which implies $v = 0$, that is $\| u(t) \| = 1$ for all t on the domain of u.

Remark 3.1. If either $n = 1$ or X is finite dimensional, then Theorem 2.1 (resp. Theorem 2.2) follows from Theorem 1.1 without any additional argument.

Remark 3.2. In Theorem 2.1 we can take instead of M any locally closed subset M_1 of it. In the case $f : [a, b) \times R_+ \to R_+$ then for each x, $y \in R$, then problem (1.1) + (1.2) has positive solution. Indeed in this case the condition of Theorem 2.1 are clearly satisfied with $D = R_+ = [0, +\infty)$, and $M_1 = \{[x, y], x \geq 0, y \geq 0\}$. A similar result is true for the problem (2.31) + (2.32).

Remark 3.3. The continuation of the local solution given by Theorem 2.1 (resp. Theorem 2.2) to the whole $[a, b)$ requires classical restrictions on f, so we don't give details in this way.

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ON A GENERAL RANDOM VOLTERRA INTEGRAL EQUATION

BY

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1. Introduction. Consider the (one-dimensional) general random non-linear (i.e. not necessarily linear) Volterra integral equation

$$x(t;\omega) = f(t;\omega) + \int_0^t h(t, s, x(s;\omega))\, ds, \quad t \geq 0. \tag{1.1}$$

Tsokos and Padgett [7] have discussed the existence and uniqueness of solutions of (1.1) in the case in which the integrand is given by

$$h(t, s, x(s;\omega)) = k(t, s;\omega)\, g(s, x(s;\omega)), \tag{1.2}$$

considering the equation on a non-compact interval of the real line. But a fundamental feature of (1.1) is that the notion of a solution is a local concept, and we shall accordingly restrict our attention in the main to compact intervals.

Furthermore, the assumptions of Tsokos and Padgett [7] result in the consideration of a situation in which T, the operator defined by

$$Tx(t;\omega) = h(t;\omega) + \int_0^t k(t, \tau;\omega)\, f(\tau, x(\tau;\omega))\, d\tau$$

is deterministic, although Bharucha-Reid, in his exposition [1] of their results, certainly regards T as a random operator. In this paper we shall exploit more fully the random nature of the operators involved, relying on a random Schauder fixed point theorem due to Mukherjea [5]. We generalize also some results of Miller [4] and Strauss [6], chiefly by the introduction of a random kernel.

After some preliminary definitions and results, presented in Section 2, we proceed, in Section 3, to a discussion of the existence, uniqueness and continuability of the solution of (1.1). Section 4 is devoted to the study of certain operator methods applied to an abstract version of (1.1). In Section 5 we compare the solution of (1.1), where h is given by $h(t, s, x(s;\omega)) = k(t, s;\omega) . [x(s;\omega) + g(s, x(s;\omega))]$, with the solution of the unperturbed random linear equation

$$(1.3)\qquad y(t;\omega)=f(t;\omega)+\int_0^t k(t,s;\omega)\,y(s;\omega)\,ds,\quad t\geqslant 0.$$

We shall show that, for suitably defined $k(t,s;\omega)$ and $|g(t,x)|\to 0$ in some sense, as $t\to\infty$ we have $\|x(t;\omega)-y(t;\omega)\|_2\to 0$ as $t\to\infty$.

It might be mentioned that, while Tsokos and Padgett [7] have con sidered certain asymptotic problems in connection with (1.1), with inte grand given by (1.2), they have not used the random kernel approach adopted here.

2. Preliminaries. In order to establish the setting for our study o (1.1), we shall present, in this Section, certain pertinent definitions an results.

Let $(\Omega,\mathcal{A},P)$ denote a complete, atomic probability space, in whic P has a countable basis (i.e. there exists a sequence $\{E_n\}$ of sets in $\mathcal{A}$ such that, for any $\varepsilon>0$ and any $A\in\mathcal{A}$, there is a set E_k of the sequenc for which $P(A\Delta E_k)<\varepsilon$). The mapping $x:R^+\times\Omega\to\mathbf{R}$ (where $R^+=[0,\infty)$ and $\mathbf{R}=(-\infty,\infty)$) will be called a random process or rando function, such a process being *regular* (or *of second order*) if, for each $t\in R^+$

$$E\,|x(t;\omega)|^2\doteq\int_\Omega|x(t;\omega)|^2\,dP(\omega)<\infty.$$

(Here, as throughout this paper, $\doteq$ denotes equality by definition $L_2\doteq L_2(\Omega,\mathcal{A},P)$ will denote the separable Banach space of all (equi valence classes of) regular random processes, with norm

$$\|x(t;\omega)\|_2\doteq\left\{\int_\Omega|x(t;\omega)|^2\,dP(\omega)\right\}^{1/2}.$$

