

INFRA-NEAR RINGS OF AFFINE TYPE

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In the papers [7], [8], we introduced and investigated a generalization of the concept of near-ring; we called it a left infra-near ring. We studied there the ideals, the homomorphisms, the left and right groups over such an algebraic system. We gave also a ternary interpretation and we discussed the possibilities to define a left infra-distributive multiplication on an additive group. Now we study a special type of left infra-near rings, which have the properties recalling those of the abstract (right) affine near-ring (see [4]). First we give basic definitions and a method for constructing a left C -infra-near ring which has the properties (A.1) – (A.3) (Proposition 1.7). In section 2, we show that is the only method to obtain such a left infra-near ring. We find the ideals and the main properties of D and W (defined in (1.4)) of a l.i.-n.r. satisfying (A.1) – (A.3) (Propositions 2.1, 2.2, 2.5). We give also the form of the ideals for a D - W -transitive left C -infra-near ring (Definition 2.8, Propositions 2.9, 2.10). We discuss the existence of the idempotents and identities in such a l.i.-n.r. (Proposition 3.1). The ternary interpretation of the definitions and conditions (A.1) – (A.3) is given in Section 4. We compare our results with the results obtained by Pilz [6], Gonschor [4] and Heathcrlly [5] on abstract affine near rings.

§.1 Definitions and notations. Recall some necessary definitions.

Definition 1.1 [7]. Let $(I, +, \cdot)$ be an ordered triple, where I is a nonempty set, $+$ and \cdot are binary operations on I . We call it a left infra-near ring (shortly, l.i.-n.r.), if it satisfies the following conditions:

(1.1) $(I, +)$ is a group (nonabelian);

(1.2) (I, \cdot) is a semigroup;

(1.3) $x \cdot (y + z) = x \cdot y + x \cdot z$, $\forall x, y, z \in I$.

If, instead of (1.3), the equality:

(1.3)' $(x + y) \cdot z = x \cdot z + y \cdot z$, $\forall x, y, z \in I$,

holds, then I is called a right infra-near ring.

An element $x \in I$, for which $x \cdot 0 = 0$, is called left distributive. If each element of I is left distributive, then I is a left near ring. If all the

elements of a right infra-near ring have the property $0 \cdot x = 0$, then I is called a *right near ring*. We denote:

$$(1.4) \quad \begin{cases} D = D(I) := \{x \mid x \in I, x \cdot 0 = 0\}, \\ W = W(I) := \{x \mid x \in I, x \cdot 0 = x\}. \end{cases}$$

the set of left distributive elements and the set of weakly left distributive elements of I . Obviously, $D \cap W = \{0\}$.

If a l.i.-n.r. (a left near ring) I has the property:

$$(1.5) \quad 0 \cdot x = 0, \quad \forall x \in I,$$

we call it a *left C-infra-near ring*—shortly, *l.C-i.-n.r.* — (resp., a *left C-ring*; see Berman and Silverman [1]). A l.i.-n.r. which is a right near ring is, obviously, a l.C-i.-n.r. If I is a l.C-i.-n.r., then (D, \cdot) is a subsemigroup of (I, \cdot) , (W, \cdot) is a two-sided ideal of the semigroup (I, \cdot) and it has the obvious property: $w \cdot x = w, \forall w \in W, \forall x \in I$. Hence all elements of W are idempotents and the multiplication in W is: $w \cdot w' = w, \forall w, w' \in W$, and then W is a right near ring. If:

$$(1.6) \quad 0 \cdot x = x, \quad \forall x \in I,$$

then we call I a *left Z-infra-near ring*, — shortly, a *l.Z.-i.-n.r.* — (resp. for a near ring, a *Z-ring*, as in [1]).

Every left near ring I is a semidirect sum (as a group) of its maximal Z-ring and C-ring, namely $I = Z(I) + C(I)$, where $Z(I) = \{x, x \in I, 0 \cdot x = x\}$, $C(I) = \{x \mid x \in I, 0 \cdot x = 0\}$ (see [1], [5]). It is the same for a l.i.-n.r. I , when $0 \cdot 0 = 0$, i.e. 0 is a left distributive element of I . If I is both a right near ring and a l.i.-n.r., then $D = C_r(I)$ and $W = Z_r(I)$ and we have $I = W + D$.

