taking into account condition b) of theorem 1 :

$$v\left(\beta y_1, \frac{1}{\beta} y_2\right) = u_0 + \int_0^{y_1}\int_0^{y_2} g\left(\eta_1, \eta_2, v\left(\beta \eta_1, \frac{1}{\beta}\eta_2\right)\right) d\eta_1\, d\eta_2 .$$

But equation (13) has a unique solution (by given u_0) so that

$$v(y_1, y_2) = v\left(\beta y_1, \frac{1}{\beta} y_2\right)$$

i.e.

$$u\left(\alpha(x - vt), \quad \alpha\left(t - \frac{v}{c^2}x\right)\right) = u(x, y).$$

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POSITIVE SOLUTIONS OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

BY

N. PAVEL and M. TURINICI

Introduction. The existence of positive solutions of Cauchy Problem for a system of ordinary differential equations has a practical value. Such a problem occurs when studying some processes in physics, technique, biology a.s.o. Actually, we shall treat a problem more general than the problem of the existence of positive solutions. Namely, we give necessary and sufficient conditions such that each component $u_i(t)$ of the solution $u(t) = (u_1(t), \ldots, u_m(t))$, belongs to a given interval $[a_i, b_i]$, $i = 1, 2, \ldots, m$ (i.e. $a_i \leqslant u_i(t) \leqslant b_i$, $i = 1 \ldots, m$, for all t in the domain of u). Here a_i and b_i are real numbers, with $a_i < b_i$.

We prove that the above problems can be derived from a result of Crandall [3] on the existence of solution on a locally closed subset F of m - dimensional euclidian space R^m.

Let us consider the Cauchy problem:

$$u'(t) = f(t, u(t)), \qquad u(a) = x, \; x \in F, \tag{1}$$

where $f : [a, b) \times F \to R^m$ is a continuous function, $a < b \leqslant +\infty$.

We say that F is forward invariant for (1) (or flow invariant set) if whenever u is a solution of (1) on $[a, a + T]$, $T > 0$ (in classical sense) and $x \in F$, then $u(t) \in F$ for $a \leqslant t \leqslant a + T$. Note that F-locally closed, means that for each $x \in F$, there is $r > 0$, such that $F_r = F \cap B_r(x)$ is closed in R^m, where $B_r(x)$ denotes the closed ball of radius r and center $x \in R^m$. Denote by $d(y; F)$ the distance from the point $y \in R^m$ to F.

The basic result on the Cauchy problem (1) (generalizing Peano's existence theorem) is the following :

Theorem 1. (Crandall [3]). *Let $t : [a, b) \times F \to R^m$ be a continuous function. A necessary and sufficient condition for the existence of a local solution $u : [a, a + T(x)] \to F$ of the problem* (1) *is the following* :

$$\lim_{h \downarrow 0} \frac{1}{h} d(x + hf(t, x); F) = 0, \text{ for all } t \in [a, b) \text{ and } x \in F, \tag{2}$$

where $T = T(x) > 0$, depends on $x \in F$.

This theorem is given by Crandall [3] and is generalized by Martin [5], Pavel [6], and Ursescu [7].

Denote by $P = \prod_{i=1}^{m} [a_i, b_i]$ the cartesian product $[a_1, b_1] \times \ldots \times [a_m, b_m]$, and by $f_1, \ldots, f_m$, $u_1, \ldots, u_m$ the components of f (resp. u).

In section 1 we shall prove that from Theorem 1 one can derive

Theorem 2. *Let $f : [a, b) \times \prod_{i=1}^{m} [a_i, b_i] \to R^m$ be a continuous function. The necessary and sufficient conditions such that for each $x = (x_1, \ldots, x_m)$, with $a_i \leqslant x_i \leqslant b_i$, $i = 1, \ldots, m$, there exists a solution $u : [a, a + T(x)] \to R^m$ (with $a_i \leqslant u_i(t) \leqslant b_i$, $i = 1, \ldots, m$, $t \in [a, a + T(x)]$) of the problem*

$$\dot{u}_i(t) = f_i(t, u_1(t), \ldots, u_n(t)), \tag{3}$$

$$u_i(a) = x_i, \tag{4}$$

are the following

$$f_i(t, x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m) \geqslant 0, \tag{5}$$

$$f_i(t, x_1, \ldots, x_{i-1}, b_i, x_{i+1}, x_m) \leqslant 0, \quad i = 1, 2, \ldots, m. \tag{6}$$

for all $t \in [a, b)$ and $x_j \in [a_j, b_j]$, $j = 1, \ldots, i-1, i+1, \ldots, m$.