A real-valued random variable $x(\cdot)$ defined on $(\Omega,\mathcal{A},P)$ is said to b *P-essentially bounded* if

$$(\exists\alpha>0)\;P(\{\omega:|x(\omega)|>\alpha\})=0.$$

The space of all such functions will be denoted by $L_\infty\doteq L_\infty(\Omega,\mathcal{A},P$ the norm in this space being given by

$$\|x(\omega)\|_\infty\doteq\inf_{\Omega_0}\sup_{\Omega-\Omega_0}|x(\omega)|,$$

where $P(\Omega_0)=0$.

Let $C(R^+,L_2)$ denote the space of all continuous, bounded function on R^+ with values in L_2 — that is, the space of all bounded, mean squa continuous, regular random processes on R^+. The norm in this space i defined by

$$\|x(t,\omega)\|_c\doteq\sup_{t\geqslant 0}\|x(t;\omega)\|_2<\infty.$$

Notice that we shall not make any notational distinction between a *set* Z and the *space* $(Z,\|\cdot\|_Z)$: it will always be clear from the context which is meant.

Let $C_r(R^+,L_2)$ denote the space of all continuous functions on R^+ with values in L_2. The topology on this space is defined by the pseudo-distances $d_\alpha(x,y)=\|x-y\|_\alpha$, where $\{\|\cdot\|_\alpha:\alpha\in I\}$ is any family of semi-norms. In particular, we shall consider

$$\|x(t;\omega)\|_n=\sup_{0<t<n}\left\{\int_\Omega|x(t;\omega)|^2\,dP(\omega)\right\}^{1/2},\quad n=1,2,3,\ldots$$

With this topology (which is compatible with the underlying vector space structure — cf. Dieudonné [2], § XII.14), $C_r(R^+,L_2)$ becomes a separable Fréchet space. Hence $C(R^+,L_2)$, being a subspace of $C_r(R^+,L_2)$, is a separable Banach space.

Definition 2.1. *The random process $x(t;\omega)$ is a random solution of (1.1) if, for every fixed $t\in R^+$, $x(t;\cdot)\in L_2$ and $x(t;\omega)$ satisfies (1.1) P-almost surely.*

Definition 2.2. *A Banach space X is said to be stronger than a Banach space Y if every norm-convergent sequence in X converges in Y.*

Definition 2.3. *If X and Y are Banach spaces which are subspaces of a Fréchet space $\mathcal{F}$, and if $T:\Omega\times\mathcal{F}\to\mathcal{F}$ is linear, the pair (X,Y) is said to be admissible with respect to T if $T(\omega)X\subset Y$ P-almost surely.*

Definition 2.4. *Let E, F be two normed linear spaces. A linear random operator $T:\Omega\times E\to F$ is compact if for any bounded sequence $\{x_n\}$ in E, there is a subsequence $\{x_{n_k}\}$ such that the sequence $\{T(\omega)x_{n_k}\}$ is almost surely convergent in F.*

The following random analogue of the Schauder fixed point theorem is due to Mukherjea [5]: the statement given here is from Bharucha-Reid [1].

Theorem 2.1. *Let $(\Omega,\mathcal{A},P)$ be a complete, atomic probability measure space, and let E be a closed, bounded and convex subset of a separable Banach space X. Let $T:\Omega\times E\to E$ be a random compact operator. Then there exists an E-valued random variable $x(\cdot)$ such that $T(\omega)x(\omega)=x(\omega)$ almost surely.*

Remark: This result remains true if X is a separable Fréchet space.

3. The general random non-linear Volterra integral equation. Consider the equation

$$(3.1)\qquad x(t;\omega)=f(t;\omega)+\int_0^t h(t,s,x(s;\omega))\,ds,\quad t\geqslant 0$$

under the following four hypotheses

H3.1. $f\in C(R^+,L_2)$,

H3.2. h satisfies the following conditions:

(i) for $0 \leqslant s \leqslant t < \infty$ and $x(s;) \cdot \in L_2$, $h(t, s, x(s; .)) \in L_2$,

(ii) $h(t, s, x(s; \omega)) = 0$ for $s > t$,

(iii) $\int_J \int_J \| h(t, s, x(s : \omega)) \|_2 \, ds \, dt$ exists and is finite

for any compact interval $J \subset R^+$.

