Further, in the case of Theorem 4, let us consider the case under Theorem 2.

Since T satisfies all the conditions of theorem 2, so T has a unique fixed point u in M. Now we shall show that u is the common fixed point of F. Let $S \in F$, then $S(u) = S(T(u)) = T(S(u))$. So $S(u)$ is a fixed point of T. By uniqueness of the fixed point of T, we must have, $S(u) = u$, so u is a fixed point of any $S \in F$. Hence u is the common fixed point of F.

The proof of Theorem 4 under the conditions of Theorem 3, can similarly be easily given.

3. Concluding remarks. 1. If we set $a_6 = 0$ in (1.3), then we get

$$d(Tx, Ty) < a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

where $\sum_{i=1}^{5} a_i = 1$, whence it follows from theorem 2, the well-known theorem 1 (b) of Hardy and Rogers [2].

2. In our theorem 2 and theorem 3, the continuity property of the mapping T can be replaced by a condition (defined by Ciric [3]) which is orbital continuity.

According to Ciric, we define orbitally continuous mapping T of a metric space M into itself as follows:

Definition: *A mapping T of metric space M into itself is said to be orbitally continuous if for every $x \in M$,*

$$\lim_{i \to \infty} T x^{n_i+1} = T \lim_{i \to \infty} T x^{n_i} = Tp \text{ when } \lim_{i \to \infty} T^{n_i} x = p \in M$$

In this connection it may be of interest to note that the theorem 1 (b) of Hardy and Rogers follows as a particular case from our theorem 2, where mapping T is orbitally continuous and further the compactness of the metric space (M, d) can be replaced by the interesting lesser condition (1.2) of Edelstein.

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NONLINEAR CONTRACTIONS AND APPLICATIONS TO VOLTERRA FUNCTIONAL EQUATIONS

BY

MIHAI TURINICI

1. Let $R =]-\infty, +\infty[$ be the real axis, $R_+ = [0, +\infty[$ the positive half-axis, $\bar{R}_+ = [0, +\infty]$ the extended positive half-axis, $N = \{0, 1, \ldots\}$ the positive integers, $F(R_+, \bar{R}_+) = \{f \mid f: R_+ \to \bar{R}_+\}$, $F(R_+, R_+) = \{f \mid f: R_+ \to R_+\}$, $K = \{f \mid f \in F(R_+, R_+), f(0) = 0, f(t) < t, \ \forall t > 0\}$, $M = \{f \mid f \in K, f$ is monotone increasing on $R_+\}$.

If $f \in K$ is given, denote

i) $f^{(n)}$ = the n-th iterate of f, $\forall n \in N$ (i.e., $f^{(0)}(t) = t$, $\forall t \in R_+$; $f^{(n+1)}(t) = f^{(n)}(f(t))$, $\forall t \in R_+$, $\forall n \in N$.

ii)) $f^{(\infty)} = \lim_{n \to \infty} f^{(n)}$ (i.e., $f^{(\infty)}(t) = \lim_{n \to \infty} f^{(n)}(t)$, $\forall t \in R_+$).

iii) $\bar{f}$ = the upper right-hand limiting function of f (i.e., $\bar{f}(t) = \limsup_{s \to t_+} f(s)$, $\forall t \in R_+$).

iv) $\sigma(f) = \{t \mid t > 0, \bar{f}(t) = t\}$.

Definition 1.1. *A function $f \in K$ will be said to have property (P) if it satisfies*

$$f^{(\infty)}(t) = 0, \quad \forall t \in R_+. \tag{1.1}$$

Denote $P = \{f \mid f \in K, f$ has the property $(P)\}$.

Theorem 1.1. *Suppose that $f \in K$ satisfies*

$$\sigma(f) \text{ is empty } (\text{i.e., } \bar{f}(t) < t, \ \forall t > 0). \tag{1.2}$$

Then, $f \in P$.

Proof. Take any $t \in R_+$ and denote $t_n = f^{(n)}(t)$, $\forall n \in N$, $r = f^{(\infty)}(t)$. Suppose that $r > 0$; then, $r \leq t_{n+1} \leq f(t_n)$, $\forall n \in N$, which gives $r \leq \bar{f}(r)$, a contradiction. Therefore, $r = 0$. Q.E.D.

