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## EXISTENCE THEOREMS FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

BY

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**1. Introduction.** Consider the Volterra integrodifferential initial value problem

$$(1.1) \quad u'(t) + \int_0^t a(t-s) Au(s) ds + \int_0^t b(t-s) Bu(s) ds \ni f(t), \quad t \geq 0;$$

$$u(0) = U_0 \in H$$

Here  $H$  is an infinite dimensional Hilbert space,  $u, f: \mathbb{R}^+ \rightarrow H$ ,  $a, b: \mathbb{R}^+ \rightarrow \mathbb{R}$ , whereas  $A, B$  are nonlinear monotone operators on  $H$ . We have two possibilities to obtain existence results for (1.1): either to restrict the class of kernels  $a, b$  or to allow a wide variety of kernels by imposing certain restrictions on the nonlinearities  $A, B$ . Both ways were already used (by London [6], [7] and Aizicovici [1] respectively) in the investigation of the special case of Eq. (1.1) where  $a \equiv b$ . In the present note we further exploit the second possibility. This is done by monotone operator methods (see [2], [3] for a systematic exposition of the theory of monotone operators). Our results apply to the hyperbolic differential equation

$$u''(t) + (A + B)u(t) \ni f(t), \quad t > 0,$$

derived from (1.1) by taking  $a \equiv b \equiv 1$ .

The plan of the paper is as follows: Section 2 contains a statement of results. The proofs are carried out in Sections 3 and 4. To illustrate the ideas, we discuss in the last section an example of a nonlinear partial differential functional equation.

**2. Main Results.** Let  $H$  be a Hilbert space over  $\mathbb{R}$  with norm  $|\cdot|$  and inner product  $(\cdot, \cdot)$ . Let  $V$  be a real reflexive separable Banach space with  $V \subset H$  algebraically and topologically, such that  $V$  is dense in  $H$ . We have the usual inclusions  $V \subset H \subset V'$ , where  $V'$  is the dual of  $V$ . If  $x \in V$  and  $f \in V'$ , we also use  $(f, x)$  to indicate the value of  $f$  at  $x$ ;  $(f, x)$  coincides with their scalar product in  $H$  whenever  $f \in H$ . The norm in  $V$  is denoted by  $\|\cdot\|$ . It is further assumed that

(2.1) The injection  $V \subset H$  is compact.

Consider a convex, lower semicontinuous (l.s.c.), proper function  $\varphi: V \rightarrow (-\infty, \infty]$  and define  $A: V \rightarrow V'$  by

(2.2)  $A = \partial\varphi$  ( $\partial =$  subdifferential).

Suppose that

(2.3)  $A$  is single-valued, everywhere defined on  $V$  and maps bounded sets in  $V$  into bounded sets in  $V'$ .

(2.4)  $\left\{ \begin{array}{l} \text{For any } T > 0 \text{ and every sequence } \{u_n\} \subset L^\infty(0, T; V) \\ \text{with } \{u'_n\} \subset L^2(0, T; H), \text{ such that } u_n \rightarrow u, \text{ weakly-star in } \\ L^\infty(0, T; V) \text{ and } u'_n \rightarrow u', \text{ weakly in } L^2(0, T; H), \text{ we have} \\ Au_{n_k} \rightarrow Au \text{ weakly-star in } L^\infty(0, T; V'), \text{ for a suitable} \\ \text{subsequence } u_{n_k}. \end{array} \right.$

There is  $k > 0$  such that

(2.5)  $\varphi(u) \geq k(\|u\|^2 - 1), \quad u \in V.$

*Remark 2.1.* Hypotheses (2.2) – (2.5) are clearly satisfied by every linear, positive, symmetric operator  $A : V \rightarrow V'$ , with  $(Au, u) \geq k\|u\|^2, k > 0, u \in V.$

Let  $B$  be a (possibly multivalued) maximal monotone operator  $H$  verifying

(2.6)  $V \subset D(B)$  and  $|B^0u| \leq c_1\|u\| + c_2, \quad u \in V,$

for some  $c_i > 0, i = 1, 2,$  where  $B^0$  denotes the minimal section of  $B$  ([3, p. 28]).