H3.3. For each real $T > 0$ and every bounded subset B of L_2 there exists a measurable function $m(.,.)$ such that

$$\| h(t, s, x(s; \omega)) \|_2 \leqslant m(t, s),$$

for $0 \leqslant s \leqslant t < \infty$ and $x(s; \cdot) \in B$, and $\sup_{0 \leq t \leq T} \int_0^t m(t, s) \, ds < \infty$.

H3.4. For each compact interval $J \subset R^+$, each bounded $B \subset L_2$ and each $t_0 \in R^+$,

$$\sup \left\{ \int_J \| h(t, s, x(s; \omega)) - h(t_0, s, x(s; \omega)) \|_2 \, ds : x(s; \omega) \in C(J, B) \right\}$$

tends to zero as $t \to t_0$. (Here $C(J, B)$ denotes the set of all continuous functions on J with values in B).

Remark: the integral in (3.1), like that of all functions in L_2, should be interpreted in the mean square sense.

Theorem 3.1. *Under the above hypotheses, there exist* $\beta > 0$ *and a random function* $x \in C([0, \beta], L_2)$ *which is a random solution of equation* (3.1).

Proof. Let B be the bounded subset of L_2 defined by

$$B = \{ x(t; .) \in L_2 : \| x(t; \omega) - f(t; \omega) \|_2 \leqslant 1, \text{ for some } t \in [0, 1] \}.$$

For $0 \leqslant t \leqslant T \leqslant 1$, it follows from H3.2 (ii) and H3.4 that

$$R(T) \doteq \sup \left\{ \int_0^t \| h(t, s, x(s; \omega)) \|_2 \, ds : x \in C([0, 1], B) \text{ and } t \in [0, T] \right\}$$

tends to zero as $T \to 0^+$. Thus we may choose $\beta \in [0, 1]$ so small that $R(T) < 1$ for $T \in [0, \beta]$.

Let $C(\beta)$ denote the Banach space $C([0, \beta], L_2)$ with norm

$$\| x(t; \omega) \|_{C(\beta)} \doteq \sup \{ \| x(t; \omega) \|_2 : t \in [0, \beta] \},$$

and define $C_0(\beta) \subset C([0, \beta], B)$ by

$$C_0(\beta) \doteq \{ x \in C(\beta) : \| x(t; \omega) - f(t; \omega) \|_2 \leqslant 1 \text{ for } t \in [0, \beta] \}.$$

Then $C_0(\beta)$ is a closed, bounded and convex subset of $C(\beta)$.

Let $U : C_0(\beta) \to C(\beta)$ be the operator defined by

$$U x(t; \omega) = \int_0^t h(t, s, x(s; \omega)) \, ds.$$

Then U is compact. For $x \in C_0(\beta)$, define

$$T x(t; \omega) \doteq f(t : \omega) + U x(t; \omega).$$

In view of the preceding discussion and the fact that $f \in C(\beta)$, it follows that T is a mapping from $C_0(\beta)$ into $C(\beta)$ and is compact. Furthermore $\| T x(t; \omega) - f(t; \omega) \|_2 < 1$ for $x \in C_0(\beta)$, and so $T : C_0(\beta) \to C_0(\beta)$. Thus, by Schauder's fixed point theorem, T has at least one fixed point in $C_0(\beta)$. This fixed point is a random solution of (3.1) on $[0, \beta]$. Q.E.D.

Remark. If h satisfies a Lipschitz condition with respect to x, the local random solution obtained above is unique („unique" here means „unique within the class $C_0(\beta)$"). The proof of this assertion parallels the discussion in Chapter 2 of T s o k o s and P a d g e t t [7], and is accordingly omitted.

Theorem 3.2. *Suppose that hypotheses H3.1 — H3.4 hold. If* $x \in C([0, \alpha], L_2)$ *is a bounded random solution of (3.1), then* x *can be extended as a random solution of (3.1) to an interval* $[0, \alpha_0]$ *where* $\alpha_0 > \alpha$.