Theorem 1.2. *Let $g \in M$, $f \in K$ be such that:*

$$g \leq f \ (\text{i.e., } g(t) \leq f(t), \ \forall t \in R_+) \tag{1.3}$$

$$f \in P \tag{1.4}$$

Then, $g \in P$.

Proof. Obviously, (1.3) is equivalent to $g^{(1)}(t) \leqslant f^{(1)}(t)$ $\forall t \in R_+$. Suppose that for some $n \in N$, $g^{(n)}(t) \leqslant f^{(n)}(t)$, $\forall t \in R_+$; as $g \in M$, we have

$$\begin{aligned} g^{(n+1)}(t) &= g^{(n)}(g(t)) = g(g^{(n)}(t)) \leqslant g(f^{(n)}(t)) \leqslant f(f^{(n)}(t)) = \\ &= f^{(n+1)}(t), \ \forall t \in R_+ . \end{aligned} \tag{1.5}$$

Therefore, $0 \leqslant g^{(n)}(t) \leqslant f^{(n)}(t)$, $\forall n \in N$, $\forall t \in R_+$. Letting n tend to infinity we get $0 \leqslant g^{(\infty)}(t) \leqslant f^{(\infty)}(t)$, $\forall t \in R^+$, that is, $g^{(\infty)}(t) = 0$, $\forall t \in R_+$, which was to be proved. Q.E.D.

In what follows, the local behaviour of $f^{(\infty)}$ (for $f \in M$) is studied.

Lemma 1.1. *Let* $f \in M$, $r \in \sigma(f)$, $t > r$, *be such that:*

$$\forall u \in\ ]r, t[,\ \exists v \in\ ]r, u[, \text{ such that } f(v) < f(u). \tag{1.6}$$

Then, $f^{(\infty)}(s) \geqslant r$, $\forall s \in\ ]r, t[$.

Proof. Let $s \in\ ]r, t[$ be given. Define $s_n = f^{(n)}(s)$, $\forall n \in N$. It is obvious that $s_0 \in\ ]r, t[$. Suppose that $s_n \in\ ]r, t[$; from (1.6), we can find $u \in\ ]r, s_n[$ such that $f(u) < f(s_n)$. As $f \in M$, this implies $r \leqslant f(r) \leqslant f(u) < f(s_n) < s_n$, i.e., $s_{n+1} \in\ ]r, s_n[\ \subset\ ]r, t[$. Therefore, $\{s_n \mid n \in N\} \subset\ ]r, t[$, which gives $f^{(\infty)}(s) = \lim_{n\to\infty} s_n \in [r, t[$. Q.E.D.

Corollary 1.1. *If* (1.6) *is replaced by*

$$f \text{ is strictly increasing on }]r, t[\tag{1.7}$$

then, the conclusion of the preceding lemma also holds.

Lemma 1.2. *Let* $f \in M$, $r \in \sigma(f)$, $t > r$, *be such that:*

$$f \text{ is constant on }]r, t[. \tag{1.8}$$

Then, $f^{(\infty)}(s) = f^{(\infty)}(r)$, $\forall s \in\ ]r, t[$.

Proof. It is easy to see that $f^{(n+1)}(s) = f^{(n)}(r)$, $\forall n \in N$, $\forall s \in\ ]r, t[$. Letting n tend to infinity we get the desired conclusion. Q.E.D.

It is well known [1] p. 331 that if $f \in M$, $\sigma(f)$ is at most countable. The following two results-closely related to this fact-may be considered as a completion of theorems 1.1 and 1.2.

Theorem 1.3. *Let* $f \in M$ *be such that*

$$\sigma(f) = \{r_1, r_2, \ldots, r_p\},\ r_1 > r_2 > \ldots > r_p,\ p \geqslant 1 \tag{1.9}$$

$$f \text{ is constant on }]r_{n+1}, r_n[,\ \forall n = 0, 1, \ldots p-1 \text{ (where } r_0 = +\infty). \tag{1.10}$$

Then, $f \in P$.

