with the boundary and initial conditions

$$u(t,0) = u(t,1) = u_x(t,0) = u_x(t,1) = 0,\ t \geq 0\ ;\ u(0,x) = u_0(x) \in H_0^2(0,1),$$

has a solution which satisfies (5.13) — (5.15).

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TIME OPTIMAL CONTROL FOR CONTINGENT EQUATIONS IN HILBERT SPACES

BY

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Introduction. The time-optimal control problem has been studied by many authors under various assumptions; and the most important results in the area above are those of Balakrishnan [1], [2], Bellman, Glicksberg, Gross [3], Blagodatskih [4], Conti [8], Falb [9], Fattorini [10], Friedman [11], Halkin [13], Kikuchi [16], [17], La Salle [18], Markus [21].

The importance of contingent equations for control problems was pointed out by the remark that in an optimal control problem with control region constant, the Pontrjagin's Maximum Principle may be put in the form of a two-point boundary value problem for a contingent equation with upper semicontinuous right hand-side. See for instance Cellina [6].

It is also obvious that the equation: $\dot{x}(t) = f(t, x(t)), u(t))$ a.e. on $[0, T]$, $x(0) = x_0$, where $u : [0, T] \to U$ is a measurable function (under suitable regularity assumptions), is equivalent to the contingent equation: $\dot{x} \in F(t, x(t))$ a.e. on $[0, T]$, $x(0) = x_0$, where $F(t, x(t)) = \bigcup_{u \in U} f(t, x(t), u)$.

The problem of existence of solutions of such equations has been studied by Cellina [7], Hermes [14] (the finite dimensional case) Castaing and Valadier [5] (the infinite dimensional case) and the conditions used (upper semicontinuity) may be derived from our conditions (continuity in the Hausdorff metric).

We would like to point out that we consider the infinite dimensional case, because this case arises in the treatement of several types of partial differential equations (see Lions [19]) and functional differential equations as differential equations of retarded type (see Hale [12]). We recall that Hale was the first who observed that every differential equation of retarded type may be rewritten as an infinite dimensional Cauchy Problem and this is the most appropriate method to study such kind of problems (see for instance [12]).

Our main result, that in the finite dimensional case is very much in the spirit of Kikuchi's works [16], [17], (see also Valadier [24]), may be formulated as follows:

Let $V \subset H \subset V'$ be three real Hilbert spaces with the norms $\|\cdot\|$, $|\cdot|$, $\|\cdot\|^*$ and suppose that H includes V algebraically and topologically and the inclusion $V \subset H$ is dense and continuous. Suppose further that V is in pairing with V' under the bilinear form $< .,. >$, such that the restriction of $< .,. >$, to $H \times H$ is the inner product in H.

Let F be a function defined on $[0, T] \times V$, valued in the class of all nonempty, closed and bounded sets in V' and let M_0, M_1 be two nonempty, closed and bounded sets of V.

Consider the equation :

$$(1) \qquad \dot{x}(t) \in F(t, x(t)) \text{ a.e. on } [0, T].$$

A solution of (Eq. 1) is a function $x \in L^2(0, T; V)$, absolutely continuous from $[0, T]$ to V' and so that $\dot{x} \in L^2(0, T; V')$.

Then, under suitable continuity assumptions on F, we can assert that every solution of (Eq. 1) that transfers M_0 in M_1 in the minimal time T_0 must necessarely satisfy the equation :

$$(2) \qquad \dot{x}(t) \in \partial F(t, x(t)) \text{ a.e. on } [0, T_0],$$

where $\partial F(t, x(t))$ is the boundary of $F(t, x(t))$.

We shall use the following notations : $[0, T_0]$, $[0, T_0]$ — closed intervals on the real line ; V, H, V', U — real Hilbert spaces ; if A is a set in a Hilbert space denote by : $\bar{A}$ — the closure of A ; int A — the interior of A ; ext A — the exterior of A ; ∂A — the boundary of A ; co A the convex hull of A ; if x, y are elements of a Hilbert space denote by : $S(x, d)$ — the open ball with center x and radius d ; $B(x, d)$ — the closed ball with center x and radius d ; $d(x, y)$ — the usual distance between x and y ; $d(x, A)$ — the usual distance between x and A ; if E is a set in $[0, T]$ denote by mes E — the Lebesgue measure of E.

