BIBLIOGRAPHY

1. Jaiyesimi, S.B. — *On The Analytic Solution of Certain Linear Difference Equations*, Anal. şt. Univ. Iaşi, S.I. a, vol. XXII, 1976, pp. 59—66.
2. Jaiyesimi, S.B. — *On The Summation of Certain Multiple Series*, An. şt. Univ. Iaşi, S. I a, vol. XXIII, 1977, pp. 43—49.
3. Trjitzinsky, W.J. — *Linear Difference Equations Containing a Parameter*, Annali di Matematica, vol. 14 (1936), pp. 181—214.

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SAINT-VENANT'S PROBLEM FOR NONLOCAL ELASTIC SOLIDS

BY

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1. Introduction. The basic equations in the linear nonlocal theory of elasticity was presented in various papers (see e.g. Kröner [1], Eringen [2]). Some boundary value problems concerning this theory are considered by Eringen [3], Edelen [4], Chiriţă [5].

In this paper, we consider Saint-Venant's problem in nonlocal theory of elasticity and we extend the results obtained in the classical elasticity by Ieşan [6]. Firstly, we define the plane strain and we prove the existence theorem for a plane boundary value problem. With the help of four plane strain problems we solve Saint-Venant's problem. In the last part of the paper we study some special problems. In the first of them it is interesting to note that the displacement distribution is the same as in classical elasticity.

2. The basic equations of nonlocal elasticity. We consider a nonlocal elastic inhomogeneous anisotropic medium occupying a region V of the three-dimensional space. Suppose the region to be referred to a fixed rectangular Cartesian coordinate system x_k $(k = 1, 2, 3)$ and let us denote by x the point (x_1, x_2, x_3). Throughout this paper, unless otherwise specified, we shall employ the usual summation and differentiation conventions; we employ also the usual indicial notation, Latin and Greek subscripts having the respective ranges (1, 2, 3) and (1, 2).

In what follows, we consider the linear nonlocal static theory of elasticity. Therefore, if u_i denote the components of the displacement vector, then the components of the strain tensor are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

The constitutive equations, which characterize the mechanical response of the nonlocal medium, are assumed as follows

$$t_{ij} = C_{ijkl}e_{kl} + \int_V C'_{ijkl}(x, z)\, e_{kl}(z)\, dV(z), \tag{2.2}$$

where t_{ij} are the components of the stress tensor. The local elastic characteristics $C_{ijkl}(x)$ and the nonlocal elastic characteristics $C'_{ijkl}(x,z)$ satisfy the symmetry relations

$$C_{ijkl} = C_{klij} = C_{jikl}, \quad C'_{ijkl}(x, z) = C'_{klij}(z, x) = C'_{jikl}(x, z). \tag{2.3}$$

If f_i are the components of the body force vector, the equilibrium equations are

$$t_{ji,j} + f_i = 0, \text{ in } V. \tag{2.4}$$

Let us assume that on the surface ∂V of the region V, we have the boundary conditions

$$t_{ji} n_j = \tilde{t}_i, \tag{2.5}$$

where $\tilde{t}_i$ are prescribed functions on ∂V, and n_i are the direction cosines of the exterior normal to ∂V at x.

3. The plane strain. Auxiliary plane strain problems. In the remainder of this paper, V denotes the interior of a right cylinder of the length l with the generic cross-section Σ and the lateral surface B. We denote by L the boundary of the generic cross-section Σ.

We assume that the x_3 — axis is parallel to the generators of V and $x_1 x_2$ - plane contains one of terminal sections. Let V be occupied by an inhomogeneous and anisotropic nonlocal elastic material for which the elastic coefficients are so that (see I e ş a n [6], E r i n g e n [3])

$$\begin{aligned} &C_{ijkl} = C_{ijkl}(x'), \\ &C'_{ijkl}(x, z) = G_{ijkl}(x', z')\, \delta(x_3 - z_3),\ x' \equiv (x_1, x_2),\ z' \equiv (z_1, z_2), \end{aligned} \tag{3.1}$$

where δ is the Dirac measure. In view of these assumptions, the constitutive equations become

$$t_{ij} = C_{ijkl}(x')\, e_{kl} + \int_{\Sigma} G_{ijkl}(x', z')\, e_{kl}(z', x_3)\, d\sigma(z'). \tag{3.2}$$