Proof. Let $t \in R_+$ be given. Without loss of generality we may suppose $t > r_1$. From (1.10) and lemma 1.2,

$$f^{(\infty)}(t) = f^{(\infty)}(r_1) = f^{(\infty)}(f(r_1)) = f^{(\infty)}(f^{(2)}(r_1)). \tag{1.11}$$

On the other hand, $r_2 \leqslant f(r_1) < r_1$, and, from this, we obtain either $f(r_1) = r_2$ or $f(r_1) \in\ ]r_2, r_1[$ (the last case giving us $f^{(2)}(r_1) = r_2$). Taking into account (1.11) we get $f^{(\infty)}(t) = f^{(\infty)}(r_2)$. Thus

$$f^{(\infty)}(t) = f^{(\infty)}(r_1) = f^{(\infty)}(r_2) = \ldots = f^{(\infty)}(r_p) = f^{(\infty)}(f(r_p)) \tag{1.12}$$

Since $f(r_p) \in [0, r_p[$ and $f(s) < s$, $\forall s \in\ ]0, r_p[$, we can apply now theorem 1.1 to obtain the conclusion that $f^{(\infty)}(f(r_p)) = 0$, and this completes the proof. Q.E.D.

Theorem 1.4. *Let* $f \in M$ *be such that*

$$\sigma(f) = \{r_n \mid n \geqslant 1\};\ r_1 > r_2 > \ldots;\ \lim_{n\to\infty} r_n = 0; \tag{1.13}$$

$$f \text{ is constant on }]r_{n+1}, r_n[,\ \forall n \in N \text{ (where } r_0 = +\infty). \tag{1.14}$$

Then, $f \in P$.

Proof. By the same way as before we get

$$f^{(\infty)}(t) = f^{(\infty)}(r_1) = f^{(\infty)}(r_2) = \ldots \tag{1.15}$$

Since $0 \leqslant f^{(\infty)}(r_n) < r_n$, $\forall n \geqslant 1$, from (1.13) this implies

$$\lim_{n\to\infty} f^{(\infty)}(r_n) = 0, \text{ i.e., } f^{(\infty)}(t) = 0,\ \forall t \in R_+.$$

Q.E.D.

2. In this paragraphe some fixed point results are established by means of property (P) given in §1.

Let (X, d) be a complete metric space; denote $F(X, X) = \{T \mid T : X \to X\}$.

Consider the mapping $\Phi : F(X, X) \to F(R_+, \overline{R}_+)$ defined by

$$\Phi(T)(t) = \sup\{d(Tx, Ty) ; d(x, y) \leqslant t\},\ \forall t \in R_+ . \tag{2.1}$$

Clearly, $\Phi(T)(0) = (0)$, $\Phi(T) =$ monotone increasing on R_+; therefore, $\Phi(T) \in K \Leftrightarrow \Phi(T) \in M$. On the other hand, (2.1) implies:

$$d(x, y) \leqslant t \Rightarrow d(Tx, Ty) \leqslant \Phi(T)(t). \tag{2.2}$$

Definition 2.1. *A mapping* $T \in F(X, X)$ *will be called a non-linear contraction if*

$$\Phi(T) \in M \cap P. \tag{2.3}$$

Denote $C(X, X) = \{T \mid T \in F(X, X),\ T = \text{non-linear contraction}\}$. The main result of this paragraphe is the following:

Theorem 2.1. *Let* $T \in C(X, X)$ *be given. Then, the following conclusions hold:*

(2.4) T *has a unique fixed point,* $z \in X$.

(2.5) $\forall x \in X,\ \exists \rho(x) \geqslant 0$, *such that* $d(T^n x, z) \leqslant \Phi^{(n)}(T)(t)$, $\forall t \geqslant \rho(x)$, $\forall n \in N$.

Proof. From (2.2) and (2.3) we obtain

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in X, \ x \neq y, \tag{2.6}$$

which shows that T is continuous on X. Now, let $x, y \in X$, $t \in R_+$ be such that $d(x, y) \leqslant t$. It is easy to see that

$$d(T^n x, T^n y) \leqslant \Phi^{(n)}(T)(t), \quad \forall n \in N, \tag{2.7}$$

and this implies (from (2.3)) $\lim\limits_{n \to \infty} d(T^n x, T^n y) = 0$. If we take $y = Tx$ in the preceding relation, then $\lim\limits_{n \to \infty} d(T^n x, T^{n+1} x) = 0$, $\forall x \in X$.