Consider next two functions  $a, b : R^+ \rightarrow R$  subject to the following conditions :

(2.7)  $a$  is locally absolutely continuous on  $[0, \infty)$

For each  $T > 0,$  there exists  $K_T > 0$  such that from

$$\int_0^t (a * v(s), v(s)) ds \leq d_1 + d_2 \max_{0 < \tau < t} \left| \int_0^\tau v(\tau) d\tau \right|, \quad 0 \leq t \leq T,$$

$$(v \in L^2(0, T; H), \quad 0 \leq d_1, d_2 < \infty, \quad a * v(t) = \int_0^t a(t-s)v(s) ds)$$

(2.8) it follows

$$\left| \int_0^t v(s) ds \right| \leq K_T (\sqrt{d_1} + d_2), \quad 0 \leq t \leq T,$$

and

$$\left| \int_0^t (a * v(s), v(s)) ds \right| \leq K_T (d_1 + d_2^2), \quad 0 \leq t \leq T,$$

(2.9)  $b, b' \in L^1_{loc}([0, \infty); R).$

*Remark 2.2.* It is known [4, Prop (a)] that if  $a$  satisfies either

$$a, a' \in L^1_{loc}([0, \infty); R), \quad a(0) > 0$$

$a'$  is of bounded variation locally on  $[0, \infty),$

or

$$a \in C^2(0, \infty) \cap C[0, \infty); \quad a(0) > 0,$$

$$(-1)^k a^k(t) \geq 0, \quad t > 0, \quad k = 0, 1, 2,$$

then condition (2.8) holds.

Finally, let  $f : [0, \infty) \rightarrow H$  satisfy

(2.10)  $f, f' \in L^1_{loc}([0, \infty); H).$

**Theorem 1.** *Let the assumptions (2.1) – (2.10) be satisfied. Then, for every  $u_0 \in V,$  the problem*

(2.11)  $\left. \begin{array}{l} \text{i) } u'(t) + a * Au(t) + b * Bu(t) \ni f(t), \quad t \geq 0, \\ \text{ii) } u(0) = u_0 \end{array} \right\}$

has a solution  $u$  in the sense

(2.12)  $u \in L^\infty_{loc}([0, \infty); V) \cap C([0, \infty); H),$

(2.13)  $u', \int_0^t Au(s) ds \in L^1_{loc}([0, \infty); H),$

(2.14)  $u'' \in L^1_{loc}([0, \infty); V'),$

(2.15)  $\left\{ \begin{array}{l} \text{There is a } y \in L^1_{loc}([0, \infty); H), \text{ with } y(t) \in Bu(t), \\ \text{a.e. } t > 0, \text{ such that the equality} \\ u'(t) + a * Au(t) + b * y(t) = f(t), \\ \text{holds in } H(V') \text{ for almost all (all) } t \geq 0. \end{array} \right.$

In the case when  $a \equiv b$  and  $B$  is the subdifferential of a l.s.c., convex, proper function it turns out that Conditions (2.5) and (2.6) can be weakened. Specifically, let

(2.16)  $\lim_{\|u\| \rightarrow \infty} \varphi(u) = \infty,$

(2.17)  $B = \partial\psi, (\psi : H \rightarrow (-\infty, \infty]); \quad D(B) \supset V;$

(2.18)  $B^0$  maps bounded subsets of  $V$  into bounded subsets of  $H.$   
We have

**Theorem 2.** Let the assumptions (2.1) – (2.4), (2.7), (2.8) and (2.10) be fulfilled. Suppose, in addition that (2.16) – (2.18) hold. Then for each  $u_0 \in V$ , equation

$$(2.19) \quad u'(t) + a * (A + B)u(t) \ni f(t), \quad t \geq 0; \quad u(0) = u_0,$$

has a solution satisfying (2.12) – (2.15).

*Remark 2.3.* If  $B$  is single-valued the above result can be obtained via [1, Remark 2.5], taking note of the fact that in this case  $A + B$  is maximal monotone from  $V$  to  $V'$  and satisfies (2.2) – (2.4) and (2.16) (where  $A, \varphi$  are replaced by  $A + B$  and  $\varphi + \psi$  respectively).

