4. Lakshmikantham, V., Richard A. Mitchell and Dolores D. Sety — *The method of quasilinearization and positivity of solutions in abstract cones*, Technical Report No. 40 (1976), The Univ. of Texas at Arlington, Dept. of Math. Texas 76 019.
5. Martin, R. Jr. — *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc. 179 (1973), 399—414.
6. Pavel N. — *Invariant sets for a class of semi-linear equations of evolution*, JNA, TMA, 1 (1977), 187—196.
7. Pavel, N. and C. Ursescu — *Flow-invariance for higher order differential equations* (to appear).

Faculty of Mathematics
University of Iaşi
R.S. România

Received 26. VII. 1977

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXIV, S. I a, f. 1, 1978

OSCILLATIONS CAUSED BY DELAYS*

BY

CH. G. PHILOS

Consider the first order (not necessarily linear) delay differential equation $x'(t)+p(t)\,\Phi[x(t-\tau)]=0$, $t \geqq t_0$, where p is a nonnegative continuous function on the interval $[t_0, \infty)$, Φ is a continuous function which is defined at least on $\mathbf{R}-\{0\}$ and has the sign property $z \neq 0 \Rightarrow z\Phi(z) > 0$ and τ is a positive constant. For this equation we give a sufficient condition for all solutions to be oscillatory. The result obtained extends a recent one by Ladas [4] concerning the case of linearity $\Phi(z) = z$. Consider also the n-th order $(n>1)$ delay differential equation $x^{(n)}(t)+(-1)^{n+1}p(t)\,\Phi[x(t-\tau)] = 0$, $t \geqq t_0$. We establish a condition under which all bounded solutions of this equation are oscillatory.

For retarded differential equations there is recently an increasing interest in studying oscillations which are caused by the retardations and which do not appear in their corresponding ordinary differential equations. We choose to refer here the papers by Gustafson [1], Koplatadze [2], Ladas [3, 4], Ladas, Ladde and Papadakis [5], Ladas and Lakshmikantham [6], Ladas, Lakshmikantham and Papadakis [7], Naito [9], Sficas and Staikos [10, 11], Shreve [12] and Staikos and Stavroulakis [13]. The study of oscillations generated by retardations is very interesting in the applications. For example, such oscillations should be seriously taken into account in studying the motion of a controlled craft moving with increasing velocities, where it is possible to have a sudden release of oscillations leading to instability (cf. Minorsky [8, p. 518]).

In a recent paper by Ladas [4] the first order linear delay differential equation

$$y'(t) + P(t)y(t-T) = 0, \quad t \geqq T_0 \tag{d}$$

is considered, where P is a nonnegative continuous function on the interval $[T_0, \infty)$ and T is a positive constant. The following theorem has been obtained:

Theorem 0. *The condition*

$$\liminf_{t\to\infty} \int_{t-T}^{t} P(s)ds > \frac{1}{e} \tag{H}$$

*) This work was supported by The National Hellenic Research Foundation.

is sufficient for all solutions of the differential equation (d) *to be oscillatory.* In the same paper it is in addition established that, in particular, the differential equation

$$(\mathrm{d}_0) \qquad y'(t) + P_0 y(t-T) = 0, \quad t \geqq T_0,$$

where P_0 is a positive constant, has at least one nonoscillatory solution, if $P_0 Te \leqq 1$. That is, the condition

$$(\mathrm{H}_0) \qquad P_0 Te > 1$$

is sufficient and also necessary for all solutions of the equation (d_0) to be oscillatory. It remains an open question if the condition (H) is also necessary in order that all solutions of the differential equation (d) be oscillatory.

It is the purpose of the present paper to extend Theorem 0 for the first order (not necessarily linear) delay differential equation

$$(\mathrm{D}) \qquad x'(t) + p(t)\,\Phi[x(t-\tau)] = 0, \quad t \geqq t_0$$

and for the n-th order ($n > 1$) delay differential equation

$$(\mathrm{E}) \qquad x^{(n)}(t) + (-1)^{n+1} p(t)\,\Phi[x(t-\tau)] = 0, \quad t \geqq t_0,$$

where p is a nonnegative continuous function on the interval $[t_0, \infty)$, Φ is a continuous function which is defined at least on $\mathbf{R} - \{0\}$ and has the sign property

$$z \neq 0 \Rightarrow z\Phi(z) > 0$$

and τ is a positive constant. Our results, which are presented below as Theorems 1 and 2, are obtained by using Theorem 0.

Throughout the paper we consider only such solutions $x(t)$ of (D) or (E) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[a, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called nonoscillatory.