Take any $x \in X$ and let $\varepsilon > 0$ be given. We can find $n(\varepsilon) \in N$, such that $n \geqslant n(\varepsilon) \Rightarrow d(T^n x, T^{n+1} x) < \varepsilon - \Phi(T)(\varepsilon) < \varepsilon$. If $n \geqslant n(\varepsilon)$ is given, then

$$\& \{d(T^n x, T^{n+1} x) < \varepsilon\} \Rightarrow d(T^{n+1} x, T^{n+2} x) \leqslant \Phi(T)(\varepsilon) \Rightarrow \tag{2.2}$$

$$\Rightarrow d(T^n x, T^{n+2} x) \leqslant d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) <$$

$$< \varepsilon - \Phi(T)(\varepsilon) + \Phi(T)(\varepsilon) = \varepsilon;$$

$$\& \{d(T^n x, T^{n+2} x) < \varepsilon\} \Rightarrow d(T^{n+1} x, T^{n+3} x) \leqslant \Phi(T)(\varepsilon) \Rightarrow \tag{2.2}$$

$$\Rightarrow d(T^n x, T^{n+3} x) \leqslant d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+3} x) <$$

$$< \varepsilon - \Phi(T)(\varepsilon) + \Phi(T)(\varepsilon) = \varepsilon \ldots.$$

Therefore (by induction) $d(T^n x, T^{n+p} x) < \varepsilon$, $\forall n \geqslant n(\varepsilon)$, $\forall p \in N$, i.e., $\{T^n x \mid n \in N\}$ is a Cauchy sequence. If $z = \lim\limits_{n \to \infty} T^n x$, then, (since T is continuous) $Tz = T \lim\limits_{n \to \infty} T^n x) = \lim\limits_{n \to \infty} T^{n+1} x = z$, i.e., z is a fixed point of T. The uniqueness of z easily follows from (2.6) and this establishes (2.4). Now, let $x \in X$ be given. Define $\rho(x) = \operatorname{diam} \{T^n x \mid n \in N\}$. Clearly, $0 \leqslant \rho(x) < +\infty$, since $\{T^n x \mid n \in N\}$ is a Cauchy sequence. Furthermore, $d(z, x) \leqslant \rho(x)$, since $z \in \overline{\{T^n x \mid n \in N\}}$. Finally, taking into account (2.7) we obtain at once (2.5). Q.E.D.

Theorem 2.2. *Let $T \in F(X, X)$, $f \in P$ be such that*

$$d(x, y) \leqslant t \Rightarrow d(Tx, Ty) \leqslant f(t) \tag{2.8}$$

Then, T satisfies (2.4) and

$$\forall x \in X, \exists \rho(x) \geqslant 0, \text{ such that } d(T^n x, z) \leqslant f^{(n)}(t), \ \forall t \geqslant \rho(x), \forall n \in N. \tag{2.9}$$

Proof. From (2.1) and (2.8) we get

$$\Phi(T) \leqslant f \text{ (i.e., } \Phi(T)(t) \leqslant f(t), \ \forall t \in R_+). \tag{2.10}$$

Now, (2.10) and lemma 1.2 imply (2.3) .i.e., $T \in C(X, X)$. Therefore, the main result is applicable. Q.E.D.

3. In this paragraphe we extend the preceding theorems by considering the case of generalized complete metric spaces.
Let (X, d) be a generalized complete metric space [2], [3]; for any $x \in X$

define $\Omega(x) = \{y \in X \mid d(x, y) < +\infty\}$, and for any $T \in F(X, X)$, define $\varphi(T) = \{x \in X \mid d(x, Tx) < +\infty\}$ $\psi(T) = \{z \in X \mid z = Tz\}$.