§2. Preliminary results. Let H be a real Hilbert space and let $\Omega(H)$ be the set :

$$\Omega(H) = \{K ;\ K \subset H,\ \exists M_K > 0,\ K \subset S(0, M_K),\ K = \bar{K} \neq \emptyset\}.$$

Denote by $\Omega^0(H)$ the set :

$$\Omega^0(H) = \{K ;\ K \in \Omega(H),\ K = \text{co}\, K\}.$$

For each $A, B \in \Omega(H)$ we put $\rho(A, B) = \inf\{d ;\ A \subset S_d(B),\ B \subset S_d(A)\}$ where for each $C \in \Omega(H) S_d(C) = \bigcup_{x \in C} S(x, d)$. It is well known that ρ is a metric on $\Omega(H)$ called the Hausdorff metric.

Lemma 2.1. *The mapping $\partial : \Omega^0(H \to \Omega(H)$, defined by $A \to \partial A$ is Lipschitz continuous on $\Omega^0(H)$ with the Lipschitz constant* 1.

Proof : Let A, B be two elements of $\Omega^0(H)$. Denote by A_1 (respectively by B_1) the set of all points of ∂A (respectively of ∂B) for which there exists a supporting hyperplane to A (respectively to B). Then by Bishop — Phelps Theorem (see [15], p. 166) it follows that $\bar{A_1} = \partial A$ and $\bar{B}_1 = \partial B$. Consider the following sets :

$P(A, B) = \{d;\ A \subset S_d(B),\ B \subset S_d(A)\}$, $Q(A, B) = \{d;\ A_1 \subset S_d(\partial B),\ B_1 \subset S_d(\partial A)\}$.

Let us remark that for proving Lemma 2.1. it is sufficient to prove the inclusion : $P(A, B) \subset Q(A, B)$.

Suppose that $P(A, B) \not\subset Q(A, B)$. We argue from this to a contradiction. $P(A, B) \not\subset Q(A, B)$ iff one of the two relations :

(i) there exist $d \in P(A, B)$ and $x_0 \in A_1$, such that $\| x_0 - y \| > d$ for all $y \in \partial B$;

(ii) there exist $b \in P(A, B)$ and $y_0 \in B_1$, such that $\| y_0 - x \| > b$ for all $x \in \partial A$, holds.

Suppose that (i) holds. Then for x_0 we distinguish three cases.

First case. $x_0 \in$ ext B. From a well known result, we may assert : $d(x_0, B) = d(x_0, \partial B)$.

We must only remark that $d \in P(A, B)$ implies $d(x_0, B) < d$ and this last relation is in contradiction with (i) and thus $x_0 \in$ ext B cannot hold.

Second case. $x_0 \in \partial B$. This condition cannot be fulfiled since $d > 0$ and x_0 satisfies the supposition (i)

Third case. $x_0 \in$ int B. As $x_0 \in A_1$, it follows that there exists a supporting hyperplane Π to A at x_0. Then, Π divides H in two halfspaces H_1 and H_2 so that $A \subset H_1$ and $B \cap H_2 \neq \emptyset$. Let y_0 be the point in which the normal to Π at x_0 intersects $\partial B \cap H_2$. Then : $d(x_0, y_0) = d(A, y_0)$. But (i) and the last relation imply $B(y_0, d) \cap A = \emptyset$. Now let us remark that $d \in P(A, B)$ and the condition above are in contradiction.

If we suppose that (ii) holds, we argue to the same contradiction using a similar proof. Thus nor (i) nor (ii) can hold and this completes the proof.

Remark 2.1. The mapping $\partial : \Omega(H) \to \Omega(H)$ defined by $A \to \partial A$ is not Lipschitz continuous, as we can state from the following exemple :

Exemple 2.1. Let us consider in the two dimensional Euclidean space the sets $A = B(0, 1)$, $B = \overline{A \setminus S(0)}$ where $S(0)$ is any circular sector of A with angle 0.

Then, for $\theta \in (0, \pi/2)$, one has : $\rho(A, B) = \sin\frac{\theta}{2}$ and $\rho(\partial A, \partial B) = 1$

But, for all $\theta = (0, \pi/2)$, $\rho(A, B) < \rho(\partial A, \partial B)$. Thus, we have proved that Lemma 2.1. cannot be extended nor in the finite dimensional case.

§3. Approximate continuity. Let H_1, H_2 be two real Hilbert spaces with the norms $\| . \|_1$ and $\| . \|_2$. Let $\Omega(H_2)$ be topologized with the Hausdorff metric.