Several corollaries of Theorem 2 are also derived. Obviously, in the case $a_i = 0$, $i = 1, \ldots, m$ the Theorem 2 gives necessary and sufficient conditions for the existence of positive solutions of the system (3).

The results of L a k s h m i k a n t h a m [4] on positivity of solutions in abstract cones don't cover ours.

Thus we can say that the result given by Theorem 2 is an application of the problem of „flow-invariance" for (1). Other applications of flow-invariance are given by B o u r g u i g n o n and B r é z i s [1].

1. Theorem 1 is proved by C r a n d a l l in [3].

For the sake of self containment we give however a proof of this theorem. This proof is not entirely different from Crandall's proof, but is much more similar to the classical method of polygonal lines.

Proof of Theorem 1. The proof of necessity is very easy. Indeed, let (t, x) be arbitrary in $[a, b) \times F$. By hypothesis there is $T = T(x, t)$ and a continuously differentiable function $u : [t, t + T] \to F$ such that

$$u'(\tau) = f(\tau, u(\tau)), \quad \tau \in [t, t + T], \ u(t) = x. \tag{7}$$

Clearly, we have

$$\frac{1}{h} d(x + h f(t, x); F) \leqslant \frac{1}{h} \| x + hf(t, x) - u(t + h) \| = \tag{8}$$
$$= \left\| \frac{u(t + h) - u(t)}{h} - f(t, x) \right\|.$$

for all $h > 0$ small enough, which implies (2).

The *sufficiency*. Let $x \in F$, $r > 0$, $M \geqslant 1$, $T = T(x) > 0$ be such that

$$F_r = F \cap B_r(x) \text{ is closed}, \ TM \leqslant r,$$

$$\| f(t, y) \| \leqslant M - 1, \text{ for all } t \in [a, a + T], \ y \in F_r. \tag{9}$$

Let n be an arbitrary natural number. Define $t_0^n = t_0 = a$, $u_0^n = u_0 = x$ (when there is no danger of confusion we drop n). Assume that we have constructed u_n on $[a, t_i]$. If $t_i < a + T$, choose the largest number $d_i \in (0, 1/n]$ such that

$$t_{i+1} = t_i + d_i \leqslant a + T, \tag{10}$$

$$d(u_i + d_i f(t_i, u_i); F) \leqslant \frac{d_i}{2n}, \tag{11}$$

where $u_n(t_i) = u_i \in F$ by hypothesis.

Note that (11) is possible in view of (2). By (11) there is $u_{i+1} \in F$ such that $\| u_i + d_i f(t_i, u_i) - u_{i+1} \| \leqslant d_i/n$. Denoting

$$P_i = \frac{1}{d_i}(u_{i+1} - u_i - d_i f(t_i, u_i)), \tag{12}$$

we have

$$u_{i+1} = u_i + (t_{i+1} - t_i)[f(t_i, u_i) + P_i], \quad \| P_i \| \leqslant \frac{1}{n}. \tag{13}$$

For $i = 0$, (13) gives $u_1 = u_0 + (t_1 - t_0)[f(t_0, u_0) + P_0]$; therefore $\| u_1 - x \| \leqslant d_1\left(M - 1 + \frac{1}{n}\right) \leqslant TM \leqslant r$; so $u_1 \in F_r = F \cap B_r(x)$. Similarly one shows that $u_i \in F_r$, for each $i = 1, 2, \ldots$ A crucial point of the proof is to show that there is a natural number i_n such that $t_{i_n} = a + T$. Indeed, if we assume by contradiction that $t_i < a + T$, $i = 1, 2, \ldots$, then let us denote $t^* = \lim_{i \to \infty} t_i$. By (13) we have $\| u_{i+1} - u_i \| \leqslant M(t_{i+1} - t_i)$ therefore $\lim_{i \to \infty} u_i = u^*$ exists, too, and $u^* \in F_r$. Let $d^* \in (0, 1/n]$ small enough, such that

$$d(u^* + d^* f(t^*, u^*); F) \leqslant \frac{d^*}{4n}. \tag{14}$$

From (14) we see that for i large enough (i.e. $i \geqslant i_0$), we have

$$d(u_i + d^* f(t_i, u_i); F) \leqslant d(u^* + d^* f(t^*, u^*); F) + \frac{d^*}{4n} \leqslant \frac{d^*}{2n}. \tag{15}$$

Taking into account that d_i is the largest number in $(0, 1/n)$ satisfying (11), from (15) it follows that $d^* \leqslant d_i$, for all $i \geqslant i_0$ (which contradicts the fact that $\lim_{i \to \infty} d_i = \lim_{i \to \infty} (t_{i+1} - t_i) = 0$). Thus, we have proved that there is i_n, such that $t_i \geqslant a + T$. We may assume that $t_{i_n} = a + T$.

