} \varphi_a \cap \operatorname{Im} \varphi_a = \{0\}$.

3°. Recall that $Na = \{na \mid n \in N\}$ is a subring of N (theorem 1) with a as identity, (see [2]). For any $x,\ y \in \operatorname{Im} \varphi_a$ and $na \in Na$, we have $(na)\,.\,x = (an)\,.\,x = a\,.\,(n\,.\,x) \in \operatorname{Im} \varphi_a$, $x + y = a\,.\,x + a\,.\,y = a\,.$

$y + a\,.\,x = y + x$ and $a\,.\,x = x$, hence $\operatorname{Im} \varphi_a$ is a unitary module over the ring Na. The isomorphism $\operatorname{Im} \varphi_a \simeq M/\operatorname{Ker} \varphi_a$ is a consequence of the theorem of homomorphism for near-modules.

Since the set of central idempotents of the distributive near-ring N is finite, we have a stronger decomposition theorem for the N — near-module M, namely:

Theorem 3. *If* N *is a distributive near-ring with a finite number of central idempotents, none of them having the order 2, and* M *is a left near-module over* N, *then, denoting by* $e_1, e_2, \ldots, e_n$ *the primitive central idempotents of* N, *the near-module* M *is the semidirect sum:*

$$M = \left(\bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i}\right) + \left(\bigoplus_{i=1}^{n} \operatorname{Im} \varphi_{e_i}\right).$$

In addition $M/\bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i} \simeq \bigoplus_{i=1}^{n} \operatorname{Im} \varphi_{e_i}$.

Proof. Let $e = e_1 + \ldots + e_n$ the maximal central idempotent of N, [2]. By applying theorem 2 for $a = e$, we have to prove that:

$$\operatorname{Ker} \varphi_e = \bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i} \text{ and } \operatorname{Im} \varphi_e = \bigoplus_{i=1}^{n} \operatorname{Im} \varphi_{e_i}.$$

For all $x \in \operatorname{Ker} \varphi_e$, $e\,.\,x = 0$, we have $(e_1 + \ldots + e_n)\,.\,x = e_1\,.\,x + \ldots + e_n\,.\,x = 0$. Multiplying this equality, in turn, by e_i, $i = 1, \ldots, n$ on the left-hand side, we obtain: $e_i\,.\,x = 0$, $i = 1, \ldots, n$ (because $e_i e_j = 0$, $i \neq j$ and $e_i\,.\,(e_j\,.\,x) = (e_i e_j)\,.\,x = 0\,.\,x = 0$, $i \neq j$), therefore $x \in \bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i}$ and $\operatorname{Ker} \varphi_e \subseteq \bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i}$. The other inclusion is obvious. Hence $\operatorname{Ker} \varphi_e = \bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i}$.

Note that $\operatorname{Im} \varphi_{e_i} \cap \operatorname{Im} \varphi_{e_j} = \{0\}$, if $i \neq j$. Indeed, if $x \in \operatorname{Im} \varphi_{e_i} \cap \operatorname{Im} \varphi_{e_j}$, then $e_i\,.\,x = e_j\,.\,x = x$. Multiplying by e_i these equations on the left-hand side, we have $e_i\,.\,(e_j\,.\,x) = (e_i e_j)\,.\,x = 0\,.\,x = 0 = e_i\,.\,x = x$. Hence $x = 0$ and $\operatorname{Im} \varphi_{e_i} \cap \operatorname{Im} \varphi_{e_j} = \{0\}$, $i \neq j$. Now, if $x \in \operatorname{Im} \varphi_e$, then $x = e\,.\,x = (e_1 + \ldots + e_n)\,.\,x = e_1\,.\,x + \ldots + e_n\,.\,x$, where $e_i\,.\,x \in \operatorname{Im} \varphi_{e_i}$, $i = 1, \ldots, n$, hence. $\operatorname{Im} \varphi_e \subseteq \operatorname{Im} \varphi_{e_1} + \ldots + \operatorname{Im} \varphi_{e_n}$. Now, let $x_i \in \operatorname{Im} \varphi_{e_i}$, $i = 1, \ldots, n$, hence $e_i\,.\,x_i = x_i$ and $e_j\,.\,x_i = 0$, $j \neq i$, $i = 1, \ldots, n$. We have $x = x_1 + x_2 + \ldots + x_n$ in $\operatorname{Im} \varphi_e$. Indeed, $e\,.\,x = e\,.\,x_1 + e\,.\,x_2 + \ldots + e\,.\,x_n = e_1.x_1 + e_2\,.\,x_2 + \ldots + e_n\,.\,x_n = x_1 + x_2 + \ldots + x_n = x$, and therefore $\operatorname{Im} \varphi_{e_1} + \ldots + \operatorname{Im} \varphi_{e_n} \subseteq \operatorname{Im} \varphi_e$. Consequently, $\operatorname{Im} \varphi_e = \operatorname{Im} \varphi_{e_1} + \ldots + \operatorname{Im} \varphi_{e_n}$ and this sum is direct, since $(\operatorname{Im} \varphi_e, +)$ is an abelian group.

