Corollaire. *La condition nécessaire et suffisante d'existence d'une structure d'espace affine sur le corps $(V, +, .)$ pour l'ensemble M est l'existence d'une opération ternaire $\sigma : M^3 \to M$ telle que :*

a — (M, σ) soit groupe ternaire.

b — Il existe une homomorphisme injectif de $(V, +, .)$ dans $(B(M, \sigma), \oplus, \otimes, S)$.

En conclusion, l'intérêt de l'étude de l'algèbre $B(M, \sigma)$ est prouvé par les théorèmes 8—11 : cette étude est facilité par le théorème 7.

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Reçu le 2.IX.1976

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXIII, s. I a, 1977, f. 1

AN EXTENSION OF CONTRACTIVE MAPPING

BY

K. M. GHOSH

1. Introduction. Let (M, d) be a metric space. A mapping $T : M \to M$ is called contractive for every $x, y \, (x \neq y)$ in M, if

$$d(Tx, Ty) < d(x, y) \tag{1.1}$$

holds.

In paper [1], M. Edelstein proved a theorem which runs as follows :

Theorem 1. *Let (M, d) be a metric space, T be a mapping of M into itself. If for every $x, y \, (x \neq y)$, T satisfies (1.1) and for every $x \in M$*

$$\{T^{n_i} x\} \subset \{T^n x\} \text{ with } \lim_{i \to \infty} T^{n_i} x = u \in M, \tag{1.2}$$

holds, then u is a unique fixed point of T.

The main purpose of the present paper is to prove some fixed point theorems, for a mapping of metric space into itself which instead of (1.1) possesses the following conditions :

Let (M, d) be a metric space, a mapping $T : M \to M$ and for every $x, y \, (x \neq y)$ in M, there exist nonnegative real numbers $a_i \, (i = 1, \ldots, 6)$ and at least one of $a_i \neq 0 \, (i = 1, \ldots, 5)$ such that

$$\begin{aligned} d(Tx, Ty)\, d(x, y) < d(x, y)\, \{a_1 d(x, y) + a_2 d(x, Tx) + \\ + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)\} + a_6 d(x, Tx)\, d(y, Ty) \end{aligned} \tag{1.3}$$

where
$$\sum_{i=1}^{6} a_i = 1.$$

To be more precise we state our theorem in the following manner :

Theorem 2. *Let T be a continuous mapping of a metric space M into itself. If for every $x, y \, (x \neq y)$ in M there exist nonnegative real numbers $a_i \, (i = 1, \ldots, 6)$ and at least one of $a_i \neq 0 \, (i = 1, \ldots 5)$ such that T satisfies (1.3) and (1.2), then T has a unique fixed point.*

Similarly if we replace the condition (1.3) by the following analogous condition :

Let (M, d) be a metric space and a mapping $T: M \to M$ and for every x, y $(x \neq y)$ there exist nonnegative real numbers a_i $(i = 1, \ldots, 7)$ and at least one of $a_i \neq 0$ $(i = 3, \ldots, 6)$ such that

$$(1.4) \quad d(x,y)\{a_1 d(x,y) + a_2 d(Tx,Ty)\} < d(x,y)\{a_3 d(x,Tx) + a_4 d(y,Ty) + a_5 d(x,Ty) + a_6 d(y,Tx)\} + a_7 d(x,Tx)\, d(y,Ty)$$

where $2a_1 < \sum_{i=3}^{7} a_i = \sum_{i=1}^{2} a_2$ $(a_2 \neq 0)$. then we can prove the following theorem:

Theorem 3. *Let T be a continuous selfmapping on a metric space M: if for every x, y, $(x \neq y)$ in M, there exist nonnegative real numbers a_i $(i = 1 \ldots 7)$ and at least one of $a_i \neq 0$ $(i = 3 \ldots 6)$, T satisfies (1.4) and (1.2), then T has a unique fixed point.*

Lastly we remark that our theorems can be extended in the case of a family of mapping as follows:

Theorem 4. *Let (M, d) be a metric space and F be a family of commutating self mappings on M. Let there exists $T \in F$ such that T satisfies the conditions of theorem 2 or theorem 3, so T must have a unique fixed point u (say). Then u is the common fixed point of F.*