Let us give some examples:

Example 1. Denote the set of the affine transformations of a vector space V , by $A(V)$. For all $f \in A(V)$, there exist $h \in \text{Hom}_k(V, V)$ and $c \in T_c(V)$ (constant functions on V), $c = f(0)$, such that $f = h + c$ (pointwise addition of functions).

Given $f = h + c, f' = h' + c'$ in $A(V)$, we define: $f + f' := (h + h') + (c + c')$ and $f \circ f' := h \circ h' + (h \circ c' + c)$. Then $(A(V), +, \circ)$ is a right near ring and a l.i.-n.r. with identity; it is not a ring. $C_r(A(V)) = D(A(V)) = \text{Hom}_k(V, V)$ and $Z_r(A(V)) = W(A(V)) = T_c(V)$. This near ring has been studied by Blackett [2], Wolfson [9] and it has been generalized by Gonshor [4], Pilz [6], Heatherly [5].

Definition 1.2. (Gonshor [4]). A right near ring $(N, +, \cdot)$ with the properties: (i) $(N, +)$ is an abelian group; (ii) $N = C_r(N) + Z_r(N)$; (iii) $C_r(N)$ is left distributive over N , is called an *abstract affine near ring*.

Example 2. A similar example is given by the set of functions on an abelian group, G , $A(G) = \{f \mid f = c + e, e \in \text{End}(G), c \in T_c(G)\}$, together with pointwise addition and composition of two functions. This example was studied as a right near ring by H.E. Heatherly ([5], §7, Lemma 7.1. Theorem 7.2).

Example 3. Let R be a left C-ring, $I = R \times R$ will be a left C-infra-near ring with respect to the binary compositions:

$$(1.7) \quad \begin{cases} x + y := (x_1 + y_1, x_2 + y_2), \\ x \cdot y := (x_1 \cdot y_1, x_2 + x_1 \cdot y_2), \quad \forall x, y \in I. \end{cases}$$

For it, $D(I) = \{(x_1, 0) \mid x_1 \in R\}$, $W(I) = \{(0, x_2) \mid x_2 \in R\}$. $I = D \oplus W$ (as groups), but its multiplication is not right distributive over the addition. We have:

$$(1.8) \quad (w + d) \cdot x = w \cdot x + d \cdot x, \quad \forall w \in W(I), \forall d \in D(I), \forall x \in I,$$

but, in general, $(d + w) \cdot x \neq d \cdot x + w \cdot x$, as one can easily see.

Definition 1.3. [7]. Let J be a normal subgroup of $(I, +)$, where I is a l.C-i.-n.r. We call J a *left ideal* of I , if it satisfies the condition:

$$(1.9) \quad x \cdot j - x \cdot 0 \in J, \quad \forall j \in J, \forall x \in I.$$

We call J a *right ideal* of I , if it satisfies the condition:

$$(1.10) \quad (j + x) \cdot y - x \cdot y \in J, \quad \forall j \in J, \forall x, y \in I.$$

If J is a left and a right ideal, we call it a *two-sided ideal* of I .

Definition 1.4. [7]. Let $(G, +)$ be a group and $(I, +, \cdot)$ be a l.i.-n.r.. If there exists a multiplication $\mu: I \times G \rightarrow G$, $\mu(x, g) = x \cdot g, \forall x \in I, \forall g \in G$, such that the following conditions hold:

$$(1.11) \quad \begin{cases} x \cdot (g + g') = x \cdot g - x \cdot 0_G + x \cdot g', \quad \forall x \in I, \forall g, g' \in G; \\ (x \cdot y) \cdot g = x \cdot (y \cdot g), \quad \forall x, y \in I, \forall g \in G, \end{cases}$$

then we call G a *left I-group*. When I is a near ring, we take $x \cdot 0_G = 0_G, \forall x \in I$. We have a dual definition for right I-groups.

Remark 1.5. Obviously, $(W, +)$ is a subgroup of the l.C-i.-n.r. I , if and only if:

$$(1.12) \quad (w + w') \cdot 0 = w \cdot 0 + w' \cdot 0, \quad \forall w, w' \in W.$$

Then we have:

Proposition 1.6. If $(W, +)$, with W given by (1.4), is a subgroup of $(I, +)$, where I is a l.C-i.-n.r., then every subgroup A of $(W, +)$ is a right I-group with the properties:

$$(1.13) \quad \begin{cases} w \cdot a - w \cdot 0 = 0, \quad \forall w \in W, \forall a \in A, \\ (a + w) \cdot w' - w \cdot w' = a, \quad \forall w, w' \in W, \forall a \in A. \end{cases}$$

Therefore, if A is a normal subgroup of W , then A is a two-sided ideal of W .