Lemma 3.1. *Let $F : [0, T] \times H_1 \to \Omega(H_2)$ be a function that satisfies the conditions :*

(a) *For each bounded set $M \subset H_1$, the families $\{F(., x)\}_{x \in M}$ and $\{\partial F(., x)\}_{x \in M}$ are equicontinuous at $t_0 \in [0, T]$;*

(b) *The functions $x \to F(t_0, x)$ and $x \to \partial F(t_0, x)$ are continuous on H_1.*

Suppose that there exist $x_0 \in H_1$ and $y \in H_2$ with the property : $y \in$ int $F(t_0, x_0)$.

Then, there exist $\varepsilon > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ that verify together : $S(y, \varepsilon) \subset$ int $F(t, x)$ for all $t \in [0, T]$, $| t - t_0 | < \delta_1$ and for all $x \in H_1$ with $\| x - x_0 \|_1 < \delta_2$.

Proof : Since $y \in$ int $F(t_0, x_0)$ it follows that there exists $\varepsilon_0 > 0$ so that :

(3) $$d(y, \partial F(t_0, x_0)) > \varepsilon_0 \text{ and } S(y, \varepsilon_0/3) \subset F(t_0, x_0).$$

The conditions (a) and (b) imply that there exist $\delta_1 > 0$ and $\delta_2 > 0$ which for one has:

(4) $$F(t_0, x_0) \subset S_{\varepsilon_0/3}(F(t, x)) ; \partial F(t_0, x_0) \subset S_{\varepsilon_0/3}(\partial F(t, x)) \text{ and}$$

(5) $$F(t, x) \subset S_{\varepsilon_0/3}(F(t_0, x_0)) ; \partial F(t, x) \subset S_{\varepsilon_0/3}(\partial F(t_0, x_0)),$$

for all $t \in [0, T]$, $|t - t_0| < \delta_1$ and for all $x \in H_1$ with $\| x - x_0 \|_1 < \delta_2$.

Suppose that there exists a pair (t, x) which for (4) ~ (5) hold and there exists $y' \in H_2$ so that:

(6) $$\| y' - y \|_2 < \varepsilon_0/3 \text{ and } y' \notin F(t, x) \text{ hold.}$$

We argue from this to a contradiction. From (4) it follows that there exists $y'' \in F(t, x)$ so that $\| y' - y'' \|_2 < \varepsilon_0/3$. Since $F(t, x)$ is bounded the last assertion is equivalent to $y' \in \text{ext}\, F(t, x)$ and from the relations above one has: $d(y', \partial F(t, x)) < \varepsilon_0/3$. Then, it follows that there exists $y_t \in \partial F(t, x)$ so that $\| y' - y_t \|_2 < \varepsilon_0/3$. From (5) one may state that there exists $y_{t_0} \in \partial F(t_0, x_0)$ with the property $\| y_t - y_{t_0} \|_2 < \varepsilon_0/3$. Now, from the last relations it follows:

(7) $$\| y' - y_{t_0} \|_2 < \varepsilon_0/3 + \varepsilon_0/3,$$

where $y_{t_0} \in \partial F(t_0, x_0)$. But (3) and (7) are in contradiction.

Thus, we have proved that the initial supposition is false and now one may conclude that for every t and x for which one has (4) and (5), and for every $y' \in H_2$ with $\| y - y' \|_2 < \varepsilon_0/3$, the relation $y' \in F(t, x)$ holds. As the condition above is equivalent to the assertion: $S(y, \varepsilon_0/3) \subset \text{int}\, F(t, x)$ for all $t \in 0, T]$, $|t - t_0| < \delta_1$ and for all $x \in H_1$ with $\| x - x_0 \|_1 < \delta_2$, the last relation completes the proof.

Let $E \subset [0, T]$ be any measurable set.

Definition 3.1. *$t_0 \in [0, T]$ is a density point of E if:*

(8) $$\lim_{\delta \to 0} \frac{\text{mes}(E \cap [t_0 - \delta, t_0 + \delta])}{2\delta} = 1.$$

Theorem 3.1. *Almost every point of a measurable set E is a density point of E* ([22] p. 229–231).

Let $f : [0, T] \to H$ be any function and $t_0 \in [0, T]$.

Definition 3.2. *The function f is approximately continuous at t_0 if there exists a measurable set $E \subset [0, T]$, so that t_0 is a density point of E and the restriction of f to E is continuous at t_0.*

Theorem 3.2. *Let $f : [0, T] \to H$ be any measurable function who in addition is almost everywhere finite. Then f is almost everywhere approximately continuous on $[0, T]$.*

Let us remark that Luzin's Theorem holds even for measurable and almost everywhere finite (in the norm) functions from $[0, T]$ to any real Hilbert space H (see [23] Theorem 2.1.). Then, the proof of Theorem 3.2. is similar to the proof of Theorem 2 from (22] p. 229–231.