Remark. If $\bigcap_{i=1}^{n} \operatorname{Ker} \varphi_{e_i} = \operatorname{Ker} \varphi_e = \{0\}$, then $M = \bigoplus_{i=1}^{n} \operatorname{Im} \varphi_{e_i}$ is a usual left module over N (which is a ring with identity). The primitive central idempotents of N form a decomposition set for the identity e.

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STRUCTURES AFFINES ET OPÉRATIONS TERNAIRES

PAR

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1. Soit M un ensemble ; ses éléments seront notés par majuscules latines. Une relation quaternaire homogène de M s'identifie à un sousensemble χ de M^4. Soient les relations binaires $\varepsilon \subseteq M^2 \times M^2$ et $\sigma \subseteq M^3 \times M$ où : $(A, B, C, D) \in \chi \leftrightarrow ((A, B), (C, D)) \in \varepsilon \leftrightarrow ((A, B, D), C) \in \sigma$. Evidemment chaque élément de l'ensemble $\{\chi, \varepsilon\ \sigma\}$ détermine univoquement les autres. Nous désignerons par $\overrightarrow{AB}$ — ou $(\overrightarrow{AB})_\sigma$ pour éviter les confusions — les relations binaires homogènes de M, définies par : $(P, Q) \in \overrightarrow{AB} \leftrightarrow (A, B, P, Q) \in \chi$. Des raisons que nous indiquerons plus loin justifient le terme de *vecteur de* (M, σ) pour AB et la notation : $V = V(M, \sigma) = \{\overrightarrow{AB} : (A, B) \in M^2\} = V(M, \chi)$.

Nous dirons que (M, χ) est un 0-*module affine* si χ est une relation quaternaire de M, telle que les axiomes suivants soient vérifiés.

0.1. Chaque vecteur de (M, χ) est une relation binaire 1-complète, [5] : $\overrightarrow{AB} \in V(M, \chi) \to \forall_{P \in M} \exists_Q (P, Q) \in \overrightarrow{AB}$. donc : $(A, B, P) \in M^3 \to \exists_Q (A, B, P, Q) \in \chi$.

0.2. Chaque vecteur de (M, χ) est une relation binaire 2-complète, [5] : $\overrightarrow{AB} \in V(M, \chi) \to \forall_{P \in M} \exists_Q (Q, P) \in \overrightarrow{AB}$, ou : $(A, B, P) \in M^3 \to \exists_Q (A, B, Q, P) \in \chi$.

0.3. Chaque vecteur de (M, χ) est une relation binaire injective : $\overrightarrow{AB} \in V(M, \chi) \to [(P, Q) \in \overrightarrow{AB} \wedge (R, Q) \in \overrightarrow{AB} \to P = R]$ ou : $(A, B, P, Q) \in \chi \wedge (A, B, R, Q) \in \chi \to P = R$.

0.4. La composition binaire des vecteurs vérifie : $\overrightarrow{BC} \circ \overrightarrow{AB} \subseteq \overrightarrow{AC}$ ou $(A, B, P, Q) \in \chi \wedge (B, C, Q, R) \in \chi \to (A, C, P, R) \in \chi$.

0.5. La relation binaire ε est reflexive : $(A, B) \in M^2 \to (A, B, A, B) \in \chi$ ou $(A, B) \in M^2 \to (A, B) \in \overrightarrow{AB}$.

0.6. La relation binaire ε est transitive : $(A, B, C, D) \in \chi \wedge (C, D, E, F) \in \chi \to (A, B, E, F) \in \chi$.