Proof; Suppose that $\| x(t; \omega) \|_2 \leqslant M$ *for* $t \in [0, \alpha)$ and let

$$B = \{ x(t; \cdot) \in L_2 : \| x(t; \omega) \|_2 \leqslant M \}.$$

For any sequence $\{ t_n \}$ from $[0, \alpha)$ with $t_n \uparrow \alpha$, we have, for $n > m$,

$$\left\| \int_0^{t_n} h(t_n, s, x(s; \omega)) \, ds - \int_0^{t_m} h(t_m, s, x(s; \omega)) \, ds \right\|_2 \leqslant$$

$$\leqslant \sup \left\{ \int_0^{t_n} \| h(t_n, s, x(s; \omega)) - h(t_m, s, x(s; \omega)) \|_2 \, ds : x \in C([0, \alpha], B) \right\},$$

the right-hand side of this inequality tending to zero as $n, m \to \infty$. Thus, from (3.1), $\| x(t_n; \omega) - x(t_m; \omega) \|_2 \to 0$ as $n, m \to \infty$. Since $\{ t_n \}$ is any increasing sequence with limit α, the L_2-limit of $x(t_n; \cdot)$ exists as $t_n \to \alpha$. Extend x to a mean square continuous function on $[0, \alpha]$ by setting

$$x(\alpha; \omega) = \text{l.i.m.}\ x(t; \omega) \text{ as } t \to \alpha^-.$$

Now let

$$y(t; \omega) = F(t; \omega) + \int_0^t h_\alpha(t, s, y(s; \omega)) \, ds, \tag{3.2}$$

where

$$F(t;\omega) = f(t+\alpha;\omega) + \int_0^{\alpha} h(t+\alpha, s, x(s;\omega))\,ds$$

and

$$h_\alpha(t, s, x(s;\omega)) = h(t+\alpha, s+\alpha, x(s;\omega)).$$

For $y(t;\cdot) \in L_2$ the integrand in (3.2) satisfies H3.2 — H3.4. Thus, by Theorem 3.1, equation (3.2) has a solution $y \in C([0,\beta], L_2)$ for some sufficiently small β.

Define $\alpha_0 \doteq \alpha + \beta$ and $x(t+\alpha;\omega) \doteq y(t;\omega)$ for $t \in [0,\beta]$. Then $x \in C([0,\alpha_0], L_2)$ and x is a random solution of (3.1) on this interval. Q.E.D.

Definition 3.1. *A random solution of (3.1) defined on an interval $[0,\alpha)$ is maximal if either (i) $\alpha = \infty$, or (ii) $\limsup_{t\to\alpha^-} \| x(t;\omega)\|_2 = \infty$.*

Corollary 3.2.1. *Under hypotheses H3.1 — H3.4, each random solution of (3.1) can be extended to the right to yield a maximal random solution.*

Proof. Let $x(t;\omega)$ be any local random solution of (3.1). By Theorems 3.1 and 3.2, x can be extended to the right, being defined either for all $t \geqslant 0$, or else on some bounded interval J. By Theorem 3.1, J must have the form $[0,\alpha)$, and hence, by Theorem 3.2, $x(t;\omega)$ must be unbounded. Q.E.D.

4. Random operator methods. Consider the separable Banach space $C_c(R^+, L_2)$ and let X, Y be linear subspaces of $C_c(R^+, L_2)$ admitting norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Suppose further that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces which are stronger than $C_c(R^+, L_2)$.

In this Section we shall consider the equations

(4.1) $$x(t;\omega) = f(t;\omega) + T(\omega)\,[x(t;\omega) + g(t, x(t;\omega))].$$

(4.2) $$y(t;\omega) = f(t;\omega) + T(w)y(t;\omega).$$

under the following assumptions

H4.1 $f \in C_c(R^+, L_2)$ and $g : R^+ \times C_c(R^+, L_2) \to C_c(R^+, L_2)$.

H4.2 $T : \Omega \times C_c(R^+, L_2) \to C_c(R^+, L_2)$ is, for each $\omega \in \Omega$, a bounded linear operator.

Notice that, since T is continuous and $C_c(R^+, L_2)$ is a separable Banach space, $T(\omega)$ is separable, and hence, if $T : \Omega \times C_c(R^+, L_2) \to E$, where E is a compact subset of $C_c(R^+, L_2)$,

$$\| T(\omega) \| \doteq \sup \{ \| T(\omega)\, x(t;\omega) \|_2 / \| x(t;\omega) \|_2 : \| x(t;\omega) \|_2 \neq 0 \}$$

is a non-negative real-valued random variable (B h a r u c h a - R e i d [1]).

Condition H4.2 and the fact that $C_c(R^+, L_2)$ is separable imply the following (H a n s [3]).:

(i) $(\forall \lambda \in \mathbf{R} - \{0\})\, \Omega(\lambda) \doteq \{ \omega : \| T(\omega) \| < |\lambda| \} \in \mathcal{A}$.