**3. Proof. of Theorem 1.** First define

$$A_H x = Ax, \quad x \in D(A_H) = \{x \in V, Ax \in H\}.$$

It is easily seen that  $A_H$  is maximal monotone in  $H$  and

$$(3.1) \quad A_H = \partial\varphi_H,$$

where

$$\varphi_H(x) = \begin{cases} \varphi(x), & x \in V, \\ +\infty, & x \in H \setminus V. \end{cases}$$

Denote by  $A_\varepsilon, B_\lambda (\varepsilon, \lambda > 0)$  the Yosida approximations of  $A_H$  and  $B$  respectively. In view of (3.1), one has

$$(3.2) \quad A_\varepsilon = \partial\varphi_\varepsilon, \quad \varphi_\varepsilon(x) = \min \{ \varphi_H(y) + (2\varepsilon)^{-1} |y - x|^2; y \in H \}$$

(See [3, Prop. 2.11] for the properties of  $\varphi_\varepsilon$ ).

Consider next the regularized problem

$$(3.3) \quad \begin{cases} \text{(i)} & u'_{\varepsilon\lambda}(t) + \lambda A_H u_{\varepsilon\lambda}(t) + a * A_\varepsilon u_{\varepsilon\lambda}(t) + b * B_\lambda u_{\varepsilon\lambda}(t) = f(t), \quad t > 0; \quad \lambda, \varepsilon > 0 \\ \text{(ii)} & u_{\varepsilon\lambda}(0) = u_0. \end{cases}$$

By [4, Lemma 2.1] it follows that (3.3) has a unique solution  $u_{\varepsilon\lambda} : [0, \infty) \rightarrow H$  satisfying

$$u_{\varepsilon\lambda} \in C([0, \infty); H), \quad u'_{\varepsilon\lambda} \in L^2_{loc}([0, \infty); H).$$

The first step is to let  $\varepsilon \rightarrow 0^+$  in (3.3) with  $\lambda > 0$  fixed. This can be easily done without involving Hypotheses (2.4) and (2.6). The idea is to take inner-product (3.3)(i) with each of  $A_\varepsilon u_{\varepsilon\lambda}(t)$  and  $u'_{\varepsilon\lambda}(t)$  and integrate the results over  $]0, t[$ ,  $0 \leq t \leq T$  (for an arbitrary  $T > 0$ ). Employing [3, Lemma 3.3], (2.7) – (2.10) and recalling that  $B_\lambda$  is Lipschitz, one finds a constant  $C_T > 0$ , independent of  $\varepsilon$ , such that for  $t \in [0, T]$ ,

$$(3.4) \quad |\varphi(u_{\varepsilon\lambda}(t))| + |u_{\varepsilon\lambda}(t)| \leq C_T,$$

$$(3.5) \quad \int_0^t |u'_{\varepsilon\lambda}(s)|^2 ds \leq C_T,$$

$$\int_0^t (|A_H u_{\varepsilon\lambda}(s)|^2 + |A_\varepsilon u_{\varepsilon\lambda}(s)|^2) ds \leq C_T.$$

Calling on (3.4), (3.5), (2.5) and (2.1) and invoking the Ascoli theorem we can assume that

$$(3.7) \quad u_{\varepsilon\lambda} \rightarrow u_\lambda (\varepsilon \rightarrow 0^+) \text{ in } C([0, T]; H) \text{ for each } T > 0.$$

Consequently,

$$B_\lambda u_{\varepsilon\lambda} \rightarrow B_\lambda u_\lambda \text{ in } C([0, T]; H), \quad \forall T > 0.$$

The maximality of  $A_H$ , (3.6) and (3.7) imply

$$A_H u_{\varepsilon\lambda} \rightarrow A_H u_\lambda \text{ and } A_\varepsilon u_{\varepsilon\lambda} \rightarrow A_H u_{\varepsilon\lambda}, \text{ weakly in } L^2(0, T; H), \quad \forall T > 0.$$

Finally (3.5) yields

$$u'_{\varepsilon\lambda} \rightarrow u'_\lambda, \text{ weakly in } L^2(0, T; H), \quad \forall T > 0.$$