Now, we are able to prove the main result of this paper, the Theorem 4.1.

§4. Time optimal control problem. Let $V \subset H \subset V'$ be three real Hilbert spaces with the norms $\| . \|$, $| . |$, $\| . \|^*$ that satisfy all the properties mentioned in Introduction. Let $F : [0, T] \times V \to \Omega(V')$ be any given function and let M_0, M_1 be two closed, bounded and nonempty sets of V.

Definition 4.1. *A time-optimal solution of (Eq. 1) that transfers M_0 in M_1 is a solution $x : [0, T_0] \to V$ that satisfies:*

(9) $$x(0) \in M_0, \; x(T_0) \in M_1$$

and so that, for every $0 \leqslant T_1 < T_0$ and for every solution $y : [0, T_1] \to V$ of (Eq. 1), one has:

(10) $$y(0) \notin M_0 \text{ or } y(T_1) \notin M_1.$$

Definition 4.2. *We say that the system defined by (Eq. 1) is normal if the following condition is fulfiled:*

(N) *If $x : [0, T_0] \to V$ is the time-optimal solution that transfers M_0 in M_1 then for each $t \in (0, T_0)$, the restriction of x to $[0, t]$ is the time-optimal solution that transfers M_0 in $\{x(t)\}$.*

Remark 4.1. It is obvious that every autonomous system is normal, but the class of all normal systems contains also non-autonomous systems.

Theorem 4.1. *Suppose that the system given by (Eq. 1) is normal and the folowing conditions hold:*

(a) *For each bounded set $M \subset V$, the families $\{F(\cdot, x)\}_{x \in M}$ and $\{\partial F(\cdot, x)\}_{x \in M}$ are equicontinuous on $[0, T_0]$.*

(b) *The functions $x \to \partial F(t, x)$, $x \to \partial F(t, x)$ are continuous on V.*

Let $x : [0, T_0] \to V$ be the time-optimal solution of (Eq. 1) that transfers M_0 in M_1. Then:

(11) $$\dot{x}(t) \in \partial F(t, x(t)) \text{ a.e. on } [0, T_0].$$

Proof: Denote by A_1 the set: $A_1 = \{t ; t \in [0, T_0], \dot{x}(t) \in \text{int}\, F(t, x(t))\}$. We shal prove that A_1 is measurable and: $\text{mes}\, A_1 = 0$. Denote A_2 the set: $A_2 = \{t ; t \in [0, T_0], x(.)$ is approximately continuous at $t\}$. Since $x \in L^2(0, T_0 ; V)$ it follows from the Theorem 3.2 that: $\text{mes}\, A_2 = T_0$. Let A_3 be the set: $A_3 = \{t ; t \in [0, T_0]$, there exists $\lim_{t' \to t} [x(t') - x(t)] (t' - t)$ in $V'\}$.

Now, one may easy state that:

(12) $$\text{mes}(A_2 \cap A_3) = T_0.$$

Let B be the set of all density points of the set $A_2 \cap A_3$ belonging to $A_2 \cap A_3$. Then, from the Theorem 3.1 it follows:

(13) $$\text{mes}\, B = T_0.$$

Let t_0 be any point of B. We shall prove that t_0 cannot belong to A_1 and thus, mes $A_1 = 0$.

Suppose that $t_0 \in A_1$. We argue from this to a contradiction. If $t_0 \in A_1$, then t_0 satisfies simultaneously : $\dot{x}(t_0) \in \operatorname{int} F(t_0)\, x(t_0))$, $\lim\limits_{t \to t_0} [x(t) - x(t_0)]/(t - t_0) = \dot{x}(t_0)$ in V' and there exists a measurable set $A_4 \subset [0, T_0]$, so that t_0 is a density point of A_4 and the restriction of x at A_4 is continuous at t_0.

Let us remark that the first relation above is equivalent to the existence of $\varepsilon_0 > 0$ for which one has : $S(\dot{x}(t_0), \varepsilon_0/3) \subset \operatorname{int} F(t_0, x(t_0))$. On the other hand, the condition (b) and Lemma 3.1 imply that there exist $\delta_1 > 0$ and $\delta_2 > 0$ so that :

$$S(\dot{x}(t_0), \varepsilon_0/3) \subset \operatorname{int} F(t, x(t)), \tag{14}$$

for all $t \in [0, T_0]$, $|t - t_0| < \delta_1$ and for all $x \in V$ with $\| x - x(t_0) \| < \delta_2$.