(ii) the inverse of $(\lambda I - T(\omega))$ exists for every $\omega \in \Omega(\lambda)$.

(iii) for every $\omega \in \Omega(\lambda)$ the resolvent operator $R_\lambda^*(\omega)$ exists and is given by $R_\lambda^*(\omega) \doteq -\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}(\omega)$.

(iv) for every $\omega \in \Omega(\lambda)$ the random solution $y(t;\omega)$ of (4.2) is of the form

(4.3) $$y(t;\omega) = R_\lambda^*(\omega) f(t;\omega).$$

(v) both R_λ^* and the solution y given in (4.3) are measurable with respect to the σ-algebra $\Omega(\lambda) \cap \mathcal{A}$.

Setting $\lambda = 1$, subtracting $T(\omega)x$ from each side of (4.1) and then applying $S(\omega) \doteq [I - T(\omega)]^{-1}$, we obtain, almost surely on $\Omega(1)$,

(4.4) $$\begin{aligned} x(t;\omega) &= [I - T(\omega)]^{-1} f(t;\omega) + [I - T(\omega)]^{-1}\, T(\omega)\, g(t, x(t;\omega)) = \\ &= S(\omega) f(t;\omega) + [I - T(\omega)]^{-1}\, T(\omega)\, g(t, x(t;\omega)). \end{aligned}$$

If we now add, to those listed above, the assumption

H4.3 $$P\left(\bigcup_{n=1}^{\infty} \{\omega : \| T^n(\omega) \| < 1\}\right) = 1,$$

it follows that $P(\Omega_0) = 1$, where

$$\Omega_0 \doteq \bigcap_{x\in\mathcal{C}} \left\{ \omega : S(\omega)\, x(t;\omega) = -\sum_{n=1}^{\infty} T^{n-1}(\omega)\, x(t;\omega) \right\},$$

with $\mathcal{C} \doteq C_c(R^+, L_2)$.

Thus, from (4.3) and (4.4), we have, almost surely on $\Omega^* \doteq \Omega(1) \cap \Omega_0$,

$$\begin{aligned} x(t;\omega) &= R_1^*(\omega) f(t;\omega) + \left[-I - \sum_{n=1}^{\infty} T^n(\omega) \right] T(\omega)\, g(t, x(t;\omega)) = \\ &= y(t;\omega) - R(\omega)\, g(t, x(t;\omega)), \end{aligned}$$

where $R(\omega) = \sum_{n=1}^{\infty} T^n(\omega)$. Since this procedure is reversible, we see that (4.1) is almost surely on Ω^* equivalent to

(4.5) $$x(t;\omega) = y(t;\omega) - R(\omega)\, g(t, x(t;\omega)),$$

where $y(t;\omega)$ satisfies (4.2) and $R(\omega)$ is as defined above.

Since Ω^* is measurable with respect to $\mathcal{A}$, we can form a new probability space $(\Omega^*, \mathcal{A}^*, P^*)$ by defining

$$\mathcal{A}^* \doteq \Omega^* \cap \mathcal{A}^*; \quad P^*(E) \doteq P(\Omega^* \cap E)/P(\Omega^*), \quad E \in \mathcal{A}^*.$$

This probability space is complete and atomic, and P^* has a countable basis. *Henceforth we shall regard $(\Omega^*, \mathcal{A}^*, P^*)$ as our underlying probability space.*

Theorem 4.1. *In addition to the above assumptions, suppose that*

H4.4. (i) $y \in Y$,

(ii) $g: R^+ \times Y \to X$,

(iii) (X, Y) *is admissible with respect to the compact operator*

$$R: \Omega \times C_c(R^+, L_2^*) \to C_c(R^+, L_2^*), \text{ where } L_2^* \doteq L_2(\Omega^*, \mathcal{A}^*, P^*).$$

For any $\rho > 0$ *let*

$$S_X(\rho) = \{x(t;\omega) : \| x(t;\omega) \|_X \leqslant \rho\}$$

$$S_Y(\rho) = \{y(t;\omega) : \| y(t;\omega) \|_Y \leqslant \rho\}.$$

Suppose further that there exist ρ_1, ρ_2 *such that* $g: R^+ \times \overline{S_Y(\rho_1)} \to \overline{S_X(\rho_2)}$ $C_c(R^+, L_2^*)$ *— continuously, where „*$\overline{}$*" denotes the* $C_c(R^+, L_2^*)$ *— closure. If* $y \in Y$ *and*

$$(*) \qquad \| y(t;\omega) \|_Y + \rho_2 \| R(\omega) \| \leqslant \rho_1,$$

then equation (4.1) *has at least one solution* $x \in Y$ *with* $\| x(t;\omega) \|_Y \leqslant \rho_1$.