Letting  $\varepsilon \rightarrow 0^+$  in (3.3) we obtain

$$(3.8) \quad \begin{cases} \text{(i)} & u'_\lambda(t) + \lambda A_H u_\lambda(t) + a * A_H u_\lambda(t) + b * B_\lambda u_\lambda(t) = f(t), \quad t > 0, \\ \text{(ii)} & u_\lambda(0) = u_0. \end{cases}$$

Now our aim is to establish bounds on  $u_\lambda$ , as  $\lambda \rightarrow 0^+$ . Let  $T > 0$  be fixed and pick any  $t_1 \in (0, T]$ . Then, multiplying (3.8)(i) by  $A_H u_\lambda(t)$ , integrating over  $]0, t[$ ,  $t \in [0, t_1]$  and using (3.8)(ii), (2.9) and (2.10) gives

$$(3.9) \quad \begin{aligned} & \varphi(u_\lambda(t)) + \lambda \int_0^t |A_H u_\lambda(s)|^2 ds + \int_0^t (a * A_H u_\lambda(s), A_H u_\lambda(s)) ds + \\ & + (b * B_\lambda u_\lambda(t), \int_0^t A_H u_\lambda(s) ds) - \int_0^t ((b * B_\lambda u_\lambda(s))', \int_0^s A_H u_\lambda(\tau) d\tau) ds = \\ & = \varphi(u_0) + (f(t), \int_0^t A_H u_\lambda(s) ds) - \int_0^t (f'(s), \int_0^s A_H u_\lambda(\tau) d\tau) ds, \\ & \quad 0 \leq t \leq t_1 \leq T. \end{aligned}$$

By virtue of (2.5), (2.9), (2.10), from (3.9) it follows:

$$(3.10) \quad \begin{aligned} & k \|u_\lambda(t)\|^2 + \lambda \int_0^t |A_H u_\lambda(s)|^2 ds + \int_0^t (a * A_H u_\lambda(s), A_H u_\lambda(s)) ds \\ & \leq k + \varphi(u_0) + C(\lambda, t_1, T) \max_{0 \leq \tau \leq t_1} \left| \int_0^\tau A_H u_\lambda(\tau) d\tau \right|, \quad 0 \leq t \leq t_1, \end{aligned}$$

where

$$(3.11) \quad C(\lambda, t_1, T) = \max_{0 \leq t \leq T} |f(t)| + \int_0^T |f'(s)| ds + \\ + (\max_{0 \leq t \leq T} |b(t)| + |b(0)| + \int_0^T |b'(s)| ds) \int_0^{t_1} |B_\lambda u_\lambda(s)| ds.$$

Since  $|B_\lambda u| \leq |B^0 u|$ ,  $u \in D(B)$ , Condition (2.6), (3.10) and (3.11) imply

$$(3.12) \quad k \|u_\lambda(t)\|^2 + \lambda \int_0^t |A_H u_\lambda(s)|^2 ds + \int_0^t (a \times A_H u_\lambda(s), A_H u_\lambda(s)) ds \\ \leq C_1 + C_2 (1 + \int_0^{t_1} \|u_\lambda(s)\| ds) \max_{0 \leq \tau \leq t} \left| \int_0^\tau A_H u_\lambda(\tau) d\tau \right|, \quad 0 \leq t \leq t_1.$$

Here and in the sequel  $C_1, C_2, \dots$ , etc, denote various finite positive constants depending on  $T$  (but not on  $\lambda, t_1$ ).

Invoking (2.8) (where  $T \rightarrow K_T$  can be assumed nondecreasing) (3.12) yields

$$(3.13) \quad \left| \int_0^t A_H u_\lambda(s) ds \right| \leq C_3 \left( 1 + \int_0^{t_1} \|u_\lambda(s)\| ds \right), \quad 0 \leq t \leq t_1,$$

and

$$(3.14) \quad \left| \int_0^t (a \times A_H u_\lambda(s), A_H u_\lambda(s)) ds \right| \leq C_3 \left( 1 + \left( 1 + \int_0^{t_1} \|u_\lambda(s)\| ds \right)^2 \right), \\ 0 \leq t \leq t_1.$$

Now use (3.13), (3.14) in (3.12) to obtain

$$k \|u_\lambda(t_1)\|^2 \leq C_4 + C_5 \left( 1 + \int_0^{t_1} \|u_\lambda(s)\| ds \right)^2, \quad 0 \leq t_1 \leq T,$$