The proof is very easy and we omit it.

If $I = W$ is a l.C-i.-n.r., we call it a *zero infra-near ring*, because its multiplication is given by:

$$(1.14) \quad x \cdot y = x, \quad \forall x, y \in I,$$

hence I is also a right Z -ring (see [1]). This proposition, together with the remark that the homomorphic image of a zero infra-near ring is also a zero infra-near ring, show that the study of zero infra-near rings is not so interesting (mainly, it is deduced from the study of the group $(I, +)$). $T_c(G)$, the set of all constant functions on a group G , is such a zero infra-near ring.

We give now a general method for constructing left C -infra-near rings; this construction will characterize a whole class of left C -infra-near rings. (It remind us the construction used by G o n s h o r ([4], Proposition 4), where D is a ring and W is a D -module.)

Proposition 1.7. *Let D be a left C -ring, W a left D -group. Then $I = D \oplus W$ (the direct sum, as groups) can be endowed with a multiplication of a l.C-i-n.r., with $D(I) = D \oplus \{0\}$, $W(I) = \{0\} \oplus W$, the elements of $D(I)$, $W(I)$, I satisfying (1.8).*

Proof. This multiplication is:

$$(1.15) \quad (d_1 + w_1) \cdot (d_2 + w_2) := d_1 \cdot d_2 + (w_1 + d_1 \cdot w_2), \quad \forall d_i + w_i \in I, \quad i = 1, 2.$$

We note that $w_1 + d_1 \cdot w_2 \in W$, since W is a left D -group, while $d_1 \cdot d_2 \in D$, which is a left C -ring. All axioms can be verified by straightforward calculations as well as the implications: " $d + w \in D(I)$ if and only if $w = 0$ " and " $d + w \in W(I)$ if and only if $d = 0$ ". We have indeed $w \cdot x + d \cdot x = w + (d \cdot d_1 + d \cdot w_1) = (w + d \cdot d_1) + d \cdot w_1 = d \cdot d_1 + (w + d \cdot w_1) = (w + d) \cdot x$, $\forall w \in W, \forall d \in D, \forall x = d_1 + w_1 \in I$, but, in general, $d \cdot x + w \cdot x \neq (d + w) \cdot x$, since the elements $w + d \cdot w_1$ and $d \cdot w_1 + w$ are not always equal. Moreover, the element $0 = 0_D + 0_W \in I$ is a right distributive element.

§ 2. Infra-near rings of affine type. Now we put another question than that in [5] and [6], namely: We give up the right distributivity of the multiplication over the addition. Instead of it, we take some weaker assumptions, for instance (1.5), (A.1), (A.3) and we assume also that $(I, +, \cdot)$ is a left infra-near ring. Which are the properties of such a system? There exists a great analogy with P i l z's results for abstract affine near rings (see [6]), as one can see in that follows. Of course, if $(I, +, \cdot)$ is also a right near ring, then $D = C_r(I)$ and $W = Z_r(I)$ and we obtain exactly the results of P i l z and H e a t h e r l y. But, in general, $w + d \cdot w' \neq d \cdot w' + w$, $\forall d \in D, \forall w, w' \in W$, as we remarked in Example 3. (This is one of the statements of the Theorem 7, [6].)

The Proposition 1.7 suggests the following conditions for a left C -infra-near ring:

$$(A.1) \quad (x + y) \cdot 0 = x \cdot 0 + y \cdot 0, \quad \forall x, y \in I;$$

$$(A.2) \quad x - x \cdot 0 = -x \cdot 0 + x, \quad \forall x \in I;$$

$$(A.3) \quad (w + d) \cdot x = w \cdot x + d \cdot x, \quad \forall w \in W(I), \forall d \in D(I), \forall x \in I.$$

Proposition 2.1. *Let $(I, +, \cdot)$ a l.C-i-n.r. satisfying (A.1). Then (using the notation (1.4)):*

(i) $(D, +, \cdot)$ is a left C -ring and a normal subgroup in $(I, +)$;

(ii) $(W, +, \cdot)$ is a zero C -infra-near ring, a left and a right I -group and a left D -group;

(iii) $I = D \dot{+} W = W \dot{+} D$ (semidirect sum as groups);

(iv) I is the direct sum of the groups D and W if and only if it satisfies (A.2).

Proof. We give only the lines of the proof.

