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THE MINIMALITY OF WEYL'S SYSTEM OF AXIOMS FOR THE AFFINE GEOMETRY OVER AN UNITARY RING

BY

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Introduction. Half century ago, Hermann Weyl in this treatise „Raum, Zeit, Materie", [10] has formulated the notion of punctual affine space, based on the properties of linearity of the associate vector space. Weyl's axiomatic gives a straight way to introduce the punctual affine spaces. But Weyl's system is not sufficiently formalised and, more than this, it is not a minimal system [4].

P. K. Raschewsky, [9], has reformulated Weyl's system and pointed out a more clear axiomatic. He replaced „the triangle axiom" with „the parallelogram axiom". Even Raschewski's construction is not built up from independent axioms, [4].

S. Golab [3] has made a very interesting study on the axiomatics of Weyl, Raschewsky etc.

In the present paper we solve the problem of minimality of Weyl's system. Namely, we weaken Weyl's axioms and drop the axiom $x + y = y + x$ for the punctual affine space on an apriori given unitary ring. This new system satisfies the minimality principle.

§ 1. The notion of punctual affine space over a ring. Let M be a set whose elements $A, B, C, \ldots$ are called points. Consider the set $M \times M$. Every element $(A, B) \in M \times M$ is called an oriented segment of M, with origin A and extremity B. We denote $(A, B) = \overline{AB}$.

Let ρ be an equivalence relation on the set $M \times M$, $TM = (M \times M)/\rho$ the quotient set and $p : M \times M \to TM$ the natural projection. We denote $p(\overline{AB}) = \overrightarrow{AB}$ and we say that $\overrightarrow{AB}$ is the vector of the set M determined by the oriented segment $\overline{AB}$ with respect to the equivalence ρ. We shall denote vectors by $\overrightarrow{AB}, \overrightarrow{CD}, \ldots, x, y, z$ etc.

Obviously : $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \overrightarrow{AB} = \overrightarrow{CD}$.

Conversely, if the set TM of vectors is given together with a surjection $p : M \times M \to TM$, then the equivalence relation ρ is well determined by $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow p(\overline{AB}) = p(\overline{CD})$.

Let Γ be an unitary ring and $1 \in \Gamma$ its identity.

Definition 1. *We call Γ-affine space structure on the nonempty set M the following data:*

a. *An equivalence relation ρ on the set $M \times M$, which defines the quotient set TM, whose elements are called the vectors of M.*

b. *A mapping*

$$* \; \Gamma \times TM \to TM, * \; (\alpha, x) = \alpha * x, \quad \forall \alpha \in \Gamma, \quad \forall x \in TM.$$

c. *The axioms of the following two groups are verified.*

Definition 2. *The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a Γ-affine space structure on the nonempty set M, is called a Γ-affine space.*

Evidently, $M \neq \Phi \Rightarrow TM \neq \Phi$.

First group of axioms.

I_1. $\forall A \in M, \ \forall x \in TM, \ \exists B \in M : \overrightarrow{AB} = x$,

I_2. $\forall A, B, A' \in M : \overrightarrow{AB} = \overrightarrow{A'B} \Rightarrow A = A'$,

I_3. $\forall A, B, C, A', B', C' \in M : \overrightarrow{AB} = \overrightarrow{A'B'}, \ \overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$

The axioms I_3 is called by prof. S. G o l a b "M i r o n-O p a i ţ axiom", [3].

Some consequences :

Proposition 1. (i) $\forall A, A' \in M ; \ \overrightarrow{AA} = \overrightarrow{A'A'}$.

(ii) $\forall A, B, A', B', \in M : \overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{BA} = \overrightarrow{B'A'}$

(iii) $\forall A \in M, \ \forall x \in TM, \ \exists_1 B \in M : \overrightarrow{AB} = x$.

The proofs are elementary and can be omitted.

Let $(x, y) \in TM \times TM$ and $A \in M$. Then $\exists_1 B \in M : \overrightarrow{AB} = x$ and $\exists_1 C \in M : \overrightarrow{BC} = y$. The vector $\overrightarrow{AC} = z$ does not depend on the point A. It is called the sum of vectors x and y. We write $z = x + y$.