Applying Gronwall's inequality, one has

$$(3.15) \quad \{u_\lambda; \lambda > 0\} \text{ is bounded in } L^\infty(0, T; V).$$

Then, from (3.13) — (3.15) we deduce that

$$(3.16) \quad \left\{ \int_0^t A_H u_\lambda(s) ds; \lambda > 0 \right\} \text{ and } \left\{ \int_0^t (a \times A_H u_\lambda(s), A_H u_\lambda(s)) ds; \lambda > 0 \right\}$$

are bounded in  $L^\infty(0, T; H)$ .  
Inasmuch as (see (2.7))

$$a \times A_H u_\lambda(t) = a(0) \int_0^t A_H u_\lambda(s) ds + \int_0^t a'(t-s) \int_0^s A_H u_\lambda(\tau) d\tau ds,$$

it follows

$$(3.17) \quad \{a \times A_H u_\lambda(t); \lambda > 0\} \text{ is bounded in } L^\infty(0, T; H).$$

Calling on (3.15) and (3.16), (3.12) implies

$$(3.18) \quad \{\sqrt{\lambda} A_H u_\lambda; \lambda > 0\} \text{ is bounded in } L^2(0, T; H).$$

In view of (3.15) and (2.6) we also have

$$(3.19) \quad \{B_\lambda u_\lambda; \lambda > 0\} \text{ is bounded in } L^\infty(0, T; H).$$

Finally, employing (3.17) — (3.19) in (3.8)(i) yields

$$(3.20) \quad \{u'_\lambda; \lambda > 0\} \text{ is bounded in } L^2(0, T; H).$$

*Remark.* Estimates (3.15) — (3.20) hold for each  $T > 0$ .

Now, by virtue of the compactness condition (2.1), (3.15) and (3.20) we may suppose that

$$(3.21) \quad u_\lambda \rightarrow u (\lambda \rightarrow 0^+) \text{ strongly in } C([0, T]; H) \text{ and weakly-star in } L^\infty(0, T; V), \quad \forall T > 0.$$

$$(3.22) \quad u'_\lambda \rightarrow u', \text{ weakly in } L^2(0, T; H), \quad \forall T > 0.$$

Since  $B$  is maximal monotone in  $H$ , (3.19) and (3.21) lead to

$$(3.23) \quad B_\lambda u_\lambda \rightarrow y \in Bu, \text{ weakly-star in } L^\infty(0, T; H), \quad \forall T > 0.$$

Consequently,

$$(3.24) \quad b \times B_\lambda u_\lambda(t) \rightarrow b \times y(t), \text{ weakly-star in } L^\infty(0, T; H), \quad \forall T > 0.$$

By (3.18) one has

$$(3.25) \quad \lambda A_H u_\lambda \rightarrow 0 \text{ in } L^2(0, T; H), \quad \forall T > 0.$$

Condition (2.4) comes now into play and enables us to conclude from (3.21) and (3.22) that

$$(3.26) \quad A_H u_\lambda \rightarrow Au, \text{ weakly-star in } L^\infty(0, T; V'), \quad \forall T > 0.$$

Combining (3.16), (3.17) and (3.26) gives

$$(3.27) \quad \int_0^t A_H u_\lambda(s) ds \rightarrow \int_0^t Au(s) ds, \text{ weakly-star in } L^\infty(0, T; H), \quad \forall T > 0.$$

(3.28)  $a \ast A_H u_\lambda(t) \rightarrow a \ast Au(t)$ , weakly-star in  $L^\infty(0, T; H)$ ,  $\forall T >$

Because of (3.8) and (3.21) – (3.28) we finally obtain (2.11) – (2.15) upon passage to the limit as  $\lambda \rightarrow 0^+$ . (The verification of (2.14) is performed by differentiating (2.11) (i) in  $V'$ ). The proof is thereby complete.