(i) Indeed, we have: $d - d', x + d - x, d' \cdot d \in D$, for all $d, d' \in D, x \in I$, since $(d - d') \cdot 0 = d \cdot 0 - d' \cdot 0 = 0 - 0 = 0, (d' \cdot d) \cdot 0 = d' \cdot (d \cdot 0) = d' \cdot 0 = 0, (x + d - x) \cdot 0 = x \cdot 0 + d \cdot 0 - x \cdot 0 = x \cdot 0 - x \cdot 0 = 0$.

(ii) Now, $w - w', w \cdot w', x \cdot w, w \cdot x \in W$, for all $w, w' \in W, x \in I$ (calculations). Moreover, $d \cdot (w + w') = d \cdot w + d \cdot w', \forall d \in D, \forall w, w' \in W$.

(iii) $D \cap W = \{0\}$, D is a normal subgroup of I and $I = D \dot{+} W = W \dot{+} D$. Indeed, for all $x \in I, x = (x - x \cdot 0) \dot{+} x \cdot 0 = x \cdot 0 \dot{+} (-x \cdot 0 + x)$, with $x \cdot 0 \in W, x - x \cdot 0, -x \cdot 0 + x \in D$.

(iv) is clear because of the above expressions of an element x in I .

We can prove a converse of the Proposition 1.7, namely:

Proposition 2.2. *If I is a l.C-i-n.r. satisfying (A.1) - (A.3), then:*

(i) D is a left C -ring and W is a left D -group with respect to the restrictions of the multiplication of I to $D \times D, D \times W$;

(ii) The multiplication of I is uniquely determined by these restrictions and $I = D \oplus W$ (direct sum as groups);

(iii) W is a two-sided ideal of I .

Proof. (i) results from the Proposition 2.1. (ii) By using the unique decompositions of the elements of $I, x_i = d_i + w_i, d_i \in D, w_i \in W, i = 1, 2$, and (A.3), we have: $x_1 \cdot x_2 = (d_1 + w_1) \cdot x_2 = (w_1 + d_1) \cdot x_2 = w_1 \cdot x_2 + d_1 \cdot x_2 = w_1 + d_1 \cdot (d_2 + w_2) = w_1 + (d_1 \cdot d_2 + d_1 \cdot w_2) = (w_1 + d_1 \cdot d_2) + d_1 \cdot w_2 = d_1 \cdot d_2 + (w_1 + d_1 \cdot w_2)$. We note that $(x_1 \cdot x_2)_D = d_1 \cdot d_2, (x_1 \cdot x_2)_W = w_1 + d_1 \cdot w_2$, and the decomposition of $x_1 \cdot x_2$ is unique, hence the product is:

$$(2.1) \quad x_1 \cdot x_2 = d_1 \cdot d_2 + (w_1 + d_1 \cdot w_2),$$

where $x_1 \cdot x_2 = w_1 + d_1 \cdot w_2$. We see that this product is uniquely determined by the products $d_1 \cdot d_2$ and $d_1 \cdot w_2$.

(iii) We know that $(W, +)$ is a normal subgroup of I (Proposition 2.1). The other relations can easily be verified: for all $x_i \in I, x_i = d_i + w_i, i = 1, 2$, for all $w \in W, x_1 \cdot w - x_1 \cdot 0 \in W$ (by applying (A.1)) and $(w + x_1) \cdot x_2 - x_1 \cdot x_2 = (d_1 \cdot d_2 + (w + w_1 + d_1 \cdot w_2)) - (d_1 \cdot d_2 + (w_1 + d_1 \cdot w_2)) = w \in W$ (by applying (2.1), (A.2)). Therefore W is a two-sided ideal of I .

Remark 2.3. If D is a ring and W is a left D -module (i.e. a left D -group, with $(d_1 + d_2) \cdot w = d_1 \cdot w + d_2 \cdot w, \forall d_i, d_2 \in D, \forall w \in W$), then (2.1) defines the unique multiplication on $I = D \oplus W$, such that the elements of D are two-sided distributive over I and I is an abstract affine (right) near ring (see P i l z [6], Theorem 7, G o n s h o r [4]).

Definition 2.4. A left C -infra-near ring which satisfies (A.1)–(A.3) will be called an infra-near ring of affine type.