We define the state of plane strain of the considered cylinder to be that state in which

$$u_i = u_i(x'). \tag{3.3}$$

With this in mind, the constitutive equations (3.2) take the form

$$t_{i\alpha} = C_{i\alpha k\beta}\, u_{k,\beta} + \int_{\Sigma} G_{i\alpha k\beta}(x', z') \frac{\partial u_k}{\partial z_\beta}\, d\sigma(z'), \tag{3.4}$$

$$t_{33} = C_{33k\beta}\, u_{k,\beta} + \int_{\Sigma} G_{33k\beta}(x', z') \frac{\partial u_k}{\partial z_\beta}\, d\sigma(z'), \tag{3.5}$$

so that t_{ij} are functions only on x_1 and x_2. The equilibrium equations become

$$t_{\alpha i,\alpha} + f_i = 0, \text{ on } \Sigma, \tag{3.6}$$

and the boundary conditions on B, take the form

$$t_{\alpha i} n_\alpha = \tilde{t}_i, \text{ on } L. \tag{3.7}$$

From (3.6), (3.7) it follows that the functions f_i and $\tilde{t}_i$ must be independent of x_3.

The plane strain problem consists in the determination of the functions $u_i(x_1, x_2)$ which satisfy the equations (3.4), (3.6) on Σ and the boundary conditions (3.7) on L.

In what follows, we suppose the functions f_i, $\tilde{t}_i$, C_{ijkl} and G_{ijkl} to belong to C^∞. We assume that the domain Σ is C^∞-smooth (F i c h e r a [7]). Using the notations

$$u \equiv (u_i),\ f \equiv (f_i),\ Au \equiv (A_i u),$$

$$A_i u \equiv -\frac{\partial}{\partial x_\alpha}\left(C_{i\alpha k\beta} \frac{\partial u_k}{\partial x_\beta}\right) - \frac{\partial}{\partial x_\alpha} \int_{\Sigma} G_{i\alpha k\beta}(x', z') \frac{\partial u_k}{\partial z_\beta}\, d\sigma(z'), \tag{3.8}$$

the equilibrium equations can be written in the form

$$Au = f, \text{ on } \Sigma. \tag{3.9}$$

If $t(u) \equiv (t_i(u))$, $t_i(u) = t_{ji} n_j$, $\tilde{t} \equiv (\tilde{t}_i)$, then the boundary conditions (3.7) become

$$t(u) = \tilde{t}, \text{ on } L. \tag{3.10}$$

The total energy

$$\begin{aligned} W(u) &\equiv \int_{\Sigma} t_{i\nu}\, e_{i\nu}\, d\sigma = \int_{\Sigma} C_{i\alpha k\beta}\, e_{i\alpha}\, e_{k\beta}\, d\sigma + \\ &+ \int_{\Sigma}\int_{\Sigma} G_{i\alpha k\beta}(x', z')\, e_{i\alpha}(x')\, e_{k\beta}(z')\, d\sigma(x')\, d\sigma(z'), \end{aligned} \tag{3.11}$$

consists of two parts: one is due to local effects and another to nonlocal effects. As in classical theory of elasticity we assume that the internal energy density due to local effects is a positive definite quadratic form, i.e.

$$C_{i\alpha k\beta}\, e_{i\alpha}\, e_{k\beta} \geq c_0 e_{i\alpha} e_{i\alpha}, \quad (c_0 = \text{const.},\ c_0 > 0). \tag{3.12}$$

Moreover we assume that the global energy due to nonlocal effects is positive.

In view of the relations (2.3), (3.8)—(3.10) and using the divergence theorem, we deduce

$$\int_{\Sigma} v A u\, d\sigma = 2W(u, v) - \int_{L} v\, t(u)\, ds, \tag{3.13}$$

$$\int_{\Sigma} (vAu - uAv)\, d\sigma = \int_{L} (u\, t(v) - v\, t(u))\, ds, \tag{3.14}$$

where

$$2W(u, v) \equiv \int_{\Sigma} t_{i\nu}(u)\, e_{i\nu}(v)\, d\sigma. \tag{3.15}$$

If u_i^0 is the solution of the homogeneous boundary value problem (3.9), (3.10) from (3.13), (3.15) and the above hypotheses it follows that

$$u_1^0 = a_1 + b x_2, \quad u_2^0 = a_2 - b x_1, \quad u_3^0 = a_3, \tag{3.16}$$

where, a_i, b are arbitrary constants.