**4. Proof of Theorem 2.** We start by considering the approximate equation

$$(4.1) \quad \begin{cases} \text{(i)} & u'_\lambda(t) + \lambda(A_H + B_\lambda)u_\lambda(t) + a \ast (A_H + B_\lambda)u_\lambda(t) = \\ & = f(t), \quad t > 0, \quad \lambda > 0. \\ \text{(ii)} & u_\lambda(0) = u_0. \end{cases}$$

Inasmuch as  $A_H + B_\lambda = \partial(\varphi_H + \psi_\lambda)$ , it follows (cf. [4 Theorem 2]) that (4.1) has a solution  $u_\lambda$  which satisfies

$$u_\lambda, u'_\lambda \in L^2_{\text{loc}}([0, \infty); H), \quad (A_H + B_\lambda)u_\lambda \in L^2_{\text{loc}}([0, \infty); H).$$

Form the inner product of (4.1)(i) with  $v_\lambda(t) = (A_H + B_\lambda)u_\lambda(t)$  and integrate over  $(0, t)$ ,  $0 \leq t \leq T$  (for any  $T > 0$ ), to obtain

$$(4.2) \quad \begin{aligned} & (\varphi + \psi)(u_\lambda(t)) + \lambda \int_0^t |v_\lambda(s)|^2 ds + \int_0^t (a \ast v_\lambda(s), v_\lambda(s)) ds \leq \\ & \leq (\varphi + \psi)(u_0) + \|f(t)\| \left| \int_0^t v_\lambda(s) ds \right| + \int_0^t |f'(s)| \left| \int_0^t v_\lambda(\tau) d\tau \right| ds, \\ & \leq t \leq T. \end{aligned}$$

Since  $\psi$  is bounded below by an affine function, i.e.,

$$\psi(u) \geq (x_1, u) + x_2, \quad u \in H, \quad \text{for some } x_1, x_2 \in H,$$

we may always assume that  $\psi(u) \geq 0$ ,  $u \in H$ . Otherwise, we set

$$\tilde{\psi}(u) = \psi(u) - (x_1, u) - x_2,$$

and replace (2.19) by

$$u'(t) + a \ast (A + \partial\tilde{\psi})u(t) \ni f_1(t), \quad t \geq 0,$$

where

$$\partial\tilde{\psi} = \partial\psi - x_1; \quad f_1(t) = f(t) - (x_1, \int_0^t a(s) ds).$$

Consequently, from (4.2) we have (use also (2.10) and (2.16))

$$\lambda \int_0^t |v_\lambda(s)|^2 ds + \int_0^t (a \ast v_\lambda(s), v_\lambda(s)) ds \leq C_1 + C_2 \max_{0 \leq s \leq t} \left| \int_0^s v_\lambda(\tau) d\tau \right|, \quad 0 \leq t \leq T.$$

Thus, in view of condition (2.8), one infers

$$(4.3) \quad \left| \int_0^t v_\lambda(s) ds \right| \leq C_3, \quad 0 \leq t \leq T, \quad \lambda > 0,$$

$$(4.4) \quad \left| \int_0^t (a \ast v_\lambda(s), v_\lambda(s)) ds \right| \leq C_4, \quad 0 \leq t \leq T, \quad \lambda > 0$$

$$(4.5) \quad \lambda \int_0^t |v_\lambda(s)|^2 ds \leq C_5, \quad \lambda > 0, \quad 0 \leq t \leq T.$$

Using (4.3), (4.4) in (4.2) yields

$$(4.6) \quad (\varphi + \psi)(u_\lambda(t)) \leq C_6, \quad \lambda > 0, \quad 0 \leq t \leq T.$$

Calling on (2.16) again, (4.6) implies

$$(4.7) \quad \{u_\lambda; \lambda > 0\} \text{ is bounded in } L^\infty(0, T; V).$$

In virtue of (2.18), it follows from (4.7) that

$$(4.8) \quad \{B_\lambda u_\lambda; \lambda > 0\} \text{ is bounded in } L^\infty(0, T; H).$$

Next observe that

$$a \ast v_\lambda(t) = a(0) \int_0^t v_\lambda(s) ds + \int_0^t a'(t-s) \int_0^s v_\lambda(\tau) d\tau ds.$$

Hence, by (4.3)

$$(4.9) \quad \{a \ast v_\lambda; \lambda > 0\} \text{ is bounded in } L^\infty(0, T; H).$$

Then use (4.1)(i), (4.5), (4.9) in conjunction with (2.10) to derive

$$(4.10) \quad \{u'_\lambda; \lambda > 0\} \text{ is bounded in } L^2(0, T; H).$$

Taking into account (4.5), (4.7) – (4.10) we can now conclude the proof by the same argument as in Section 3.