We shall describe the two-sided ideals of such a l.C-i.n.r. by:

Proposition 2.5. Let I be an infra-near ring of affine type. Then:

(i) Every two-sided ideal J of I is of the form $J = D_1 \oplus W_1$, where D_1 is a two-sided ideal of the left C -ring D , W_1 is a D -subgroup of W including D_1 , $W = \{d_1 \cdot w \mid d_1 \in D_1, w \in W\}$. D_1 satisfies the condition:

$$(2.2) \quad (d_1 + d) \cdot w - d \cdot w \in D_1, \quad \forall w \in W, \quad \forall d \in D, \quad \forall d_1 \in D_1.$$

(ii) J is closed to the left multiplication by elements of I if and only if $W_1 = W$.

(iii) If $(J, +)$ is a left D -group and a normal subgroup of W , then J is a two-sided ideal of I .

Proof: (i) If D_1 and W_1 satisfy the hypotheses of the proposition, and $J = D_1 \oplus W_1$, then J is a two-sided ideal of I , as we can show by straightforward calculation. For the converse, let J be a two-sided ideal of I . Then $j \cdot 0 = (j + 0) \cdot 0 = 0 \cdot 0 = w_1 \in J \cap W$, $\forall j \in J$, $j = d_1 + w_1$ hence $d_1 = j - w_1 \in J \cap D$. We have $J = D_1 \oplus W_1$, with D_1, W_1 normal subgroups of $(I, +)$, because of $J \cap D = D_1, J \cap W = W_1$. For all $d_1 \in D_1$, $x, y \in I$, $d, d' \in D$, we have: $x \cdot d_1 - x \cdot 0 \in J \cap D = D_1$ and $(d_1 + d) \cdot d' - d \cdot d' \in J \cap D = D_1$. Therefore D_1 is a two-sided ideal in D and a left ideal in I . Similarly, if $w_1 \in W_1$, $d \in D$, then $d \cdot w_1 \in W \cap J = W_1$, hence W_1 is a left D -group. The other statement is easy to prove.

(ii) Indeed, if $I \cdot J \subseteq J$, then, by taking $0 \in J$ and $w \in W$, we have: $w \cdot 0 \in J$, ($w \cdot 0 = w$), hence $W \subseteq J$, $W_1 = W$. For the converse, take $x = d + w \in I$, $j = d_1 + w_1 \in J$; then $x \cdot j = d \cdot d_1 + (w + d \cdot w_1) \in J$, since $d \cdot d_1, d \cdot w_1, w \in J$.

(iii) We apply (i) for $J = W_1, D_1 = \{0\}$.

Remark 2.6. If J is a two-sided ideal of I (which is an infra-near ring of affine type), then J is a right I -group, hence $J \cdot I \subseteq J$.

Remark 2.7. If I is an infra-near ring of affine type, which is also a right near ring, then its two-sided ideals are two-sided ideals of the right near ring and Proposition 2.5 is just a result obtained by G o n s h o r [4], but here we did not make the assumption of the commutativity of the addition.

Definition 2.8. Let I be a l.C-i.n.r. satisfying (A.1). It will be called D - W -transitive, if, for all $w, w' \in W, w \neq 0$, there exists $d \in D$, such that $d \cdot w = w'$, i.e. $D \cdot w = W$.

We remark that $D \cdot w \subseteq W$, but the equality is not always satisfied (Example 3). We have:

Proposition 2.9. Let I be a D - W -transitive l.C-i.n.r. Then:

(i) D is a maximal left ideal and a maximal subinfra-near ring of I ;
 (ii) Each left (two-sided) ideal J , containing an element $w_0 \in W, w_0 \neq 0$, contains also all elements of W . If $J \cap D = \{0\}$, then $J = W$ or $J = \{0\}$.

Proof: We prove first (ii): Let J be a left ideal of $I, w_0 \in J \cap W, w_0 \neq 0$. Then $D \cdot w_0 = W \subseteq J$, since J is a left ideal and D is left distributive over I . The other statement is trivial. Now, to prove (i), let T be a left ideal of I with the properties: $D \subseteq T \subseteq I, T \neq D$. Then there exists $w_0 \in T \setminus D, w_0 \neq 0, w_0 \in W$. But I is D - W -transitive, therefore $D \cdot w_0 = W \subseteq T$ (because T is a left ideal of I and D is left distributive over I), hence $T = I$. Similar arguments are used to prove that D is a maximal subinfra-near ring of I .

We have found, this way, in more general conditions, some of the results obtained by D. W. Blackett [2] and K.G. Wolfson [9], for $A(V)$, where V is a vector space over a field k . It is sufficient to remark that $A(V)$ is a D - W -transitive left C -infra-near ring and we have the form and the properties of its ideals.