Theorem 3.1. *Let $T \in C(X, X)$ be such that*

$$\varphi(T) \text{ is not empty} \tag{3.1}$$

Then, the following conclusions hold:

(3.2) *$\psi(T)$ is not empty (i.e., T has at least a fixed point).*

(3.3) *If $z \in \psi(T)$ is given, then, $\forall x \in \varphi(T) \cap \Omega(z) \, \exists \rho(x) \geqslant 0$, such that, $d(T^n x, z) \leqslant \Phi^{(n)}(T)(t)$, $\forall t \geqslant \rho(x)$, $\forall n \in N$.*

Proof. Let $x \in \varphi(T)$ be given. If $y \in \Omega(x)$ is an arbitrary element, then, by definition, $t = d(x, y) < +\infty$. As $T \in C(X, X)$, this implies $d(Tx, Ty) \leqslant \Phi(T)(t) < +\infty$. From the triangle inequality, we have $d(x, Ty) \leqslant d(x, Tx) + d(Tx, Ty) < +\infty$, i.e., $Ty \in \Omega(x)$. Moreover, if $y, z \in \Omega(x)$, $d(y, z) \leqslant d(y, x) + d(x, z) < +\infty$. Therefore, $\Omega(x)$ is an ordinary complete metric space, invariant under T, such that, $T \in C(\Omega(x), \Omega(x))$ and theorem 2.1 is applicable. Q.E.D.

Theorem 3.2. *Let $T \in F(X, X)$, $f \in P$ be such that (3.1) is satisfied and suppose in addition that*

$$d(x, y) \leqslant t \Rightarrow d(Tx, Ty) \leqslant f(t). \tag{3.4}$$

Then, T satisfies (3.2) and

$$\begin{aligned} &\text{if } z \in \psi(T) \text{ is given, then, } \forall x \in \varphi(T) \cap \Omega(z), \exists \rho(x) \geqslant 0, \\ &\text{such that } d(T^n x, z) \leqslant f^{(n)}(t), \ \forall t \geqslant \rho(x), \ \forall n \in N. \end{aligned} \tag{3.5}$$

Proof. The result easily follows from the preceding result and theorem 2.2. Q.E.D.

Remark 3.1. *If $u, v \in \psi(T)$ are such that $d(u, v) < +\infty$, then $u = v$ (or, in other words, $u, v \in \psi(T)$, $u \neq v \Rightarrow d(u, v) = +\infty$).*
Indeed, if $t = d(u, v) \in]0, +\infty[$, we have (from $T \in C(X, X)$) $t = d(u, v) = d(Tu, Tv) \leqslant \Phi(T)(t) < t$, a contradiction. Q.E.D.

Remark 3.2. *Suppose that T satisfies*

$$\forall x, y \in X, \exists n = n(x, y) \in N, \text{ such that } d(T^n x, T^n y) < +\infty. \tag{3.6}$$

Then, $\psi(T)$ consists of exactly one element $z \in X$, (i.e., T has a unique fixed point, $z \in X$).

For, if $u, v \in \psi(T)$, we have $d(u, v) = d(T^n u, T^n v) < +\infty$, and thus, (from remark 3.1) $u = v$. Q.E.D.

4. In this paragraphe a special class of functional equations will be considered: the so-called Volterra functional equations, which represent an extension of the classical Volterra integral equations and of some kind of integro-differential equations.

Let R_+ be the positive half-axis, $D = \{(t, s) \mid 0 \leqslant s \leqslant t < +\infty\}$, $(R^n, \|\cdot\|)$, the euclidean n-dimensional space. Denote $C(R_+, R^n) = \{x \mid x : R_+ \to R^n, x$ is continuous$\}$, $C(D \times R_+, R_+) = \{x \mid x : D \times R_+ \to R_+, x$

is continuous}, $C([0,t], R^n) = \{x \mid x : [0,t] \to R^n,\ x$ is continuous$\}$, $\forall t \in R_+$. For a given $x \in C(R_+, R^n)$, x_t stands for the restriction $x_t(s) = x(s)$, $\forall s \in [0,t]$; clearly, $x_t \in C([0,t], R^n)$, $\forall t \in R_+$.

Let us consider a family of mappings

(4.1) $$(t,s) \to k(t,s) : C([0,s], R^n) \to R^n,\ (t,s) \in D,$$

such that

(4.2) $\forall x \in C(R_+, R^n)$, the mapping $(t,s) \to k(t,s)(x_s)$, of $[0,t]$ into R^n is continuous on D.