**5. An Example.** For simplicity, we confine our attention to the one-dimensional case. Using standard notations for the Sobolev spaces we set:

$$(5.1) \quad V = H^2_0(0, 1), \quad H = L^2(0, 1), \quad V' = H^{-2}(0, 1).$$

Let  $p \in [2, \infty)$ ,  $W = W^{1,p}_0(0, 1)$  and observe that the imbedding theorems imply

$$(5.2) \quad V \subset W \subset H,$$

with dense and compact injections.

$$\text{Define } A_1: V \rightarrow V' \text{ and } A_2: W \rightarrow W' = W^{-1,p}(0,1), \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

by

$$(5.3) \quad A_1 = \frac{d^4 u}{dx^4}, u \in V; A_2 u = -\frac{d}{dx} \left( \left| \frac{du}{dx} \right|^{p-2} \frac{du}{dx} \right), u \in W, x \in (0,1).$$

Here  $du/dx$  denotes the distributional derivative of  $u$ .

It is easy to see that  $A_1 = \hat{c}\varphi_1$ ,  $A_2 = \hat{c}\varphi_2$ , where

$$(5.4) \quad \varphi_1(u) = \frac{1}{2} \int_0^1 \left| \frac{d^2 u}{dx^2}(x) \right|^2 dx, u \in V; \varphi_2(u) = \frac{1}{p} \int_0^1 \left| \frac{du}{dx}(x) \right|^p dx, u \in W.$$

We then define  $A: V \rightarrow V'$  by

$$(5.5) \quad A = A_1 + A_2.$$

In view of (5.1) – (5.5) we deduce that the assumptions (2.1) – (2.3) and (2.5) hold (note that  $\varphi = \varphi_1 + \varphi_2$ ). Invoking Remark 2.1, (5.2), [5, Thm. 5.1, Ch. 1] and the demicontinuity of  $A_2$  from  $W$  to  $W'$  we can also verify (2.4).

*Remark.* Instead of (5.5) one could take  $Au = A_1 u + \beta(u)$ ,  $u \in V$ , with  $\beta: R \rightarrow R$  satisfying

$$\beta \in C(-\infty, \infty), \beta(0) = 0, \beta \text{ nondecreasing,} \\ |\beta(r)| \leq k(|r|^{p-1} + 1), r \in R, k > 0, p \geq 2.$$

Next let  $g: R \rightarrow R$  satisfy

$$(5.6) \quad g \in C^1(-\infty, \infty),$$

$$(5.7) \quad g' \geq 0; |g'(x)| \leq \lambda_1(|x| + 1),$$

$$(5.8) \quad xg'(x) \geq \lambda_2(x^2 - 1),$$

for some  $\lambda_1, \lambda_2 > 0$  and any  $x \in R$ .

Consequently, the operator  $B$  given by

$$(5.9) \quad Bu = -\frac{d}{dx} g \left( \frac{du}{dx} \right), \\ u \in D(B) = \left\{ u \in H_0^1(0,1); \frac{d}{dx} g \left( \frac{du}{dx} \right) \in L^2(0,1) \right\}$$

is maximal monotone in  $H$ . Moreover

$$(5.10) \quad B = \hat{c}\psi, \psi(u) = \begin{cases} \int_0^1 \left( \int_0^1 g(s) ds \right) dx, u \in H_0^1(0,1), \\ -\infty, u \in L^2(0,1) \setminus H_0^1(0,1). \end{cases}$$

From (5.6) – (5.10) it then follows that  $B$  fulfills (2.17), (2.18). If (5.7) is strengthened to

$$(5.11) \quad 0 \leq g' \leq M < \infty,$$

then Condition (2.6) also holds.