Proof. For any $\Phi \in \overline{S_Y(\rho_1)}$ define the random operator M by

$$M(\omega)\,\Phi(t;\omega) = y(t;\omega) + R(\omega)\,g(t, \Phi(t;\omega))$$

where $y \in Y$. Clearly $M(\omega)\Phi(t;\omega) \in Y$. Since $\| M(\omega)\,\Phi(t;\omega) \|_Y \leqslant \rho_1$, it follows that $M(\omega)\,\Phi(t;\omega) \in S_Y(\rho_1)$, and so $M: \Omega \times \overline{S_Y(\rho_1)} \to \overline{S_Y(\rho_1)}$.

If $\{y_n(t;\omega)\}$ is any bounded sequence from $\overline{S_Y(\rho_1)}$ and $y \in Y$, then $M(\omega)\,y_n(t;\omega) = y(t;\omega) - R(\omega)\,g(t, y_n(t;\omega))$. Now g is $C_c(R^+, L_2^*)$-continuous, and hence $\{g(t, y_n(t;\omega))\}$ is bounded. Since $R: \Omega \times S_X(\rho_2) \to$ $\to S_Y(\rho_1)$ is compact, there exists a subsequence $\{g(t, y_{n_k}(t;\omega))\}$ such that $\{R(\omega)\,g(t, y_{n_k}(t;\omega))\}$ converges to a limit in $S_Y(\rho_1)$, {and hence $\{M(\omega)\,.\,y_{n_k}(t;\omega)\}$ converges to a limit in $S_Y(\rho_1)$}. Thus M is compact.

Since $\overline{S_Y(\rho_1)}$ is closed, bounded and convex, it follows by the random Schauder fixed point theorem that there is an $\overline{S_Y(\rho_1)}$-valued random function $x(t;\omega)$ such that $M(\omega)\,x(t;\omega) = x(t;\omega)$ P^*-almost surely. Since $M: \Omega \times \overline{S_Y(\rho_1)} \to S_Y(\rho_1)$, it follows that $x(t;\omega) \in S_Y(\rho_1)$. In view of the discussion preceding this theorem x is seen to be a random solution of (4.1). Q.E.D.

5. Some perturbation problems. We now specialize the abstract random operators in equations (4.1), (4.2) to obtain the random Volterra integral equations

$$x(t;\omega) = f(t;\omega) + \int_0^t k(t, s;\omega)\,[x(s;\omega) + g(s, x(s;\omega))]\,ds, \tag{5.1}$$

$$y(t;\omega) = f(t;\omega) + \int_0^t k(t, s;\omega)\,y(s;\omega)\,ds. \tag{5.2}$$

Here the random kernel $k(t, s;\omega)$ is a random variable defined and P^*-essentially bounded for $0 \leqslant s \leqslant t < \infty$ (and identically zero for $s > t$), and k, as a mapping from $\{(t, s): 0 \leqslant s \leqslant t < \infty\}$ into $L_\infty^* \doteq L_\infty(\Omega^*, \mathcal{A}^*, P^*)$, is continuous, i.e.

$$P^* - \operatorname*{ess.\,sup}_{\omega \in \Omega^*} | k(t_n, s_n;\omega) - k(t, s;\omega) | \to 0$$

as $n \to \infty$ whenever $(t_n, s_n) \to (t, s)$ as $n \to \infty$.

The random resolvent equation corresponding to (5.2) is

$$r(t, s;\omega) = -k(t, s;\omega) + \int_s^t k(t, u;\omega)\,r(u, s;\omega)\,du,$$

and its solution is called the *random resolvent kernel*. We shall assume throughout that this equation possesses a solution such that the random operator $R: \Omega \times C_c(R^+, L_2^*) \to C_c(R^+, L_2^*)$ defined by

$$R(\omega)\,x(t;\omega) \doteq \int_0^t r(t, s;\omega)\,x(s;\omega)\,ds, \quad t \geqslant 0$$

is compact.