Definition 3. *The mapping $+ : (x, y) \in TM \times TM \to x + y \in TM$, $\forall x, y$ is called the opperation of addition of vectors from M.*

We have $z = x + y$ or

(1) $$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \ (x = \overrightarrow{AB}, \ y = \overrightarrow{BC}, \ A \in M)$$

Theorem 1. *The pair $(TM, +)$ is a group.*

Proof. From (1), $\forall x, y, z \in TM : x = \overrightarrow{AB}, \ y = \overrightarrow{BC}, \ z = \overrightarrow{CD}$, $x + (y + z) = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) = \overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = (x + y) + z$. Let $\vec{0} = \overrightarrow{AA} = \overrightarrow{BB}$, $\forall A, B \in M$. We have $\forall x \in TM : x + \vec{0} = \overrightarrow{AB} + \overrightarrow{BB} = \overrightarrow{AB} = x = \overrightarrow{AA} + \overrightarrow{AB} = \vec{0} + x$. Let $-x = -\overrightarrow{AB} = \overrightarrow{BA}$. Then $x + (-x) = \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0} = \overrightarrow{BA} + \overrightarrow{AB} = (-x) + x$. q.e.d.

We denote $x + (-y) = x - y$.

Generally the group $(TM, +)$ is not abelian.

Consider the mapping :

$$t : (x, y) \in TM \times TM \to -(x + y) + (y + x) \in TM.$$

The group $(TM, +)$ is abelian iff M has the property $t(x, y) = \vec{0}$ $\forall x, y \in TM$.

Proposition 2. *If $x = \overrightarrow{AB} = \overrightarrow{CD'}$, $y = \overrightarrow{AC} = \overrightarrow{BD}$, then $t(x, y) = \overrightarrow{DD'}$.*

Proposition 3. $\forall A, B, A', B' \in M : \ \{\overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{AA'} = \overrightarrow{BB'}\} \Leftrightarrow \{t(x, y) = \vec{0}, \ \forall x, y \in TM\}$.

The second group of axioms contains compatibility conditions between the mapping $*$ and the applications $+$ and . from Γ and M.

Second group of axioms.

If the opperation $+ : TM \times TM \to TM$ is given by definition 3, then :

II_1. $\forall \alpha \in \Gamma, \ \forall x, y \in TM :$ $\quad \alpha * (x + y) = \alpha * x + \alpha * y$,

II_2. $\forall \alpha, \beta \in \Gamma, \ \forall x \in TM :$ $\quad (\alpha + \beta) * x = \alpha * x + \beta * x$,

II_3. $\forall \alpha, \beta \in \Gamma, \ \forall x \in TM :$ $\quad \alpha * (\beta * x) = (\alpha . \beta) * x$,

II_4. $\forall x \in TM \setminus \{\vec{0}\} :$ $\quad 1 * x \neq \vec{0}$.

With this the list of axioms for the Γ-affine space structure is finished.

Consequences of the axioms of the first and second group.

Proposition 4. $\forall \alpha \in \Gamma, \ \forall x, y \in TM :$

(i) $\alpha * (x - y) = \alpha * x - \alpha * y$,

(ii) $\alpha * \vec{0} = \vec{0}, \ \alpha * (-x) = -\alpha * x$.

Proposition 5. $\forall \alpha, \beta \in \Gamma, \ \forall x \in TM :$

(i) $(\alpha - \beta) * x = \alpha * x - \beta * x$,

(ii) $0 * x = \vec{0}, \ (-\alpha) * x = -\alpha * x$.

Theorem 2. $\forall x \in TM$, $1 * x = x$, *1 being the identity of Γ.*

Indeed, we have $1 * [x - 1 * x] = 1 * x - (1.1) * x = 1 * x - 1 * x = \vec{0}$. From the axiom II_4 it follows $x - 1 * x = \vec{0}$. Then $x = 1 * x$, $\forall x \in TM$. q.e.d.

Theorem 3. $\forall x, y \in TM : x + y = y + x$.

Indeed, $(1 + 1) * (x + y) = 1 * (x + y) + 1 * (x + y) = (x + y) + (x + y) = (1 + 1) * x + (1 + 1) * y = (x + x) + (y + y)$. But $(x + x) + (y + y) = (x + y) + (y + x)$ and theorem 1 given $x + y = y + x$. q.e.d.

Theorem 4. *In the Γ-affine space M is verified the following „parallelogram property" :*

$$\forall A, B, A', B' \in M : \overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{AA'} = \overrightarrow{BB'}.$$

Proposition 3 and theorem 3 yield the statement.

Of course, on this way we can study the Γ-affine spaces, which R. Permutti [8], A. Barlotti [1] etc. [2, 6, 7] have studied.

In another paper we shall consider the general case, where the axiom II_4 is omitted.

§ 2. Noncontradiction and independence of the axioms of Γ-affine spaces. Using Hilbert's method of models we shall prove the noncontradiction and independence of the axioms from the both groups of axioms. Some of the following models are taken from a joint paper with F. Radó, [5].