By definition, a Volterra functional equation is of the form

(4.3) $$x(t) = x^0(t) + \int_0^t k(t,s)(x_s)\, ds,\quad \forall t \in R_+$$

(where $x^0 \in C(R_+, R^n)$ is fixed), or, in an abstract form

(4.4) $$x(t) = Tx(t),\quad \forall t \in R_+,$$

where $T : C(R_+, R^n) \to C(R_+, R^n)$ is defined by

(4.5) $$Tx(t) = x^0(t) + \int_0^t k(t,s)(x_s)\, ds,\quad \forall t \in R_+.$$

In order to apply the methods of §2 and §3, we must define a metric structure on $C(R_+, R^n)$.

Let $g : R_+ \to R_+$ be such that g = continuous, $g(t) > 0$, $\forall t \in R_+$. Define a generalized norm $\|\cdot\|_g$ on $C(R_+, R^n)$ as follows:

(4.6) $$\|x\|_g = \sup\{\|x(t)\|/g(t) : t \in R_+\},\ x \in C(R_+, R^n).$$

Denote $C_g(R_+, R^n) = \{x \mid x \in C(R_+, R^n),\ \|x\|_g < +\infty\}$. In what follows we suppose that it is possible to construct a family of mappings

(4.7) $$t \to \|\cdot\|_t = \text{seminorm on } C([0,t], R^n),\ \forall t \in R_+$$

Of course, $\|\cdot\|_t$ may be considered as a seminorm on $C(R_+, R^n)$, if we put $\|x\|_t = \|x_t\|_t$, $\forall x \in C(R_+, R^n)$. Suppose in addition that

(4.8) the family $\{\|\cdot\|_t ; t \in R_+\}$ is a sufficient family of seminorms on $C(R_+, R^n)$.

(4.9) $$\|x\|_t g(t) \leq \|x\|_g,\quad \forall x \in C(R_+, R^n),\ \forall t \in R_+.$$

The main result about (4.3) can be stated as follows

Theorem 4.1. *Suppose that there exist $h \in C(D) \times R_+, R_+)$, $f \in P$, such that*

(4.10) $(t,s) \in D$, $x, y \in C(R_+, R^n)$, $\sigma \in R_+$, $\|x - y\|_s \leq \sigma$ *imply that*
$$\|k(t,s)(x_s) - k(t,s)(y_s)\| \leq h(t,s,\sigma)$$

(4.11) $$\int_0^t h(t,s,\tau g(s))\, ds \leq g(t) f(\tau),\quad \forall t \in R_+,\ \forall \tau \in R_+$$

(4.12) $$\int_0^t k(t,s)(x^0_s)\, ds \in C_g(R_+, R^n)$$

Then, the following conclusions hold:

(4.13) *the Volterra functional equation* (4.3) *has at least one solution in $C(R_+, R^n)$ and exactly one solution in $x^0 + C_g(R_+, R^n)$.*

(4.14) *the sequence of succesive approximations $\{x^{[n]} ; n \in N\}$, $x^{[0]}(t) = x^0(t)$, $\forall t \in R_+$, $x^{[n+1]}(t) = x^0(t) + \int_0^t k(t,s)(x^{[n]}_s)\, ds$, $\forall t \in R_+$, $\forall n \in N$, converges to the unique solution $z \in x^0 + C_g(R_+, R^n)$;*

(4.15) *the error of approximation is given by $\|x^{[n]} - z\|_g \leq f^{[n]}(\tau)$, $\forall \tau \geq \rho$ where*

(4.16) $$\rho = \sup\{\|x^{[p]} - x^{[q]}\|_g : p, q \in N\}.$$

Proof. If T is given by (4.5) then, from (4.12) we obtain at once

(4.17) $$Tx^0 - x^0 \in C_g(R_+, R^n),\ (\text{respectively, } \|Tx^0 - x^0\|_g < +\infty).$$

Besides, let $x, y \in C(R_+, R^n)$, $\tau \in R_+$ be such that $\|x - y\|_g \leq \tau$. (4.9) this implies $\|x - y\|_t \leq \tau g(t)$, $\forall t \in R_+$. Taking into account (4.10), (4.11), we have