Remark 2.10. In the hypotheses of the Proposition 2.9, D is not a two-sided ideal of I . Indeed, $D \cdot W = W$ and $D \cap W = \{0\}$, hence $D \cdot W \not\subseteq D$. This contradicts the fact that D is a right ideal.

Proposition 2.11. If I is a l.C-i.n.r. which is D - W -transitive and D is a simple ring, then I does not have proper two-sided ideals or its only two-sided ideal is W .

Proof: Let J be a two-sided ideal of I ; as D is a simple ring, either $D \cap J = \{0\}$ or $D \cap J = D$. But the last case implies that $J = I$, because $D \neq J$ (D cannot be a two-sided ideal of I ; see Remark 2.10) and there exists an element $w_0 \in W \cap J, w_0 \neq 0, I$ being D - W -transitive. If $D \cap J = \{0\}$, then either W is a two-sided ideal of I and then $J = W$, or $J = \{0\}$.

Many theorems giving the forms of the left I -subgroups, left ideals, D -groups of D and W can be obtained by similar arguments, but there is nothing new in the manner to state and to prove them and therefore we omit them.

3. Idempotents. Which are the idempotents and the left and right identities of a l.C-i.n.r. of affine type? Some facts are almost obvious:

1. Every $w \in W$ is a nondistributive idempotent of I , because of $w \cdot w = w, w \cdot 0 = w$.
2. The elements of I are right identities for W ; indeed, $w \cdot x = (w \cdot 0) \cdot x = w \cdot (0 \cdot x) = w \cdot 0 = w, \forall x \in I, \forall w \in W$.
3. Every left identity e of I is contained in D , since $e \cdot 0 = 0$.
4. A right identity of $D, r, r \in D$, is a right identity of I . Indeed, $d \cdot r = d, \forall d \in D$, implies that $(d + w) \cdot r = d \cdot r + w = d + w, \forall d + w \in I$.
5. If $r \in D$ is a right identity of D and $a \in W$ is a right annihilator of D , i.e. $d \cdot a = 0, \forall d \in D$, then $b = r + a$ is a right identity of I and conversely.

Indeed, for all $x = d + w \in I, (d + w) \cdot b = d \cdot r + w + d \cdot a = d + w$, if and only if $d \cdot a = 0$ and $d \cdot r = d, \forall d \in D$ (because of the uniqueness of the decomposition of $(d + w) \cdot b$ over D and W).

The existence of a left identity in a l.i.n.r. which is also a right near ring is a strong condition. We prove it with the:

Proposition 3.1. If I is a l.i.n.r. which is also a right near ring, and there exists a left identity e of I , then:

- (i) $(I, +)$ is an abelian group;

(ii) When $I = I_1 \oplus I_2$, with I_1 and I_2 right ideals of I , then there exist $e_1 \in I_1$ and $e_2 \in I_2$, $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$, $e_1 \cdot e_1 = e_1$, $e_2 \cdot e_2 = e_2$, e_1 being a left identity of I_1 and a left annihilator of I_2 , e_2 being a left identity of I_2 and a left annihilator of I_1 ;

(iii) If e is a two-sided identity of I , a right ideal I_1 of I is a direct summand in I if and only if there exists a distributive idempotent $e_1 \in I_1$, such that $I_1 = e_1 \cdot I$.

Proof: (i) By developing in two ways the product $(e + c) \cdot (x + y)$, for x, y arbitrary elements of I , we find that $x + y = y + x$.

(ii) We have: $e = e_1 + e_2$, $e_1 \in I_1$, $e_2 \in I_2$. Then $e \cdot e_k = (e_1 + e_2) \cdot e_k = e_1 \cdot e_k + e_2 \cdot e_k$, $k = 1, 2$. But $e_1 \cdot e_k \in I_1$, $e_2 \cdot e_k \in I_2$, $e \cdot e_k = e_k$, therefore $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$, $e_1 \cdot e_1 = e_1$, $e_2 \cdot e_2 = e_2$. For all $x_k \in I_k$, $k = 1, 2$, we have: $e \cdot x_k = x_k = e_1 \cdot x_k + e_2 \cdot x_k$, therefore $e_1 \cdot x_2 = 0$, $e_1 \cdot x_1 = x_1$, $e_2 \cdot x_1 = 0$, $e_2 \cdot x_2 = x_2$; we have also $e \cdot 0 = 0$, $k = 1, 2$, because of $e \cdot 0 = 0$.