Define now the following polygonal line (approximate solution)

$$u_n(t) = u_i + (t - t_i)[f(t_i, u_i) + P_i], \quad t_i \leqslant t \leqslant t_{i+1}, \ i = 1, 2, \ldots, i_n. \tag{16}$$

It is easy to prove that

(17) $$u_n(t) \in B_r(x), \| u_n(t) - u_n(s) \| \leqslant M \mid t - s \mid, \quad u_n(a) = x,$$

for all $t, s \in [a, a + T]$.

The approximate solution u_n can be written under the form:

(18) $$u_n(t) = x + \sum_{j=0}^{i-1} (t_{j+1} - t_j) f(t_j, u_j) + (t - t_i) f(t_i, u_i) + \sum_{j=0}^{i-1} (t_{j+1} - t_j) P_j + (t - t_i) P_i, \quad t_i \leqslant t \leqslant t_{i+1}.$$

Define the step function a_n as follows

(19) $$a_n(s) = t_i, \text{ for } s \in [t_i, t_{i+1}), \ a_n(a + T) = a + T.$$

It follows that

$$u_n(a_n(s)) = u_n(t_i) = u_i, \quad \text{for all } s \in [t_i, t_{i+1}).$$

Set

(20) $$g_n(t) = \sum_{j=0}^{i-1} (t_{j+1} - t_j) P_j + (t - t_i) P_i, \qquad t_i \leqslant t \leqslant t_{i+1}.$$

It follows that u_n can be written under the following convenient form

(21) $$u_n(t) = x + \int_a^t f(a_n(s), u_n(a_n(s)))\, ds + g_n(t), \qquad t \in [a, a + T]$$

where $\| g_n(t) \| \leqslant T/n$ (by (20)).

In view of (17) we may assume (without loss of generality) that u_n is uniformly convergent on $[a, a + T]$ as $n \to \infty$, to a function u,

We may show that

(22) $$(a_n(s), u_n(a_n(s))) \to (s, u(s)) \text{ as } n \to \infty, \text{ uniformly on } [a, a + T].$$

Indeed, let s be arbitrary in $[a, a + T]$, and let i_n^* be the corresponding integer for which $s \in [t_{i_n^*}, t_{i_n^*+1}]$, where $t_{i_n^*} \in \{t_i\}_{i=1}^{i_n}$. Thus $s - t_{i_n^*} < t_{i_n^*+1} - t_{i_n^*} \leqslant 1/n$, $a_n(s) = t_{i_n^*}$, $u_n(a_n(s)) = u_{i_n^*}$.

Therefore

(23) $$\begin{aligned} | a_n(s) - s | &= | t_{i_n^*} - s | \leqslant \frac{1}{n}, \quad \| u_n(a_n(s)) - u(s) \| = \| u_n(t_{i_n^*}) - \\ &- u(s) \| \leqslant \| u_n(t_{i_n^*}) - u_n(s) \| + \| u_n(s) - u(s) \| \leqslant \\ &\leqslant \frac{M}{n} + \| u_n(s) - u(s) \|, \end{aligned}$$

which implies (22). Passing to the limit in (21) (for $n \to \infty$), we get

$$u(t) = x + \int_a^t f(s, u(s))\, ds, \quad t \in [a, a + T],$$

which shows that u is a solution of (1).

The fact that $u(t) \in F$ for each $t \in [a, a + T]$ follows from (23), i.e. from the inequality

(24) $$\| u_{i_n^*} - u(s) \| \leqslant \frac{M}{n} + \| u_n(s) - u(s) \|, \quad u_{i_n^*} \in F.$$

This last inequality implies

$$u(s) = \lim_{n \to \infty} u_{i_n^*} \in F.$$

This completes the proof of Theorem 1.