Theorem 1. *If*

$$(\mathrm{c}) \qquad \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e} \max\left\{\limsup_{z\to 0+} \frac{z}{\Phi(z)},\ \limsup_{z\to 0-} \frac{z}{\Phi(z)}\right\},$$

then all solutions of the differential equation (D) *are oscillatory.*

Proof. Let x be a nonoscillatory solution on an interval $[t_x, \infty)$, $t_x \geqq t_0$, of the equation (D). We suppose, without loss of generality, that $x(t) \neq 0$ for every $t \geqq t_x$. Furthermore, since the substitution $w = -x$ transforms (D) into an equation of the same form satisfying the assumptions posed for (D), we assume that x is positive. Then it is obvious that the restriction of x on the interval $[T_0, \infty)$, $T_0 = t_x + \tau$, is a (nonoscillatory) solution on this interval of the (linear) differential equation (d) with

$$P(t) = p(t)\frac{\Phi[x(t-\tau)]}{x(t-\tau)}, \quad t \geqq T_0$$

and $T = \tau$. Thus, by Theorem 0, we must have

$$(1) \qquad \liminf_{t\to\infty} \int_{t-\tau}^{t} P(s)ds \leqq \frac{1}{e}.$$

Consequently, for every $t \geqq T_0 + \tau$ we obtain

$$\int_{t-\tau}^{t} p(s)ds = \int_{t-\tau}^{t} P(s)\frac{x(s-\tau)}{\Phi[x(s-\tau)]}ds \leqq \left[\sup_{t-\tau \leqq s \leqq t} \frac{x(s-\tau)}{\Phi[x(s-\tau)]}\right] \int_{t-\tau}^{t} P(s)ds.$$

But, from equation (D) it follows that x is decreasing on $[T_0, \infty)$ and hence

$$(2) \qquad \int_{t-\tau}^{t} p(s)ds \leqq \left[\sup_{0 < z \leqq x(t-2\tau)} \frac{z}{\Phi(z)}\right] \int_{t-\tau}^{t} P(s)ds$$

for all $t \geqq T_0 + 2\tau$. Furthermore, we observe that

$$(3) \qquad \lim_{t\to\infty} x(t) = 0.$$

Indeed, in the opposite case there exists a positive constant K such that

$$\Phi[x(t-\tau)] \geqq K \text{ for every } t \geqq T_0$$

and consequently from equation (D) for every $t \geqq T_0 + \tau$ we get

$$x(t-\tau) - x(t) = \int_{t-\tau}^{t} p(s)\,\Phi[x(s-\tau)]ds \geqq K \int_{t-\tau}^{t} p(s)ds.$$

So, since $\lim_{t\to\infty} x(t)$ is finite,

$$\lim_{t\to\infty} \int_{t-\tau}^{t} p(s)ds = 0.$$

This contradicts (c) and hence (3) holds true. Now, in view of (1) and (3), (2) gives

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds \leqq \frac{1}{e} \limsup_{z\to 0+} \frac{z}{\Phi(z)},$$

which contradicts the condition (c).

Remark 1. It is known (cf. Shreve [12]) that

$$(\mathrm{c}^*) \qquad \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \max\left\{\limsup_{z\to 0+} \frac{z}{\Phi(z)},\ \limsup_{z\to 0-} \frac{z}{\Phi(z)}\right\}$$

is also a sufficient condition for all solutions of the differential equation (D) to be oscillatory. We must note here that the conditions (c) and (c*) are independent. That is, it is possible to have one of the conditions (c) and (c*) valid while the other fails. For example, in the case of the delay differential equation

$$x'(t) + (1 + 1/t)\, x(t - 1/2) = 0, \quad t \geqq 1$$

the condition (c) is satisfied while (c*) fails; for the differential equation

$$x'(t) + (1 + \cos t)\, x(t - \pi/3) = 0, \quad t \geqq 0$$

the condition (c*) holds while (c) is not valid.

Now, in order to obtain our second theorem we need the following lemma, which is an adaptation of a lemma due to S f i c a s and S t a i k o s [11].

Lemma. *Let h be a positive bounded and n-times differentiable function on an interval $[t_h, \infty)$ such that*

$$(-1)^n h^{(n)}(t) \geqq 0 \text{ for every } t \geqq t_h$$

and $h^{(n)}$ is not identically zero on any interval of the form $[\tau_h, \infty)$, $\tau_h \geqq t_h$. Then

$$(-1)^k h^{(k)}(t) > 0 \text{ for every } t \geqq t_h \quad (k = 1, 2, \ldots, n-1)$$

and for any u and v with $t_h \leqq u \leqq v$

$$h(u) \geqq \frac{(-1)^{n-1} h^{(n-1)}(v)}{(n-1)!} (v-u)^{n-1}.$$

Theorem 2. *If*

$$\text{(C)} \qquad \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{2^n (n-1)!}{e\tau^{n-1}} \max\left\{\limsup_{z\to 0+} \frac{z}{\Phi(z)},\ \limsup_{z\to 0-} \frac{z}{\Phi(z)}\right\},$$

then all bounded solutions of the differential equation (E) are oscillatory.