(iii) The necessity follows from (ii). To prove that the condition is sufficient, we show that $e_2 = e - e_1$ is a left distributive idempotent: $e_2 \cdot e_2 = (e - e_1) \cdot e_2 = e \cdot e_2 - e_1 \cdot e_2 = e_2 - e_1 \cdot (e - e_1) = e_2 - (e_1 \cdot e - e_1 \cdot e_1) = e_2 - (e_1 - e_1) = e_2$ and $e_2 \cdot 0 = (e - e_1) \cdot 0 = e \cdot 0 - e_1 \cdot 0 = 0$. We take $I_2 = e_2 \cdot I$ and then $I = I_1 \oplus I_2$.

Remark 3.2. The statement (i) of the Proposition 3.12 is a generalization of a sufficient condition given by Ph. Fürtwängler and Olga Tauskvy [3] for a group $(I, +)$, having an associative and (two-sided) distributive multiplication over the addition, to be abelian. The other two statements are evident generalizations of facts being valid in the theory of rings.

Remark 3.3. If $(I, +, \cdot)$ is a 1.C-i.n.r. of affine type which is also a right near ring, then, by developing the equal products $(d_1 + w_1) \cdot w_2$ and $(w_1 + d_1) \cdot w_2$, we obtain the identity:

$$(3.1) \quad w_1 + d_1 \cdot w_2 = d_1 \cdot w_2 + w_1, \quad \forall w_1, w_2 \in W, \quad \forall d \in D.$$

Therefore, by defining:

$$(3.2) \quad x \circ y := x \cdot y - x \cdot 0, \quad \forall x, y \in I,$$

(the left distributive multiplication „canonically” associated to the multiplication of I — see [8]), we have:

$$(3.3) \quad x \circ y := d_1 \cdot d_2 + d_1 \cdot w_2, \quad \forall x = d_1 + w_1, \quad y = d_2 + w_2 \in I, \text{ hence}$$

$$x \cdot y = d_1 \cdot y.$$

But this multiplication satisfies the condition (4.1) from the Proposition 4.1 [8], and then $(I, +, \circ)$ is a distributive near ring. We reobtain a remark used by Blackett [2] in his main theorem. So we can study the infra-near ring of affine type which is also a right near ring by means of the distributive near ring associated to it by (3.2).

§4. Ternary interpretation. Now we look at the left infra-near rings of affine type from the point of view of the ternary operation of the form:

$$[x, y, z] := x - y + z, \quad \forall x, y, z \in I.$$

Recall the following:

Definition 4.1. (Definition 2.3, [7]). A triple $(I, [., ., .], \cdot)$ is called a left ternary near ring of first type, if it satisfies the axioms:

$$(4.1) \quad (I, \cdot) \text{ is a semigroup};$$

$$(4.2) \quad [., ., .]: I \times I \times I \rightarrow I \text{ is a ternary operation on } I;$$

$$(4.3) \quad x \cdot [y, z, t] = [x \cdot y, x \cdot z, x \cdot t], \quad \forall x, y, z, t \in I;$$

$$(4.4) \quad [x, 0, [y, z, t]] = [[x, 0, y], z, t], \quad \forall x, y, z, t \in I;$$

$$(4.5) \quad [x, 0, 0] = x, \quad \forall x \in I;$$

$$(4.6) \quad [x, x, 0] = 0, \quad \forall x \in I.$$

We add the axiom:

$$(4.7) \quad 0 \cdot x = 0, \quad \forall x \in I.$$

We have proved in [7] the following:

Proposition 4.2. ([7], Proposition 2.3). If $(I, [., ., .], \cdot)$ is a left ternary near ring of first type, then $(I, +, \cdot)$, with $+$ given by:

$$(4.8) \quad x + y := [x, 0, y], \quad \forall x, y \in I,$$

is a l.i.-n.r.

The converse of this proposition is also true ([7], Remark 2.4).