2. To prove the Theorem 2, we apply Theorem 1 with $F = \prod_{i=1}^{n} [a_i, b_i]$.

Assume by contradiction that (5) is not true, that is

(2.1) $$f_i(t, x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m) < 0.$$

Then it is easy to prove that

(2.2) $$\begin{aligned} &\frac{1}{h} d\, [\, (x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m) + h\, (f_1, \ldots, f_i \ldots, f_m)\,; F] = \\ &= | f_i(t, x_1, x_{i-1}, a_i, x_{i+1}, \ldots, x_m) |, \end{aligned}$$

for $x_j \in (a_j, b_j)$, $j = 1, \ldots, i - 1, i + 1, \ldots, m$, where $f_k = f_k(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n)$, $k = 1, 2, \ldots, m$. Thus the hypothesis (2.1), implies (2.2), which contradicts the condition (2) of Theorem 1. Similarly, we prove that if (6) is not true, then (2) is also not true. Therefore we have proved that (2) implies (5) + (6). Conversely, it is obvious that (5) + (6) implies (2), so (2) is equivalent (in the case $F = \prod_{i=1}^{m} [a_i, b_i]$) to (5) + (6). This last fact proves the theorem.

To obtain positive solution, we choose obviously $a_i = 0$, $i = 1, \ldots, m$ and $F = R_+^m = \underbrace{R_+ \times \ldots \times R_+}_{m \text{ times}}$ where $R_+ = [0, +\infty)$. From Theorem 2 it follows

Corollary 2.1. *Let $f : R \times R_+^m \to R^m$ be a continuous function. A necessary and sufficient condition, in order that for each $x_0 \in R_+^n$ the problem*

(P) $$\dot{x}_i(t) = f_i(t, x_1(t), \ldots, x_n(t)), \quad i = 1, 2, \ldots, m, \ x_i(a) = x_0^i, \quad x_0 = (x_0^1, \ldots, x_0^m) \in R_+^m$$

admits a positive solution (i.e.

$$x_i(t) \geqslant 0 \ \textit{for all} \ t \in [a, a + T(x_0)], \ i = 1, \ldots, m)$$

is the following

$$f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_m) \geq 0, \tag{2.3}$$

for all $t \in R$, $x_j \geq 0$, $j = 1, \dots, i-1, i+1, \dots, m$.

Remark 2.1. In the particular case of the linear system

$$\dot{x}(t) = A(t)\,x(t), \quad x(a) = x_0 \in R^m_+, \tag{2.4}$$

where $A(t)$ is a $m \times m$ matrix of continuous functions $a_{ij}(t)$, the condition (2.3) is equivalent with nonnegativity of all elements a_{ij} with $i \neq j$ (i.e.) (2.4) has positive solutions for all $x_0 \in R^m_+$, iff all elements a_{ij} which don't belong to the first diagonal, are nonnegative). This is a classical result.

Remark 2.2. The same result remains true if instead of (2.4) we consider a more general case

$$\left|\begin{array}{l} \dot{x}_i(t) = \sum_{j=1}^{m} a_{ij}\,\varphi_j(x_j), \\ x_i(a) = x_0^i, \ x_0^i \geq 0, \ i = 1, 2, \dots, m, \end{array}\right. \tag{2.5}$$

where φ_j are continuous functions from R_+ to R, with $\varphi_i(0) = 0$, $i = 1, \dots, m$.

In other words, (2.5) has positive solutions for each $x_0 = (x_0^i) \in R^n_+$, iff $a_{ij} \geq 0$, for all $i, j = 1, \dots, m$, $i \neq j$.

Remark 2.3. In the scalar case

$$\dot{x}(t) = f(t, x(t)) \qquad x(a) = x_0 \in R_+ \tag{2.6}$$

where $f: R_+ \times R_+ \to R$ is a continuous function, (2.6) has positive solutions for each $x_0 \in R_+$, iff

$$f(t, 0) \geq 0, \text{ for all } t \geq 0. \tag{2.7}$$

This condition (2.7) follows from (2.3).

Remark 2.4. We give a direct proof of the sufficiency of the conditions (2.3), interesting also by itself.

Define the function $g: R^{m+1} \to R^{m+1}_+$, by

$$g(t, x_1, \dots, x_m) = (\max(t, 0), \max(x_1, 0), \dots, \max(x_m, 0)) \tag{2.8}$$

This function g is continuous and

$$g(t, x) = (t, x) \text{ for all } (t, x) \in R^{m+1}_+ \tag{2.9}$$

where $x = (x_1, \dots, x_m)$.

Define now the function F, by

$$F(t, x) = f(g(t, x)), \quad \forall (t, x) \in R^{m+1}. \tag{2.10}$$

The function F is continuous on R^{m+1}, too, and in addition

$$F(t, x) = f(t, x), \text{ for all } (t, x) \in R^{m+1}_+. \tag{2.11}$$

Let us consider now the Cauchy problem

$$\dot{x}(t) = F(t, x(t)), \quad x(t_0) = x_0 \in R^m_+, \quad x_0 = (x_0^1, \dots, x_0^m). \tag{2.12}$$

By the classical Peano's existence theorem the problem (2.12) admits a solution $x = x(t, t_0, x_0)$ defined on an interval $I = [t_0, t_0 + T]$ with $T = T(t_0, x_0)$.