The following assumptions will be held throughout

H5.1. (i) $\| k(t, s;\omega) \|_\infty$ is locally L^1 in (t, s),

(ii) for any $T > 0$, $\sup\limits_{0 \leqslant t < T} \int_0^t \| k(t, s;\omega) \|_\infty\,ds < \infty$,

(iii) $\lim\limits_{h \to 0} \int_0^t \| k(t+h, s;\omega) - k(t, s;\omega) \|_\infty\,ds = 0$,

(iv) $\lim\limits_{h \to 0} \int_t^{t+h} \| k(t+h, s;\omega) \|_\infty\,ds = 0$,

H5.2. (i) $\| r(t, s;\omega) \|_\infty$ is locally L^1 in (t, s),

(ii) $r^* \doteq \sup\limits_{t > 0} \int_0^t \| r(t, s;\omega) \|_\infty\,ds < \infty$,

H5.3. (i) $g(t, x)$ is continuous for $0 \leqslant t < \infty$, uniformly in scalars x

(ii) $g(t, 0) \equiv 0$

(iii) $(\forall \varepsilon > 0)\,(\exists \delta \equiv \delta(\varepsilon)) \Rightarrow | g(t, x) | \leqslant \varepsilon | x |$, for $t \geqslant 0$ and $\| x \| \leqslant \delta$.

Remarks: (1) These assumptions, while sufficient for the results of this Section, are, for certain classes of functions, not all necessary (cf. Miller [4], pp. 334—5).

(2) It follows from H5.3 (iii) that for $x \in C_c(R^+, L_2^*)$, $g(t, x(t;\omega)) \in L_2^*$. Recalling the discussion earlier in this Section, it follows that, for fixed t and s in R^+, the product of $k(t, s; \omega)$ and $g(s, x(s;\omega))$ will be in L_2^*.

(3) Under the above assumptions, equations (5.1) and (5.2) can respectively be written in the „variation of constants" forms

$$x(t;\omega) = y(t;\omega) - \int_0^t r(t, s; \omega)\, g(s, x(s;\omega))\, ds, \tag{5.3}$$

$$y(t;\omega) = f(t;\omega) - \int_0^t r(t, s; \omega) f(s;\omega)\, ds. \tag{5.4}$$

Equation (5.3) is „equivalent" to (5.1) in the sense that, for any $T > 0$, $x(t;\omega)$, as defined by (5.3), satisfies (5.1) P^*-almost surely on $[0, T]$. Similar remarks apply to the pair of equations (5.2) and (5.4).

We pass on now to demonstrate the existence of a random solution to equation (5.1). This result differs from Theorem 3.1 in its reliance on the random resolvent kernel $r(t, s; \omega)$.

Theorem 5.1. *If*

(i) $f, y \in C_c(R^+, L_2^*)$,

(ii) $g: R^+ \times C(R^+, L_2^*) \to C(R^+, L_2^*)$,

(iii) $\| f(t;\omega) \|_C$ *is sufficiently small,*

then (5.1) has a random solution $x \in C(R^+, L_2^*)$. *Furthermore,*

$$\| x(t;\omega) \|_C \to 0 \text{ as } \| f(t;\omega) \|_C \to 0.$$

Proof. We show that the assumptions of Theorem 4.1 are satisfied with $X = Y = C(R^+, L_2^*)$. Notice firstly that (X, Y) is admissible with respect to the operator $R: \Omega^* \times X \to Y$ defined by

$$R(\omega)\, x(t;\omega) = \int_0^t r(t, s; \omega)\, x(s;\omega)\, ds.$$

(cf. Tsokos and Padgett [7], p. 38). While we are not assuming H4.3 here, all that is required for Theorem 4.1 is the *existence of* $r(t, s; \omega)$ which we are assuming. Finally, the defined operator R is by assumption compact. Thus the conditions of Theorem 4.1 are satisfied. Moreover

$$y(t;\omega) = (I - R(\omega)) f(t;\omega) \in C(R^+, L_2^*), \text{ and}$$

$$\| y(t;\omega) \|_C \leq (1 + \| R(\omega) \|)\ \| f(t;\omega) \|_C.$$

The rest of the proof parallels that of Miller [4], pp. 277/8, and is therefore omitted. Q.E.D.

Remark: If g is locally Lipschitz, the random solution is unique.

This result can be viewed as a (local) stability result in the following sense: for every sufficiently small $\varepsilon > 0$, there exists a $\delta \equiv \delta(\varepsilon) > 0$ such that, for every $f \in C(R^+, L_2^*)$ with $\| f(t;\omega) \|_C < \delta$, the solution $x(t;\omega)$ of (5.1) is in $C(R^+, L_2^*)$ and satisfies $\| x(t;\omega) \|_C < \varepsilon$.