Theorem 5. *The axioms of the Γ-affine space are noncontradictory.*

Proof. The unitary ring Γ being given, consider $M = \Gamma$. We have $\overline{AB} = (\alpha, \beta)$, $\overline{CD} = (\gamma, \delta)$ and we put $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \beta - \alpha = \delta - \gamma$. The vector $\overrightarrow{AB}$ can be identified with $x = \beta - \alpha \in \Gamma$. Define the mapping $*$ by $\alpha * x = \alpha \,.\, x$, $\forall \alpha \in \Gamma$, $\forall x \in TM$. All axioms from the both groups are verified. q.e.d.

Theorem 6. *Let Γ be a given unitary ring.*

(i) *The axioms of the first group are independent.*

(ii) *If the axioms of the first group are verified, then the axioms of the second group are independent in the system of the first and second group of axioms.*

Proof. 1°. We take $M = \Gamma$, $\overline{AB} = (\alpha, \beta)$, $\overline{CD} = (\gamma, \delta)$ and put $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \alpha = \gamma$, $\beta = \delta$. We have $\overrightarrow{AB} = (\alpha, \beta)$, $\vec{0} = (0, 0)$. Define $\alpha * (x_1, x_2) = (\alpha \,.\, x_1, \alpha \,.\, x_2)$, $\forall \alpha \in \Gamma$, $\forall x = (x_1, x_2) \in TM$. All axioms of the first group are verified, except the axiom I_1.

2°. Consider $M = \Gamma$, $\overline{AB} = (\alpha, \beta)$, $\overline{CD} = (\gamma, \delta)$ and $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \beta = \delta$. The vector $x = \overrightarrow{AB}$ is given by the element $\beta \in \Gamma$ and the vector nul $\vec{0}$ is given by $0 \in \Gamma$. Then we define $\alpha * x = \alpha \,.\, \beta$, $\forall \alpha \in \Gamma$, $\forall x \in TM$. The axioms I_1, I_3 are verified and I_2 is not verifies.

3°. Let $M = \Gamma$, $\overline{AB} = (\alpha, \beta)$, $\overline{CD} = (\gamma, \delta)$ be and $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \alpha + \beta = \gamma + \delta$. If $\overrightarrow{AB}$ is given by $\alpha + \beta$ and the vector $\vec{0}$ is given by $\vec{0} = \alpha - \alpha$, then we define $\gamma * \overrightarrow{AB} = \gamma \,.\, (\alpha + \beta)$. There are verified the axioms of the first group less the axiom I_3.

4°. Let $M = \Gamma \times \Gamma$ be and $\overline{AB} = [(\alpha_1, \alpha_2), (\beta_1, \beta_2)]$, $\overline{CD} = [(\gamma_1, \gamma_2), (\delta_1, \delta_2)]$, $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \{\beta_1 - \alpha_1 = \delta_1 - \gamma_1,\ \beta_2 - \alpha_2 = \delta_2 - \gamma_2\}$. The vector $\overrightarrow{AB}$ is given by (x_1, x_2), $x_1 = \beta_1 - \gamma_1$, $x_2 = \beta_2 - \gamma_2$. The vector nul $\vec{0}$ has its components $(0, 0)$. Then, we have $x + y = (x_1 + y_1, x_2 + y_2)$ for $x = (x_1, x_2)$, $y = (y_1, y_2)$. Define, [5], $\alpha * x$ by :

$$\alpha * (x_1, x_2) = \begin{cases} (0, \alpha \,.\, x_2), & \text{if } x_2 \neq 0, \\ (\alpha \,.\, x_1, 0), & \text{if } x_2 = 0. \end{cases}$$

The axiom II_1 fails, but the other axioms of the first and second group are verified.

5°. $M = \Gamma$, $\overline{AB} = (\alpha, \beta)$, $\overline{CD} = (\gamma, \delta)$, $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \beta - \alpha = \delta - \gamma$. Then $\overrightarrow{AB} = x = \beta - \alpha$ and $\vec{0} = 0 \in \Gamma$. We have $\overrightarrow{AB} + \overrightarrow{BC} = (\beta - \alpha) + (\gamma - \beta) = \gamma - \alpha$. Define $\alpha * x = x$, $\forall \alpha \in \Gamma$, $\forall x \in TM$. Except the axiom II_2 the other axioms are verified.

6°. Let $M = \Gamma \times \Gamma$ be and the vectors $\overrightarrow{AB}$ as in the model 4°. Then $x = (x_1, x_2)$, $y = (y_1, y_2)$ we have $x + y = (x_1 + y_1, x_2 + y_2)$. Define $\alpha * x = \alpha * (x_1, x_2) = (\alpha \,.\, x_2, \alpha \,.\, x_1)$. All axioms of the first and second group are verified, except II_3.

7°. Let $M = \Gamma$ be and the vectors $\overrightarrow{AB}$ as in the model 5°. Define $\alpha * x = \vec{0}$, $\forall \alpha \in \Gamma$, $\forall x \in TM$. The axiom II_4 is not verified and the other axioms of the system hold.

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