Finally let  $f: [0, \infty) \rightarrow L^2(0,1)$  verify

$$(5.12) \quad f, f_t \in L_{loc}^1([0, \infty); L^2(0,1)).$$

Thus all the assumptions of Theorem 1 (Theorem 2) hold provided  $a, b$  satisfy (2.7) – (2.9). Accordingly, we obtain the following

**Corollary (i)** *Let Hypotheses (5.1) – (5.6) and (5.8) – (5.12) be satisfied. Further assume that (2.7) – (2.9) hold. Then for every  $u_0 \in H_0^2(0,1)$  there exists at least one solution  $u(t, x)$  ( $t \geq 0, 0 < x < 1$ ) to the problem*

$$\begin{cases} u_t(t, x) + \int_0^t a(t-s) \left( \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left( \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) \right) (s, x) ds - \\ - \int_0^t b(t-s) (g(u_x(s, x)))_x ds = f(t, x), t \geq 0, 0 < x < 1, \\ u(0, x) = u_0(x), u(t, 0) = u(t, 1) = u_x(t, 0) = u_x(t, 1) = 0, t \geq 0, \end{cases}$$

such that

$$(5.13) \quad u \in L_{loc}^\infty([0, \infty); H_0^2(0,1)) \cap C([0, \infty); L^2(0,1)),$$

$$(5.14) \quad u_t \in L_{loc}^\infty([0, \infty); L^2(0,1)),$$

$$(5.15) \quad u_{tt} \in L_{loc}^1([0, \infty); H^{-2}(0,1)).$$

(ii) *If Conditions (5.1) – (5.10), (5.12) and (2.7), (2.8) hold, the equation*

$$\begin{aligned} u_t(t, x) + \int_0^t a(t-s) \left( \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left( \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} g \left( \frac{\partial u}{\partial x} \right) \right) (s, x) ds = \\ = f(t, x), t \geq 0, 0 < x < 1, \end{aligned}$$

with the boundary and initial conditions

$$u(t, 0) = u(t, 1) = u_x(t, 0) = u_x(t, 1) = 0, \quad t \geq 0; \quad u(0, x) = u_0(x) \in H_0^2(0, 1),$$

has a solution which satisfies (5.13) — (5.15).

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## TIME OPTIMAL CONTROL FOR CONTINGENT EQUATIONS IN HILBERT SPACES

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**Introduction.** The time-optimal control problem has been studied by many authors under various assumptions; and the most important results in the area above are those of Balakrishnan [1], [2], Bellman, Glicksberg, Gross [3], Blagodatskih [4], Conti [8], Falb [9], Fattorini [10], Friedman [11], Halkin [13], Kikuchi [16], [17], La Salle [18], Markus [21].

The importance of contingent equations for control problems was pointed out by the remark that in an optimal control problem with control region constant, the Pontrjagin's Maximum Principle may be put in the form of a two-point boundary value problem for a contingent equation with upper semicontinuous right hand-side. See for instance Cellina [6].

It is also obvious that the equation:  $\dot{x}(t) = f(t, x(t), u(t))$  a.e. on  $[0, T]$ ,  $x(0) = x_0$ , where  $u: [0, T] \rightarrow U$  is a measurable function (under suitable regularity assumptions), is equivalent to the contingent equation:  $\dot{x} \in F(t, x(t))$  a.e. on  $[0, T]$ ,  $x(0) = x_0$ , where  $F(t, x(t)) = \bigcup_{u \in U} f(t, x(t), u)$ .

The problem of existence of solutions of such equations has been studied by Cellina [7], Hermes [14] (the finite dimensional case) Castaing and Valadier [5] (the infinite dimensional case) and the conditions used (upper semicontinuity) may be derived from our conditions (continuity in the Hausdorff metric).

We would like to point out that we consider the infinite dimensional case, because this case arises in the treatment of several types of partial differential equations (see Lions [19]) and functional differential equations as differential equations of retarded type (see Hale [12]). We recall that Hale was the first who observed that every differential equation of retarded type may be rewritten as an infinite dimensional Cauchy Problem and this is the most appropriate method to study such kind of problems (see for instance [12]).

Our main result, that in the finite dimensional case is very much in the spirit of Kikuchi's works [16], [17], (see also Valadier [24]), may be formulated as follows:

Let  $V \subset H \subset V'$  be three real Hilbert spaces with the norms  $\|\cdot\|$ ,  $\|\cdot\|_*$  and suppose that  $H$  includes  $V$  algebraically and topologically