Next we prove that $x(t, t_0, x_0) \in R^m_+$, for all $t \in I$. Indeed if we assume that there is $i \in \{1, \dots, m\}$ and $t_1 \in [t_0, t_0 + T]$, such that $x_i(t_1, t_0, x_0) < 0$ then we get a contradiction, as follows:

First of all $t_0 < t_1$, since $x_i(t_0, t_0, x_0) \geq 0$. Let t_2 be the number in $[t_0, t_1)$, with the property

$$x_i(t_2, t_0, x_0) = 0, \quad x_i(t, t_0, x_0) < 0, \quad \text{for all } t \in (t_2, t_1). \tag{2.13}$$

By the Lagrange formula, there is $\tau \in (t_2, t_1)$ such that

$$x_i(t_1, t_0, x_0) - x_i(t_2, t_0, x_0) = (t_1 - t_2)\,\dot{x}_i(\tau, t_0, x_0), \tag{2.14}$$

which implies (by (2.12))

$$\begin{array}{l} x_i(t_1, t_0, x_0) = (t_1 - t_2)\,F_i(\tau, x_1, \dots, x_i(\tau, t_0, x_0), \dots, x_m), \\ \qquad \text{with } x_j = x_j(\tau, t_0, x_0),\ j = 1, 2, \dots, m. \end{array} \tag{2.15}$$

In view of (2.13), $g(t, x_1, \dots, x_i(\tau, t_0, x_0), \dots, x_m) =$
$= (\max(t, 0), \dots, \max(x_{i-1}, 0), 0, \max(x_{i+1}, 0), \dots, \max(x_m, 0)) =$
$= (t, y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_m)$

with

$$t = \max(t, 0), \ y_j = \max(x_j(\tau, t_0, x_0), 0), \ j = 1, \dots, i-1, \ i+1, \dots, m.$$

Thus, (2.15) can be written as

$$x_i(t_1, t_0, x_0) = (t_1 - t_2)\,f_i(t, y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_m) \tag{2.16}$$

since $y_j \geq 0$.

The formula (2.16) is a contradiction, since the left hand side is (strictly) negative, while the right hand side is nonnegative (by (2.3)). Thus we have proved that (2.3) implies the existence of positive solutions for the problem (P) in Corollary 2.1.

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OSCILLATIONS CAUSED BY DELAYS*

BY

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Consider the first order (not necessarily linear) delay differential equation $x'(t)+p(t)\,\Phi[x(t-\tau)]=0,\ t\geqq t_0$, where p is a nonnegative continuous function on the interval $[t_0, \infty)$, Φ is a continuous function which is defined at least on $\mathbf{R}-\{0\}$ and has the sign property $z\neq 0 \Rightarrow z\Phi(z)>0$ and τ is a positive constant. For this equation we give a sufficient condition for all solutions to be oscillatory. The result obtained extends a recent one by Ladas [4] concerning the case of linearity $\Phi(z)=z$. Consider also the n-th order $(n>1)$ delay differential equation $x^{(n)}(t)+(-1)^{n+1}p(t)\,\Phi[x(t-\tau)]=0,\ t\geqq t_0$. We establish a condition under which all bounded solutions of this equation are oscillatory.

For retarded differential equations there is recently an increasing interest in studying oscillations which are caused by the retardations and which do not appear in their corresponding ordinary differential equations. We choose to refer here the papers by Gustafson [1], Koplatadze [2], Ladas [3, 4], Ladas, Ladde and Papadakis [5], Ladas and Lakshmikantham [6], Ladas, Lakshmikantham and Papadakis [7], Naito [9], Sficas and Staikos [10, 11], Shreve [12] and Staikos and Stavroulakis [13]. The study of oscillations generated by retardations is very interesting in the applications. For example, such oscillations should be seriously taken into account in studying the motion of a controlled craft moving with increasing velocities, where it is possible to have a sudden release of oscillations leading to instability (cf. Minorsky [8, p. 518]).

In a recent paper by Ladas [4] the first order linear delay differential equation

$$y'(t)+P(t)y(t-T)=0,\quad t\geqq T_0 \tag{d}$$

is considered, where P is a nonnegative continuous function on the interval $[T_0, \infty)$ and T is a positive constant. The following theorem has been obtained:

Theorem 0. *The condition*

$$\liminf_{t\to\infty}\int_{t-T}^{t}P(s)ds>\frac{1}{e} \tag{H}$$

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