Theorem 5.2. *In addition to the above assumptions, let r satisfy*

$$\lim_{t \to \infty} \int_0^T \| r(t, s; \omega) \|_\infty\, ds = 0, \textit{ for each } T > 0.$$

Let $| g(t, x) | \leq \Phi_M(t)$ *for* $t \leq 0$ *and* $| x | \leq M$, *where* $\Phi_M(t) \to 0$ *as* $t \to \infty$ *for each* $M > 0$. *If* $x \in C(R^+, L_2^*)$, *then* $\| x(t;\omega) - y(t;\omega) \|_2 \to 0$. *as* $t \to \infty$.

Proof. Let $\| x(t;\omega) \| = M$, and let $\varepsilon > 0$ be given. Choose $T > 0$ so that $\Phi_M(t) \leq \varepsilon/(2r^*)$ for all $t \geq T$, and then choose $\tau > T$ so that

$$\int_0^T \| r(t, s; \omega) \|_\infty\, ds \leq \frac{\varepsilon}{2 \| \Phi_M \|} \quad \text{for } t \geq \tau,$$

where $\| \Phi_M \| \doteq \sup_{t \geq 0} | \Phi_M(t) |$. Then, for $t \geq \tau$,

$$\| x(t;\omega) - y(t;\omega) \|_2 \leq \left\| \int_0^t r(t, s; \omega)\, g(s, x(s;\omega))\, ds \right\|_2 \leq$$

$$\leq \int_0^T \| r(t, s; \omega) \|_\infty\, \Phi_M(s)\, ds + \int_T^t \| r(t, s; \omega) \|_\infty\, \Phi_M(s)\, ds \leq$$

$$\leq \| \Phi_M \| \frac{\varepsilon}{2 \| \Phi_M \|} + \frac{\varepsilon}{2r^*} \int_0^t \| r(t, s; \omega) \|_\infty\, ds \leq \varepsilon.$$

Thus $\| x(t;\omega) - y(t;\omega) \|_2 \to 0$ as $t \to \infty$. Q.E.D.

Remark. It follows that, if

$$| g(t, x) | \leq \Phi(t)\,(1 + | x |), \text{ for } t \geq 0,$$

where $\Phi(t) \to 0$ as $t \to \infty$, then $\| x(t;\omega) - y(t;\omega) \|_2 \to 0$ as $t \to \infty$.

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EXISTENCE THEOREMS FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

BY

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1. Introduction. Consider the Volterra integrodifferential initial value problem

$$(1.1)\quad u'(t) + \int_0^t a(t-s)\,Au(s)\,ds + \int_0^t b(t-s)\,Bu(s)\,ds \ni f(t),\ t \geqslant 0;$$

$u(0) = U_0 \in H$

Here H is an infinite dimensional Hilbert space, $u, f: R^+ \to H$, $a, b: R^+ \to R$, whereas A, B are nonlinear monotone operators on H. We have two possibilities to obtain existence results for (1.1): either to restrict the class of kernels a, b or to allow a wide variety of kernels by imposing certain restrictions on the nonlinearities A, B. Both ways were already used (by Londen [6], [7] and Aizicovici [1] respectively) in the investigation of the special case of Eq. (1.1) where $a \equiv b$. In the present note we further exploit the second possibility. This is done by monotone operator methods (see [2], [3] for a systematic exposition of the theory of monotone operators). Our results apply to the hyperbolic differential equation

$$u''(t) + (A + B)\,u(t) \ni f(t),\ t > 0,$$

derived from (1.1) by taking $a \equiv b \equiv 1$.

The plan of the paper is as follows: Section 2 contains a statement of results. The proofs are carried out in Sections 3 and 4. To illustrate the ideas, we discuss in the last section an example of a nonlinear partial differential functional equation.

2. Main Results. Let H be a Hilbert space over R with norm $|\,.\,|$ and inner product $(\,,\,)$. Let V be a real reflexive separable Banach space with $V \subset H$ algebraically and topologically, such that V is dense in H. We have the usual inclusions $V \subset H \subset V'$, where V' is the dual of V. If $x \in V$ and $f \in V'$, we also use (f, x) to indicate the value of f at x; (f, x) coincides with their scalar product in H whenever $f \in H$. The norm in V is denoted by $\|\,.\,\|$. It is further assumed that

(2.1) The injection $V \subset H$ is compact.

Consider a convex, lower semicontinuous (l.s.c.), proper function $\varphi: V \to (-\infty, \infty]$ and define $A: V \to V'$ by