Proof. Let x be a bounded nonoscillatory solution on an interval $[t_x, \infty)$, $t_x \geqq t_0$, of the equation (E). We assume, without loss of generality that x is positive on the whole interval $[t_x, \infty)$. Then from equation (E) it follows that $(-1)^n x^{(n)} \geqq 0$ on $[t_x + \tau, \infty)$. Moreover, $x^{(n)}$ is not identically zero on any interval of the form $[\tau_x, \infty)$ $\tau_x \geqq t_x + \tau$, since, by condition (C) the same holds for the function p. Thus, the lemma ensures that

$$\text{(4)} \qquad (-1)^k x^{(k)}(t) > 0 \text{ for every } t \geqq t_x + \tau \ (k = 1, 2, \ldots, n-1).$$

Next, we observe that the restriction of the function $(-1)^{n-1} x^{(n-1)}$ on the interval $[T_0, \infty)$, $T_0 = t_x + (3/2)\tau$, is a (nonoscillatory) solution on this interval of the (first order linear) differential equation (d) with

$$P(t) = p(t) \frac{\Phi[x(t-\tau)]}{(-1)^{n-1} x^{(n-1)}\left(t - \frac{\tau}{2}\right)}, \quad t \geqq T_0$$

and $T = \frac{\tau}{2}$. Hence, by Theorem 0, we must have $\liminf_{t\to\infty} \int_{t-\frac{\tau}{2}}^{t} P(s)\,ds \leqq \frac{1}{e}$, i.e.

$$\text{(5)} \qquad \liminf_{t\to\infty} \int_{t-\tau}^{t} P(s)\,ds \leqq \frac{2}{e}.$$

Consequently, the lemma gives

$$x(t-\tau) \geqq \frac{\tau^{n-1}}{2^{n-1}(n-1)!} (-1)^{n-1} x^{(n-1)}\left(t - \frac{\tau}{2}\right), \quad t \geqq T_0 + \frac{\tau}{2}$$

and so

$$p(t) \leqq \frac{2^{n-1}(n-1)!}{\tau^{n-1}} P(t) \frac{x(t-\tau)}{\Phi[x(t-\tau)]}$$

for all $t \geqq T_0 + \frac{\tau}{2}$. But, from (4) it follows that x is decreasing on $[t_x + \tau, \infty)$ and hence for every $t \geqq T_0 + \frac{3}{2}\tau$, we obtain

$$\text{(6)} \qquad \int_{t-\tau}^{t} p(s)\,ds \leqq \frac{2^{n-1}(n-1)!}{\tau^{n-1}} \left[\sup_{0 < z \leqq x(t-2\tau)} \frac{z}{\Phi(z)}\right] \int_{t-\tau}^{t} P(s)\,ds.$$

Furthermore, (3) holds. Indeed, in the opposite case we have

$$\Phi[x(t-\tau)] \geqq K \text{ for every } t \geqq t_x + \tau,$$

where K is a positive constant, and consequently equation (E) gives

$$(-1)^n [x^{(n-1)}(t) - x^{(n-1)}(t-\tau)] = \int_{t-\tau}^{t} p(s)\Phi[x(s-\tau)]\,ds \geqq K \int_{t-\tau}^{t} p(s)\,ds, \quad t \geqq t_x + 2\tau.$$

This, since $\lim_{t\to\infty} x^{(n-1)}(t)$ is finite, yields

$$\lim_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds = 0,$$

which contradicts (C). Finally, by (3) and (5), from (6) it follows that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds \leqq \frac{2^n(n-1)!}{\tau^{n-1}} \limsup_{z\to 0+} \frac{z}{\Phi(z)},$$

which contradicts the condition (C).

Remark 2. It is known (cf. Naito [9], Sficas and Staikos [11] and Staikos and Stavroulakis [13]) that each of the conditions

$$(C^*) \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} (s-t+\tau)^{n-1} p(s)ds > (n-1)! \max\left\{\limsup_{z\to 0+} \frac{z}{\Phi(z)}, \limsup_{z\to 0-} \frac{z}{\Phi(z)}\right\},$$

$$(C^{**}) \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} (t-s)^{n-1} p(s)ds > (n-1)! \max\left\{\limsup_{z\to 0} \frac{z}{\Phi(z)}, \limsup_{z\to 0} \frac{z}{\Phi(z)}\right\}$$

is also sufficient for all bounded solutions of the differential equation (E) to be oscillatory. We must note here that it is possible to have the condition (C) valid while (C*) and (C**) fail. For example, in the case of the delay differential equation

$$x''(t) - (1/8)\, x(t-4) = 0, \; t \geq 0$$

the condition (C) holds while (C*) and (C**) are not satisfied.