(4.18)
$$\|Tx(t) - Ty(t)\| = \left\| \int_0^t [k(t,s)(x_s) - k(t,s)(y_s)]\, ds \right\| \leq$$
$$\leq \int_0^t \|k(t,s)(x_s) - k(t,s)(y_s)\|\, ds \leq \int_0^t h(t,s,\tau g(s))\, ds \leq$$
$$\leq g(t) f(\tau),\ \forall t \in R_+$$

or, in an equivalent form, $\|Tx(t) - Ty(t)\|/g(t) \leq f(\tau)$, $\forall t \in R_+$. Taking the supremum we find

(4.19) $$\|Tx - Ty\|_g \leq f(\tau)$$

Therefore, theorem 3.2 is entirely applicable, and this copletes the proof. Q.E.D.

Remark 4.1. If we suppose that i) $k(t,s)$ *is independent on t;* ii) $x^0(t) = x^0 \in R^n$; iii) $\|x\|_t = \sup\{\|x(s)\| ;\ s \in [0,t]\}$, $\forall x \in C(R_+, R^n)$,

$\forall t \in R_+$; iv) g is monotone increasing and continuous on R_+ ; v) $h(t, s, u) = \lambda(s)\, u$, where $\lambda : R_+ \to R_+$ is continuous on R_+ ; vi) $f(t) = \alpha t$, $0 \leqslant \alpha < 1$, then, the above theorem reduces to theorem 1 of [4].

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COINCIDENCE POINTS AND APPLICATIONS TO VOLTERRA FUNCTIONAL EQUATIONS

BY

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1. Let $R_+ = [0, +\infty[$ be the positive half-axis, $N = \{0, 1, \ldots\}$, the positive integers, $F(R_+, R_+) = \{f; f : R_+ \to R_+\}$. Denote:

i) $M = \{f \in F(R_+, R_+) ; f(0) = 0, f(t) < t, \ \forall t > 0\}$,

ii) $P = \{f \in M ; \ \lim\limits_{n \to \infty} f^{[n]}(t) = 0, \ \forall t > 0\}$ (where, $f^{[n]}$ stands for the n-th iterate of f, $\forall n \in N$).

iii) $Q = \{f \in M : \limsup\limits_{s \to t+} f(s) < t, \ \forall t > 0$, (i.e., $\forall t > 0, \exists a(t) \in]0, t[$, such that $s \in]t, t + a(t)[\Rightarrow f(s) \leqslant t - a(t)$). Let us denote by P^* (respectively Q^*) the set all (φ, Ψ), $\varphi, \Psi \in F(R_+, R_+)$, for which there exists $f \in P$ (resp. $f \in Q$), such that, the following implication holds

$$u > 0, \ v > 0, \ \varphi(u) \leqslant \Psi(v) \Rightarrow u \leqslant f(v). \tag{1.1}$$

Note [3] that

$$Q \subset P \ (\Rightarrow Q^* \subset P^*), \tag{1.2.}$$

In what follows, a partial characterization of the class Q^* is given.

Theorem 1.1. *Suppose that*

$$\forall t > 0, \ \exists s \in [0, t[\ \textit{such that} \ \varphi(s) \leqslant \Psi(t), \tag{1.3}$$

$$u > 0, \ v > 0, \ \varphi(u) \leqslant \Psi(v) \Rightarrow u < v, \tag{1.4}$$

$$\liminf_{s \to t} \varphi(s) > \limsup_{s \to t+} \Psi(s), \ \forall t > 0, \ (\textit{i. e.}, \ \forall t > 0, \ \exists a(t) \in]0, t[, \tag{1.5}$$
such that $\varphi(u) > \Psi(v)$, $\forall u \in]t - a(t), t + a(t)[$, $\forall v \in]t, t + a(t)[$).

Then, $(\varphi, \Psi) \in Q^*$.

Proof. Define $f \in F(R_+, R_+)$ by $f(0) = 0$, and

$$f(t) = \sup\{s \in R_+ ; \ \varphi(s) \leqslant \Psi(t)\}, \ \forall t > 0. \tag{1.6}$$

From (1.3) and (1.4), $f(t)$ is well defined and $f(t) \leqslant t$, $\forall t \in R_+$. Now, let