Since $t_0 \in A_2 \cap A_3$, using (12) one may assert that there exists $t_1 > t_0$ so that : $|t_1 - t_0| < \delta_1$, $\| x(t_1) - x(t_0) \| < \delta_2$ and $[x(t_1) - x(t_0)], (t_1 - t_0) \in S(\dot{x}(t_0), \varepsilon_0/3)$. From the last relations it follows :

$$S(x(t_0), \varepsilon_0/3) \subset F(t_1, x(t_1)). \tag{15}$$

Denote by W the set : $W = S(\dot{x}(t_0), \varepsilon_0/3)$. Then, (15) may be rewritten as follows : $W \subset \operatorname{int} F(t_1, x(t_1))$.

Let us consider the problem :

$$\begin{cases} \dot{y}(t) \in W \text{ a.e. on } [t_0, t_1], \\ y(t_0) = x\, t_0) \end{cases} \tag{16}$$

Let $y_1 : [t_0, t_1] \to V$, be the function :

$$y_1(t) = \left(1 - \frac{t - t_0}{t_1 - t_0}\right) x(t_0) + \frac{t - t_0}{t_1 - t_0}\, x(t_1). \tag{17}$$

Then y_1 is a solution of (16) and $y_1(t_1) = x(t_1)$.

Le us remark that

$$\left\| \frac{x(t_1) - x(t_0)}{t_1 - t_0} - \frac{x(t_1) - x(t_0)}{t_2 - t_0} \right\|^* = \left| \frac{t_2 - t_1}{t_2 - t_0} \right| \cdot \left\| \frac{x(t_1) - x(t_0)}{t_1 - t_0} \right\|^* \tag{18}$$

Then, from (13) and (18) it follows that there exists $t_0 < t_2 < t_1$ such that :

$$\frac{x(t_1) - x(t_0)}{t_2 - t_0} \in W. \tag{19}$$

Let $y_2 : [t_0, t_2] \to V$ be the function defined by :

$$y_2(t) = \left(1 - \frac{t - t_0}{t_2 - t_0}\right) x(t_0) + \frac{t - t_0}{t_2 - t_0}\, x(t_1). \tag{20}$$

As y_2 is a solution of the problem (16) on $[t_0, t_2]$, as $\| x(t_0) - y_2(t) \| < \delta_2$, the condition (14) implies that y_2 is a solution of (Eq. 1) on $[t_0, t_2]$. Now, let us remark that the existence of y_2 is in contradiction with the normality of the system defined by (Eq. 1). Indeed, one may conclude that y_2 is a solution of (Eq. 1) that transfers $x(t_0)$ in $x(t_1)$ on $[t_0, t_2]$, where $t_2 < t_1$. Thus, it follows that the supposition $t_0 \in A_1$ is false and consequently mes $A_1 = 0$ holds, as claimed.

Theorem 4.2. *Let $F : [0, T] \times V \to \Omega^0(V')$ be any given function. Suppose that the system given by the* (Eq. 1) *is normal and the following conditions hold :*

(a′) *For each bounded set $M \subset V$, the family $\{F(., x)\}_{x \in M}$ is equicontinuous on $[0, T_0]$.*

(b′) *The function $x \to F(t, x)$ is continuous on V for all $t \in [0, T_0]$*

Let $x : [0, T_0] \to V$ be the time-optimal solution of the (Eq. 1) *that transfers M_0 in M_1.*

Then x satisfies the (Eq. 11).

Proof : As F satisfies (a′) and (b′) it follows from Lemma 2.1. that F satisfies all the hypothesis of the Theorem 4.1.

Now, we are able to give a Theorem that improves the Theorem 4.1.

Theorem 4.3. *Let $F_i : [0, T_0] \times V \to \Omega(V')$, $i = 1, 2$, be two given functions. Suppose that the system : $\dot{x}(t) \in F_i(t, x(t))$ is normal and the following conditions hold :*

(a″) *F_1 satisfies all the conditions of the* Theorem 4.1.

(b″) *$F_2(t, x) \subset \operatorname{int} F_1(t, x)$ for all $t \in [0, T_0]$ and $x \in V$.*

Let $x : [0, T_0] \to V$ be the time-optimal solution of the equation : $\dot{x}(t) \in F_1(t, x(t))$ a.e. on $[0, T_0]$ that transfers M_0 in M_1.