2. Discussions of the theorems. First we consider theorem 2. Let $x_0 \in M$. Define $Tx_0 = x_1$, $x_2 = Tx_1 = Tx_0^2$, ..., $x_n = T^n x_0$, ... Let us assume that $\{T^n x_0\}$ has a convergent subsequence $\{T^{n_i} x_0\}$ and $\lim_{i \to \infty} Tx_0^{n_i} = u$. We now suppose that $Tx_n \neq x_n$ for any finite n, if not, then x_n is a fixed point of T. So, for points x_0, x_1, we have,

$$d(x_1, x_0)\, d(Tx_1, Tx_0) < d(x_1, x_0)\{a_1 d(x_1, x_0) + a_2 d(x_1, Tx_1) + a_3 d(x_0, Tx_0) + a_4 d(x_1, Tx_0) + a_5 d(x_0, Tx_1)\} + a_6 d(x_1, Tx_1)\, d(x_0, Tx_0)$$

$$\because \quad d(x_1, x_0) = d(Tx_0, x_0) \neq 0.$$

$$\therefore \quad d(x_2, x_1) < a_1 d(x_1, x_0) + a_2 d(x_2, x_1) + a_3 d(x_0, x_1) + a_5 d(x_0, x_2) + a_6 d(x_1, x_2)$$

i.e. $(1 - a_2 - a_5 - a_6)\, d(x_2, x_1) < (a_1 + a_3 + a_5)\, d(x_1, x_0)$ ... (A).

By the symmetry property of metric, we have also,

$(1 - a_3 - a_4 - a_6)\, d(x_2, x_1) < (a_1 + a_2 + a_4)\, d(x_1, x_0)$ .... (B).

Adding (A) and (B), we get,

$$d(x_2, x_1) < \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - (a_2 + a_3 + a_4 + a_5 + 2a_6)}\, d(x_1, x_0)$$

$$\therefore \quad d(x_2, x_1) < d(x_1, x_0) \quad \because \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - (a_2 + a_3 + a_4 + a_5 + 2a_6)} = 1$$

Similarly, $d(x_3, x_2) < d(x_2, x_1)$
By induction, it can be shown that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1})$$

i.e.

$$d(T^{n+1} x_0, T^n x_0) < d(T^n x_0, T^{n-1} x_0)$$

So, $\{d(T^{n+1} x_0, T^n x_0)\}$ is a decreasing sequence of positive real numbers. Hence $\{d(T^{n+1} x_0, T^n x_0)\}$ is convergent.

$\therefore$ $d(T^{n_i+1} x_0, T^{n_i} x_0) \to d(Tu, u)$ as $i \to \infty$. ($\because$ T is continuous

$$\therefore \quad \lim_{i \to \infty} T^{n_i+1} x_0 = T \lim_{i \to \infty} T^{n_i} x_0 = Tu).$$

Since, $\{d(T^{n_i+1} x_0, T^{n_i} x_0)\}_{i=1}^{\infty} \subset \{d(T^{n+1} x_0, T^n x_0)\}_{n=1}^{\infty}$

$$\therefore \quad \lim_{n \to \infty} d(T^{n+1} x_0, T^n x_0) = d(Tu, u).$$

Also, as $\{d(T^{n_i+2} x_0, T^{n_i+1} x_0)\}_{i=1}^{\infty} \subset \{d(T^{n+1} x_0, T^n x_0)\}_{n=1}^{\infty}$

and, $\lim_{i \to \infty} d(T^{n_i+2} x_0, T^{n_i+1} x_0) = d(T^2 u, Tu)$

$$\therefore \quad \lim_{n \to \infty} d(T^{n+1} x_0, T^n x_0) = d(T^2 u, Tu).$$

$$\therefore \quad d(T^2 u, Tu) = d(Tu, u).$$

Now let us suppose that $d(Tu, u) \neq 0$, then from (1.3) it can be shown by putting $x = Tu$ and $y = u$, that $d(T^2 u, Tu) < d(Tu, u)$ which is a contradiction. Hence

$$d(Tu, u) = 0 \text{ i.e. } Tu = u.$$

If possible, let v be a another fixed point other than u: then by (1.3) we have

$$d(u, v)\, d(Tu, Tv) < d(u, v)\{a_1 d(u, v) + a_2 d(u, Tu) + a_3 d(v, Tv) + a_4 d(u, Tv) + a_5 d(v, Tu)\} + a_6 d(u, Tu)\, d(v, Tv)$$

$$\therefore \quad (d(u, v))^2 < (d(u, v))^2 (a_1 + a_4 + a_5);$$

$$\therefore \quad 1 < a_1 + a_4 + a_5 \; (\because d(u, v) \neq 0)$$

which is impossible as $\sum_{i=1}^{6} a_i = 1$. Hence $u = v$. This completes the proof of the Theorem 2.

Since the proof of Theorem 3 can be easily carried in exactly the same way as that of Theorem 2, we omit the proof.

Further, in the case of Theorem 4, let us consider the case under Theorem 2.