Acknowledgement. The author would like to thank Professors Y.G. Sficas and V.A. Staikos for their helpful suggestions concerning this paper.

REFERENCES

1. Gustafson G.B. — *Bounded oscillations of linear and nonlinear delay-differential equations of even order*, J. Math. Anal. Appl., 46 (1974), 175—189.
2. Koplatadze R.G. — *Remarks on the oscillation of solutions of higher order differential inequalities and equations with retarded argument* (Russian), Differential'nye Uravnenija, 10 (1974), 1 400—1 405.
3. Ladas G. — *Oscillatory effects of retarded actions*, J. Math. Anal. Appl., (to appear).
4. Ladas G *Sharp conditions for oscillations caused by delays*, University of Rhode Island Technical Report No. 64, September 1976.
5. Ladas G., Ladde G., Papadakis J.S. — *Oscillations of functional differential equations generated by delays*, J. Differential Equations, 12 (1972), 385—395.
6. Ladas G. and Lakshmikantham V. — *Oscillations caused by retarded actions*, Applicable Anal., 4 (1974), 9—15.
7. Ladas G., Lakshmikantham V. and Papadakis J.S. — *Oscillations of higher-order retarded differential equations generated by the retarded argument*, Delay and Functional Differential Equations and their Applications, Academic Press New York, 1972, pp. 219—231.
8. Minorsky N. — *Nonlinear Oscillations*, D. Van Nostrand Co., Inc., Princeton, N.J. 1962.
9. Naito M. — *Oscillations of differential inequalities with retarded arguments*, Hiroshima Math. J., 5 (1975), 187—192.
10. Sficas Y.G. and Staikos V.A. — *The effect of retarded actions on nonlinear oscillations*, Proc. Amer. Math. Soc., 46 (1974), 259—263.
11. — *Oscillations of differential equations with deviating arguments*, Funkcial. Ekvac. 19 (1976), 35—43.
12. Shreve W.E. — *Oscillation in first order nonlinear retarded argument differential equations*, Proc. Amer. Math. Soc., 41 (1973), 565—568.
13. Staikos V.A. and Stavroulakis I.P. — *Bounded oscillations under the effect of retardations for differential equations of arbitrary order*. Proc. Roy. Soc. Edinburgh Sect. A, 77 (1677), 129—136.

Received 13. VII. 1977

Department of Mathematics
University of Ioannina
Ioannina, Greece

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXIV, S. I a, f. 1, 1978

ON THE BEHAVIOR OF SOLUTIONS OF A CERTAIN CLASS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

BY

B.G. PACHPATTE

1. Introduction. In 1961, V. M. Alekseev [1] developed a variation of constants formula which has led to the development of many interesting results on the behavior of solutions of perturbed nonlinear differential equations. In [3], F. Brauer has developed a formula analogous to the Alekseev formula for solutions of perturbed nonlinear integrodifferential and integral equations. Recently, S. R. Bernfeld and M. E. Lord [2] obtained another variation of constants formula for perturbed nonlinear integrodifferential and integral equations by assuming the existence of the inverse of the fundamental matrix corresponding to the unperturbed systems and applied it to some important problems in the theory of integral equations. In this paper we wish to study the behavioral relationships between the solutions of the integrodifferential system

$$(1) \qquad x'(t) = f\left(t, x(t), \int_{t_0}^{t} K(t, s, x(s))\, ds\right), \qquad x(t_0) = x_0,$$

and its perturbed system

$$(2) \quad y'(t) = f\left(t, y(t), \int_{t_0}^{t} K(t, s, y(s))\, ds\right) + g\left(t, y(t), \int_{t_0}^{t} h(t, s, y(s))\, ds,\right) y(t_0) = x_0,$$

by using the variant of the variation of constants formula developed by Bernfeld and Lord in [2] and the integral inequality recently established by the present author in [9]. In equations (1) and (2), x, y, k, h, f, g, and x_0 are the elements of R^n, an n-dimensional Euclidean space. Let I be the interval $0 \leqslant t < \infty$ and $C[X, Y]$ denote the space of continuous functions from X to Y where X and Y are convenient spaces. We shall assume that $K, h \in C[I \times I \times R^n, R^n]$ and $f, g \in C[I \times R^n \times R^n, R^n]$. We use $x(t) = x(t, t_0, x_0)$ to denote the solution of (1) passing through the point (t_0, x_0) and $y(t) = y(t, t_0, x_0)$ to denote a solution of (2) passing through the point (t_0, x_0) for $t_0 > 0$. The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix norm.