Then x is also the time-optimal solution of the equation : $\dot{x}(t) \in F_1(t, x(t)) \setminus F_2(t, x(t))$ a.e. on $[0, T_0]$ that transfers M_0 in M_1.

The proof being very simple is left to the reader.

Remark 4.2. Now, it is obvious that the conditions (a) and (b) in the Theorem 4.1. may be substituted by the following conditions : "the exterior boundary" of $F(., x)$ is continuous on $[0, T_0]$ uniformely on every bounded set $M \subset V$ and "the exterior boundary" of $F(t, .)$ is continuous on V for all $t \in [0, T_0]$.

§5. Time optimal control for evolution equations in Hilbert spaces. Let V be a real Hilbert space with the norm $\| . \|$ and U another real Hilbert space with the norm $\| . \|_U$. Denote by $L(V, V')$ the class of all linear, continuous operators from V to V'. Let $U_0 \in \Omega(U)$ be any given set and consider the equation :

$$\dot{x}(t) + A(t)x(t) \in BU_0 \text{ a.e. on } [0, T_0], \tag{21}$$

where : $A(t) \in L(V, V')$, while B is a linear, continuous, surjective and invertible operator from U to V' whose inverse is continuous.

Suppose that $A(t)$ satisfies the conditions :

(22) *$t \to A(t)$ is continuous in the topology of $L(V, V')$ on $[0, T_0]$.*

(23) *$\| A(t)x \|^* \leqslant A \| x \|$ for all $x \in V$, where $A > 0$ is independent of t.*

Consider the equation:

(24) $$\dot{x}(t) + A(t)x(t) = Bu(t) \text{ a.e. on } [0, T_0],$$

where u may be any measurable function from $[0, T_0]$ to U_0.

Theorem 5.1. *Let the system defined by the* (Eq. 21) *be normal and sup- pose that U is separable and $A(t)$ verifies* (22)~(23).

Let $x : [0, T_0] \to V$ be the time-optimal solution of the (Eq. 21) *that tran- fers M_0 in M_1. Then:*

(i) *There exists a measurable function $u : [0, T_0] \to U_0$ such that the pa- (x, u) verifies the* ((Eq. 24) *and:*

(ii) $u(t) \in \partial U_0$ *a.e. on* $[0, T_0]$.

Proof: (i) As U is separable and $A(t)$ satisfies (22) ~ (23), one ma- use the Theorem 2.2. from [20] and we state that (i) is a direct consequen- of the above Theorem. (ii) From the Theorem 4.1. it follows: $\dot{x}(t) \in$ $\in \partial(-A(t)x(t) + BU_0)$ a.e. on $[0, T_0]$. But the last relation is equiva- lent to:

(25) $$\dot{x}(t) \in \partial BU_0 - A(t)x(t) \text{ a.e. on } [0, T_0].$$

Now, it is easy to see that (25) implies (ii) and this completes the proo-

Exemple 5.1. Let Ω be any open and bounded set in R^n with suffi- cient smooth boundary Γ and take $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$ and $A(t) = -\Delta$ (the notations being those of Lions [19]).

It is well known that $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the natural iso- morphism between $H_1^0(\Omega)$ and $H^{-1}(\Omega)$. Let $U_0 \subset H^{-1}(\Omega)$ be any bounde- and closed set with nonempty interior and consider the problem:

(26) $$\begin{cases} \dfrac{\partial x(t)}{\partial t} - \Delta x(t) \in U_0 \text{ a.e. on } [0, T_0], \\ x(0, \xi) = x_0(\xi),\ x(T_0, \xi) = x_1(\xi), \\ x(t, \xi)|_\Gamma = 0. \end{cases}$$

Let x be the time-optimal solution of 26) that transfers x_0 in x_1 o- $[0, T_0]$. Then there exists a measurable function $u : [0, T_0] \to \partial U_0$ such that the pair (x, u) verifies:

(27) $$\begin{cases} \dfrac{\partial x(t)}{\partial t} - \Delta x(t) = u(t) \text{ a.e. on } [0, T_0], \\ x(0, \xi) = x_0(\xi),\ x(T_0, \xi) = x_1(\xi), \\ x(t, \xi)|_\Gamma = 0. \end{cases}$$

For similar results in the case in which U_0 is in H see for instance [10] and [19] Ch. 3 §17.3, pp. 273 ~280.

The most part of the results of this paper have been anounced withou- proof in [25].

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