It is easy to prove that D and W are closed under the ternary operation and their semigroupal properties with respect to the multiplication are valid too. We can translate (A.1), (A.2), (A.3), the fact that $I = D + W$ and the Definition 2.4 in terms of the ternary operation by:

$$(A.1)' \quad [x, 0, y] \cdot 0 = [x \cdot 0, 0, y \cdot 0], \quad \forall x, y \in I;$$

$$(A.2)' \quad [[0, x \cdot 0, 0], 0, x] = [x, 0, [0, x \cdot 0, 0]], \quad \forall x \in I;$$

$$(A.3)' \quad [w \cdot 0, d] \cdot x = [w \cdot x, 0, d \cdot x], \quad \forall w \in W, \quad \forall d \in D, \quad \forall x \in I;$$

$$(4.9) \quad I = [D, \{0\}, W].$$

Note. We can study the left Z-infra-near rings in the same manner as the left C-infra-near rings. We shall be concerned with this matter and with methods for constructing infra-near rings with fixed D and W , $C(I)$ and $Z(I)$ in a future paper. We shall study also a class of infra-near rings which are left or two-sided self-distributive (i.e. the identities: $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$ hold for all x, y, z in I).

REFERENCES

1. Berman G. and Silverman R.J. — *Near rings*, Amer. Math. Monthly, 66 (1959) 28—34.
2. Blackett D.W. — *The near ring of affine transformations*, Proc. Amer. Math. Soc. 7 (1956), 517—519.
3. Fürtwängler Ph. and Taussky Olga — *Über Schieferringe*, S.B. Akad. Wiss. Wien, 145(1936), 525.
4. Gonsior H. — *On abstract affine near rings*, Pacific J. Math., 14(1964), 1237—1240.
5. Heatherly H.B. — *C-Z-transitive and C-Z-decomposable near rings*, J. Algebra, 19 (1971), 496—508.
6. Pilz G. — *A construction method for near rings*, Acta Math. Acad. Sci. Hungar, 24 (1973), 97—105.
7. Ștefănescu Mirela — *A generalization of the concept of near ring: Infra-near rings (to appear)*
8. Ștefănescu Mirela — *Multiplications infra-distributives sur un groupe* (in print at Publ. Math. Debrecen).
9. Wolfson K.G. — *Two-sided ideals of the affine near rings*, Amer. Math. Monthly, 65 (1958), 29—30.

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THE MINIMALITY OF WEYL'S SYSTEM OF AXIOMS FOR THE AFFINE GEOMETRY OVER AN UNITARY RING

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Introduction. Half century ago, Hermann Weyl in his treatise „Raum, Zeit, Materie”, [10] has formulated the notion of punctual affine space, based on the properties of linearity of the associate vector space. Weyl's axiomatic gives a straight way to introduce the punctual affine spaces. But Weyl's system is not sufficiently formalised and, more than this, it is not a minimal system [4].

P. K. Raschewsky, [9], has reformulated Weyl's system and pointed out a more clear axiomatic. He replaced „the triangle axiom” with „the parallelogram axiom”. Even Raschewski's construction is not built up from independent axioms, [4].

S. Golab [3] has made a very interesting study on the axiomatics of Weyl, Raschewsky etc.

In the present paper we solve the problem of minimality of Weyl's system. Namely, we weaken Weyl's axioms and drop the axiom $x + y = y + x$ for the punctual affine space on an apriori given unitary ring. This new system satisfies the minimality principle.

§1. The notion of punctual affine space over a ring. Let M be a set whose elements A, B, C, \dots are called points. Consider the set $M \times M$. Every element $(A, B) \in M \times M$ is called an oriented segment of M , with origin A and extremity B . We denote $(A, B) = \overrightarrow{AB}$.

Let ρ be an equivalence relation on the set $M \times M$, $TM = (M \times M) / \rho$ the quotient set and $p: M \times M \rightarrow TM$ the natural projection. We denote $p(\overrightarrow{AB}) = \overrightarrow{AB}$ and we say that \overrightarrow{AB} is the vector of the set M determined by the oriented segment \overrightarrow{AB} with respect to the equivalence ρ .

We shall denote vectors by $\overrightarrow{AB}, \overrightarrow{CD}, \dots, x, y, z$ etc.

Obviously: $\overrightarrow{AB} \rho \overrightarrow{CD} \Leftrightarrow \overrightarrow{AB} = \overrightarrow{CD}$.

Conversely, if the set TM of vectors is given together with a surjection $p: M \times M \rightarrow TM$, then the equivalence relation ρ is well determined by $\overrightarrow{AB} \rho \overrightarrow{CD} \Leftrightarrow p(\overrightarrow{AB}) = p(\overrightarrow{CD})$.

Let Γ be an unitary ring and $1 \in \Gamma$ its identity.

Definition 1. We call Γ -affine space structure on the nonempty set M the following data: