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ON THE EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF A BOUNDARY VALUE PROBLEM

BY

N. CALISTRU

1. Introduction. We consider the problem

$$\mathcal{L} u \equiv \frac{\partial^2}{\partial x \partial t}\left[p(x)\frac{\partial u}{\partial x}\right] + q(x)\frac{\partial u}{\partial t} = f(t, x, u, u_t, u_x, u_{xt}, u_{xx}), \tag{1}$$

$$u(0, x) = u_0(x), \qquad 0 \leqslant x \leqslant a, \tag{2}$$

$$\alpha_1 u(t, 0) + \beta_1 u_x(t, 0) = 0, \quad \alpha_2 u(t, a) + \beta_2 u_x(t, a) = 0, \quad t \geqslant 0, \tag{3}$$

where t, x are independent variables; $u = u(t, x)$ is the unknown function, u_t, u_x, u_{xt}, u_{xx} are the partial derivatives of u; p, q, f, u_0 are given functions and α_1, α_2, β_1, β_2 are real constants which satisfy the condition

$$(|\alpha_1| + |\beta_1|)(|\alpha_2| + |\beta_2|) > 0.$$

In an earlier note, [4], we have proved an existence and uniqueness theorem for (1) — (3).

In this paper we shall give an existence and global uniqueness theorem for the solution of problem (1) — (3), in which the behaviour of the solution for $t \to +\infty$ is a priori given. Among the first results in this way have been stated and proved two decades ago, in connection with the Cauchy Problem for ordinary differential equations [1], [2], and for Pfaff's systems [3].

2. Existence, uniqueness and asymptotic behaviour of the solution of problem (1) — (3).

First, we specify some notations and hypotheses which will be used throughout this paper. We note by a and $\bar{\tau}$ two positive numbers, the first is arbitrarily fixed (but finite), the second is arbitrary, finite or not. By $\mathbf{R}^5$ we denote the 5-dimensional Euclidian space, endowed with the norm $\|.\| = \|.\|_1$ defined by $\|z\|_1 = \sum_{i=1}^{5} |z_i|$, for every $z = (z_1, z_2, z_3, z_4, z_5) \in \mathbf{R}^5$.

For every function $\varphi(t, x)$ continuous together with its partial derivatives φ_t, φ_x, φ_{xt}, φ_{xx}, we denote

$\| \varphi(t, x) \|_1 = | \varphi(t, x) | + | \varphi_t(t, x) | + | \varphi_x(t, x) | + | \varphi_{xt} (t, x) | + | \varphi_{xx} (t, x) |$,
for every (t, x) fixed. Finally, with A and B we denote the sets,

$$A = \{0 \leqslant t < \bar{\tau}\,;\ 0 \leqslant x \leqslant a\} \text{ and } B = A \times \mathbf{R}^5 .$$

Now, we consider the problem (1) — (3) under the following hypotheses:

(II$_1$) a. $p \in C^1([0, a])$, $q \in C([0, a])$, $u_0 \in C^2([0, a])$ and

$$\alpha_1 u_0 (0) + \beta_1 u_0' (0) = 0, \quad \alpha_2 u_0 (a) + \beta_2 u_0' (a) = 0, \quad \left(\,' = \frac{d}{dx} \right);$$

b. The problem

$$Qy \equiv \frac{d}{dx}\left[p (x) \frac{dy}{dx}\right] + q(x) y = 0, \tag{4}$$

$$\alpha_1 y (0) + \beta_1 y' (0) = 0, \quad \alpha_2 y (a) + \beta_2 y' (a) = 0 \tag{5}$$

has only the trivial solution, and the operator Q subject to the conditions, (5), admits a Green's function $G (x\,; \xi)$ for $x, \xi \in [0, a]$.

c. $$p (x) \geqslant P_0 > 0, \quad | p' (x) | \leqslant P_1 , \quad | q (x) | \leqslant Q_0 ,$$
$$| G (x\,; \xi) | \leqslant H, \quad | G_x (x\,; \xi) | \leqslant H,$$

or all $x, \xi \in [0, a]$. (P_0, P_1, Q_0, H are nonnegative constants).

(II$_2$) $f \in C (B)$ and furthermore:

a. There exists a nonnegative and continuous function $\lambda (s)$ defined on $[0, \bar{\tau} + a)$, such that

$$| f (t, x, z) - f (t, x, \bar{z}) | \leqslant \lambda (t + x) \| z - \bar{z} \|_1 ,$$

for all $z, \bar{z} \in \mathbf{R}^5$;

b. There exist three real numbers $\alpha > \sigma$, $K \geqslant 0$ and T, $0 \leqslant T < \bar{\tau} + a$, such that

$$| f (t, x, 0) | \leqslant K \lambda (t + x) \exp\left(\alpha \int_0^{t+x} \lambda (s)\, ds\right),$$

for every $x \in [0, a]$ and $t \geqslant T$. (σ is a real constant which will be defined from below by the relation (21)).

Theorem. *If the hypotheses* (II$_1$), (II$_2$) *are satisfied, then there exists a unique function* $\varphi(t, x)$ *continuous, with partial derivatives* φ_t, φ_x, φ_{xt}, φ_{xx}, φ_{xxt} continuous for $(t, x) \in A$, *which fulfils* (1) — (3). *In addition, the estimation*

$$\| \varphi (t, x) \|_1 = O\left(\exp\left(\alpha \int_0^{t+x} \lambda(s)\, ds\right)\right), \tag{6}$$

holds, for $x \in [0, a]$ *and* $t \to +\infty$.

Proof. Let $S_\alpha (A)$ be the set of functions φ, $\varphi : A \to \mathbf{R}$, continuous together with their partial derivatives φ_t, φ_x, φ_{xt}, φ_{xx}, and satisfy the relation

$$\| \varphi (t, x) \|_1 = O\left(\exp\left(\alpha \int_0^{t+x} \lambda (s)\, ds\right)\right), \tag{7}$$

for all $x \in [0, a]$ and $t \to \bar{\tau}$.

The set $S_\alpha (A)$ is a linear space. As in [2], we consider the norm

$$\| \varphi \| = \sup_{(t,x) \in A} \left\{ \| \varphi (t, x) \|_1 \exp\left(- \alpha \int_0^{t+x} \lambda (s)\, ds \right)\right\}. \tag{8}$$

It is easy to verify that $S_\alpha (A)$ is a Banach space. We have shown, (see [4]), that problem (1) — (3) is equivalent to the problem of finding continuous solutions $u (t, x)$ which admit partial derivatives u_t, u_x, u_{xt}, u_{xx} continuous, of the integro-differential equation

$$u (t, x) = u_0 (x) + \int_0^a G (x\,; \xi) \int_0^t F (\tau, \xi, u (\tau, \xi))\, d\tau d\xi , \tag{9}$$

where

$$F (t, x, u (t, x)) = f (t, x, u (t, x), u_t (t, x), u_x (t, x) (u^{\xi}_{k}, t, x), u_{xx} (t, x)).$$

Now, we consider the operator $\mathcal{T}$ defined by

$$(\mathcal{T} \varphi) (t, x) = u_0(x) + \int_0^a G (x\,; \xi) \int_0^t F (\tau, \xi, \varphi (\tau, \xi))\, d\tau d\xi , \tag{10}$$

for every $\varphi \in S_\alpha (A)$, and we shall show that all hypotheses of Banach's fixed point theorem hold.

First, we shall prove that $\mathcal{T}$ maps $S_\alpha (A)$ into $S_\alpha (A)$. Let $\varphi \in S_\alpha (A)$ and let $\Phi (t, x) = (\mathcal{T} \varphi) (t, x)$ be the corresponding image, i.e.

$$\Phi (t, x) = u_0 (x) + \int_0^a G (x\,; \xi) \int_0^t F (\tau, \xi, \varphi (\tau, \xi))\, d\tau d\xi . \tag{11}$$

At every $(t, x) \in A$, the function $\Phi (t, x)$ is continuous together with its partial derivatives:

$$\Phi_t (t, x) = \int_0^a G (x\,; \xi) F (t, \xi, \varphi (t, \xi))\, d\xi , \tag{12}$$

$$\Phi_x(t,x) = u_0'(x) + \int_0^a G_x(x;\xi) \int_0^t F(\tau,\xi,\varphi(\tau,\xi))\,d\tau\,d\xi, \tag{13}$$

$$\Phi_{xt}(t,x) = \int_0^a G_x(x;\xi)\,F(t,\xi,\varphi(t,\xi))\,d\xi, \tag{14}$$

$$\Phi_{xx}(t,x) = u_0''(x) + \frac{1}{p(x)}\int_0^t F(\tau,x,\varphi(\tau,x))\,d\tau - \tag{15}$$

$$-\int_0^a \frac{p'(x)\,G_x(x;\xi) + q(x)\,G(x;\xi)}{p(x)} \int_0^t F(\tau,\xi,\varphi(\tau,\xi))\,d\tau d\xi.$$

By virtue of hypothesis (H_1), we have

$$\|\Phi(t,x)\|_1 = \|u_0(x)\|_1 + 2H\int_0^a |F(t,\xi,\varphi(t,\xi))|\,d\xi + \tag{16}$$

$$+ H(2 + P_0^{-1}(P_1 + Q_0))\int_0^a\int_0^t |F(\tau,\xi,\varphi(\tau,\xi))|\,d\tau d\xi +$$

$$+ P_0^{-1}\int_0^t |F(\tau,x,\varphi(\tau,x))|\,d\tau,$$

for every $(t,x) \in A$ fixed

For the estimation of integrals from (16), we shall use (8), hypothesis (H_2) and the functions $g_1(t)$, $g_2(t)$, $h(t,x)$ defined by

$$g_1(t) = \exp\left(\alpha\int_0^t \lambda(s)\,ds\right),\quad g_2(t) = \exp\left(\alpha\int_t^{t+a}\lambda(s)\,ds\right) - 1,$$

$$h(t,x) = \int_0^a \exp\left(\alpha\int_{t+x}^{t+\xi}\lambda(s)\,ds\right)d\xi,$$

for $t \geq 0$, $x \in [0,a]$. Evidently, $g_1(t)$ is continuous, nonnegative and nondecreasing for $t \geq 0$, $g_2(t)$ and $h(t,x)$ are continuous and nonnegative for $t > 0$, $x \in [0,a]$. Since

$$\lim_{t\to+\infty} g_2(t) = 0 \text{ and } \lim_{t\to\infty} h(t,x) = a$$

for all $x \in [0,a]$, there exist constants L_1, L_2, $0 \leq L_1, L_2 < +\infty$, such that $g_2(t) \leq L_1$, $h(t,x) \leq L_2$, for $t \geq 0$, $x \in [0,a]$.

Finally, because of

$$|F(t,x,\varphi(t,x))| \leq \lambda(t+x)\,\|\varphi(t,x)\|_1 + K\lambda(t+x)\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right) \leq$$

$$\leq (\|\varphi\| + K)\,\lambda(t+x)\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right)$$

for every $(t,x) \in A$, we have:

$$\int_0^a |F(t,\xi,\varphi(t,\xi))|\,d\xi \leq L_1\frac{\|\varphi\| + K}{\alpha}\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right), \tag{17}$$

$$\int_0^t |F(\tau,x,\varphi(\tau,x))|\,d\tau \leq \frac{\|\varphi\| + K}{\alpha}\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right), \tag{18}$$

$$\int_0^a\int_0^t |F(\tau,\xi,\varphi(\tau,\xi))|\,d\tau d\xi \leq L_2\frac{\|\varphi\| + K}{\alpha}\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right). \tag{19}$$

The last three relations permit to estimate the right-hand part of (16) in the following form

$$\|\Phi(t,x)\|_1 \leq \left(\|u_0\| + \frac{\sigma}{\alpha}(\|\varphi\| + K)\right)\exp\left(\alpha\int_0^{t+x}\lambda(s)\,ds\right), \tag{20}$$

for every $(t,x) \in A$, where σ is the constant defined by

$$\sigma = 2H(L_1 + L_2) + P_0^{-1}(1 + HL_2(P_1 + Q_0)). \tag{21}$$

(This is the constant which appears in hypothesis (H_2)). The relations (11) — (15) and (20) show that $\Phi \in S_\alpha(A)$.

The proof will be finished if we show that the operator $\mathcal{T}$ is a contraction. Indeed, let φ, $\psi \in S_\alpha(A)$, and $\Phi = \mathcal{T}\varphi$, $\Psi = \mathcal{T}\psi$. Then,

$$\|\Phi(t,x) - \Psi(t,x)\|_1 \leq 2H\int_0^a |F(t,\xi,\varphi(t,\xi)) - F(t,\xi,\psi(t,\xi))|\,d\xi +$$

$$+ P_0^{-1}\int_0^t |F(\tau,x,\varphi(\tau,x)) - F(\tau,x,\psi(\tau,x))|\,d\tau +$$

$$+ H(2 + P_0^{-1}(P_1 + Q_0)) \int_0^a \int_0^t |F(\tau, \xi, \varphi, (\tau, \xi)) - F(\tau, \xi, \psi(\tau, \xi))| \, d\tau \, d\xi \leqslant$$

$$\leqslant \frac{\sigma}{\alpha} \| \varphi - \psi \| \exp\left(\alpha \int_0^{t+x} \lambda(s)\, ds\right).$$

Hence, for every $t \geqslant 0$, $x \in [0, a]$,

$$\| \Phi(t, x) - \Psi(t, x) \|_1 \leqslant \exp\left(-\alpha \int_0^{t+x} \lambda(s)\, ds\right) \leqslant \frac{\sigma}{\alpha} \| \varphi - \psi \|,$$

and by (8)

$$\| \Phi(t, x) - \Psi(t, x) \| \leqslant \frac{\sigma}{\alpha} \| \varphi - \psi \|.$$

Since $\alpha > \sigma$, $\mathcal{T}$ is a contraction.

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Faculty of Mathematics
University of Iași
R.S. România

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MAJORANT PROPERTIES OF SOLUTIONS OF DIFFERENCE EQUATIONS

BY

S.B. JAIYESIMI

1. Introduction. Consider the $(r+1)^{\text{th}}$ order equations

(1.1) $\Delta y_{n-1}(\lambda) = \lambda \beta_n y_{n+r}(\lambda)$, $r \geqslant 0$, $y_n(0) = C$, $y_0(\lambda) = f(\lambda)$, λ a parameter.

A purely analytic solution of this equation has been found [1].
A rapidly convergent computational techniques was developed, transforming the given equation (1.1) to

$$\Delta y_{n-1}^{(k)} = \lambda \beta_n y_{n+r}^{(k-1)}, \quad r \geqslant 0, \quad k \geqslant 1. \tag{1.2}$$

In order to generalize the results it is desirable, first to discusss the characteristics of certain majorant equations and their solutions. We therefore consider an arbitrarily high order equation, which is a generalized form of (1.1), given by

$$y_n(\lambda) - \sum_{i=1}^{m} \mu_i(n) y_{n-i}(\lambda) = \lambda y_{n+r}(\lambda), \tag{1.3}$$
$$r > 0, \ \lambda \geqslant 0, \ y_n(0) = C, \ y_0(\lambda) = f(\lambda).$$

We proceed to show that the solution of (1.1), in general, is a majorant of that of (1.3).

2. Majorized equations. We again consider an $(m+r)^{\text{th}}$ order difference equation in a more general form but with a single parameter λ. This is of the form (1.3). Suppose $\exists$ constants $\mu_i > 0$ $\mu_i(n) \leqslant \mu_i$, $1 \leqslant i \leqslant m$. Then we can write

$$y_n^{(k)} - \sum_{i=1}^{m} \mu_i y_{n-i}^{(k)} = \lambda y_{n+r}^{(k-1)}, \quad r > 0, \quad \lambda \geqslant 0. \tag{2.1}$$

For $i = 1, 2, \ldots, m-1$, let

$$y_i^k = \begin{cases} P_i > 0, & k = 0, \\ 0, & k < 0. \end{cases} \tag{2.2}$$

We note that, since all quantities are non-negative,

i) $y_n^{(k)} \geqslant 0 \ \forall k, \ n \geqslant 0$;