Since T satisfies all the conditions of theorem 2, so T has a unique fixed point u in M. Now we shall show that u is the common fixed point of F. Let $S \in F$, then $S(u) = S(T(u)) = T(S(u))$. So $S(u)$ is a fixed point of T. By uniqueness of the fixed point of T, we must have, $S(u) = u$, so u is a fixed point of any $S \in F$. Hence u is the common fixed point of F.

The proof of Theorem 4 under the conditions of Theorem 3, can similarly be easily given.

3. Concluding remarks. 1. If we set $a_6 = 0$ in (1.3), then we get

$$d(Tx, Ty) < a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

where $\sum_{i=1}^{5} a_i = 1$, whence it follows from theorem 2, the well-known theorem 1 (b) of Hardy and Rogers [2].

2. In our theorem 2 and theorem 3, the continuity property of the mapping T can be replaced by a condition (defined by Ćirić [3]) which is orbital continuity.

According to Ćirić, we define orbitally continuous mapping T of a metric space M into itself as follows:

Definition: *A mapping T of metric space M into itself is said to be orbitally continuous if for every $x \in M$,*

$$\lim_{i\to\infty} T x^{n_i+1} = T \lim_{i\to\infty} T x^{n_i} = Tp \text{ when } \lim_{i\to\infty} T^{n_i} x = p \in M$$

In this connection it may be of interest to note that the theorem 1 (b) of Hardy and Rogers follows as a particular case from our theorem 2, where mapping T is orbitally continuous and further the compactness of the metric space (M, d) can be replaced by the interesting lesser condition (1.2) of Edelstein.

I am indebted to Dr. S. K. Chatterjea, Department of Pure Mathematics, Calcutta University, for his helpful suggestions in the preparattion of the paper.

I am also extremely grateful to the Referee for his helpful suggestions for improvement of the paper.

Received 20. IV. 1975

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Analele ştiinţifice ale Universităţii „Al. I. CUZA" din Iaşi
Tomul XXIII, s. I a, 1977, f. 1

NONLINEAR CONTRACTIONS AND APPLICATIONS TO VOLTERRA FUNCTIONAL EQUATIONS

BY

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1. Let $R =]-\infty, +\infty[$ be the real axis, $R_+ = [0, +\infty[$ the positive half-axis, $\bar{R}_+ = [0, +\infty]$ the extended positive half-axis, $N = \{0, 1, \ldots\}$ the positive integers, $F(R_+, R_+) = \{f \mid f : R_+ \to R_+\}$, $F(R_+, \bar{R}_+) = \{f \mid f : R_+ \to \bar{R}_+\}$, $K = \{f \mid f \in F(R_+, R_+), f(0) = 0, f(t) < t, \forall t > 0\}$, $M = \{f \mid f \in K, f \text{ is monotone increasing on } R_+\}$.

If $f \in K$ is given, denote

i) $f^{(n)}$ = the n-th iterate of f, $\forall n \in N$ (i.e., $f^{(0)}(t) = t$, $\forall t \in R_+$, $f^{(n+1)}(t) = f^{(n)}(f(t))$, $\forall t \in R_+$, $\forall n \in N$).

ii)) $f^{(\infty)} = \lim_{n\to\infty} f^{(n)}$ (i.e., $f^{(\infty)}(t) = \lim_{n\to\infty} f^{(n)}(t)$, $\forall t \in R_+$).

iii) $\bar{f}$ = the upper right-hand limiting function of f (i.e., $\bar{f}(t) = \lim\sup_{s\to t_+} f(s)$, $\forall t \in R_+$).

iv) $\sigma(f) = \{t \mid t > 0, \bar{f}(t) = t\}$.

Definition 1.1. *A function $f \in K$ will be said to have property (P) if it satisfies*

$$f^{(\infty)}(t) = 0, \quad \forall t \in R_+. \tag{1.1}$$

Denote $P = \{f \mid f \in K, f \text{ has the property } (P)\}$.

Theorem 1.1. *Suppose that $f \in K$ satisfies*

$$\sigma(f) \text{ is empty } (i.e., \bar{f}(t) < t, \forall t > 0). \tag{1.2}$$

Then, $f \in P$.

Proof. Take any $t \in R_+$ and denote $t_n = f^{(n)}(t)$, $\forall n \in N$, $r = f^{(\infty)}(t)$. Suppose that $r > 0$; then, $r \leqslant t_{n+1} \leqslant f(t_n)$, $\forall n \in N$, which gives $r \leqslant \bar{f}(r)$, a contradiction. Therefore, $r = 0$. Q.E.D.

Theorem 1.2. *Let $g \in M$, $f \in K$ be such that:*

$$g \leqslant f \text{ (i.e., } g(t) \leqslant f(t), \forall t \in R_+) \tag{1.3}$$