Let us consider that we have reduced the boundary condition (3.10) to the homogeneous condition

$$t(u) = 0 \text{ on } L. \tag{3.17}$$

To prove the existence theorem for the boundary value problem (3.9), (3.17), as in [7], we consider the system

$$Au + p_0 u = f, \tag{3.18}$$

where p_0 is any positive constant. We consider the problem (3.18), (3.17) and we denote by $H_1(\Sigma)$ the function space obtained by the functional completion of $C^1(\bar{\Sigma})$ with respect to the scalar product

$$(u, v)_1 = \int_\Sigma D^s u \, D^s v \, d\sigma, \quad (0 \leqslant s \leqslant 1).$$

It is easily seen that the inequality to be proven in this case is the following (C h i r i ţ ă [5]):

$$\int_\Sigma C_{i\alpha k\beta}\, e_{i\alpha}(v)\, e_{k\beta}(v)\, d\sigma + \int_\Sigma \int_\Sigma G_{i\alpha k\beta}(x', z')\, e_{i\alpha}(v(x'))\, e_{k\beta}(v(z'))\, d\sigma(x') d\sigma(z') + p_0 \int_\Sigma v_i v_i \, d\sigma \geqslant c_1 \| v \|_1^2, \quad \forall v \in H_1(\Sigma). \tag{3.19}$$

If we use the second Korn inequality

$$\int_\Sigma e_{\alpha\beta}(v^{(1)})\, e_{\alpha\beta}(v^{(1)})\, d\sigma + \int_\Sigma v_\alpha v_\alpha \, d\sigma \geqslant c_2 \| v^{(1)} \|_1^2, \quad v^{(1)} \equiv (v_1, v_2, 0), \tag{3.20}$$

and the inequality

$$\int_\Sigma e_{3\alpha}(v^{(2)})\, e_{3\alpha}(v^{(2)})\, d\sigma + \int_\Sigma v_3^2 \, d\sigma \geqslant \frac{1}{4} \| v^{(2)} \|_1^2, \quad v^{(2)} \equiv (0, 0, v_3), \tag{3.21}$$

from (3.12) it follows (3.19). Thus, the boundary value problem (3.18), (3.17) has only one solution which is C^∞ in $\bar{\Sigma}$ (C h i r i ţ ă [5]). The operator (3.8) is formally self-adjoint, so that a C^∞-solution in $\bar{\Sigma}$ of the following system

$$Au + p_0 u - \lambda u = f, \tag{3.22}$$

with the boundary condition (3.17) exists if and only if

$$\int_\Sigma f_i \rho_i \, d\sigma = 0, \tag{3.23}$$

where ρ_i is any solution belonging to $C^\infty(\bar{\Sigma})$ of the problem (3.22), (3.17) with $f = 0$. In particular case, when $\lambda = p_0$, the only $C^\infty(\bar{\Sigma})$ solution of the homogeneous system is (3.16), so that the boundary value problem (3.9), (3.17) has solutions belonging to $C^\infty(\bar{\Sigma})$ if and only if

$$\int_\Sigma f_i \, d\sigma = 0, \quad \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha f_\beta \, d\sigma = 0, \tag{3.24}$$

where ε_{ijk} is the alternating symbol.

It is easy to show that, in the case of inhomogeneous conditions (3.10), the conditions (3.24) are replaced by

$$\int_\Sigma f_i \, d\sigma + \int_L \tilde{t}_i \, ds = 0, \quad \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha f_\beta \, d\sigma + \int_L \varepsilon_{3\alpha\beta}\, x_\alpha \tilde{t}_\beta \, ds = 0. \tag{3.25}$$

Thus, the boundary value problem (3.9), (3.10) has solutions belonging to $C^\infty(\bar{\Sigma})$, if and only if the conditions (3.25) are satisfied.

In what follows, we will use four special problems $P^{(s)}$ $(s = 1, 2, 3, 4)$ of plane strain, characterized by the equations

$$\sigma_{i\alpha}^{(s)} = C_{i\alpha k\beta} \frac{\partial v_k^{(s)}}{\partial x_\beta} + \int_\Sigma G_{i\alpha k\beta}(x', z') \frac{\partial v_k^{(s)}}{\partial z_\beta}\, d\sigma(z'), \quad (s = 1, 2, 3, 4), \tag{3.26}$$

$$\sigma_{\alpha i,\alpha}^{(\beta)} + (C_{i\alpha 33} x_\beta)_{,\alpha} + \frac{\partial}{\partial x_\alpha} \int_\Sigma G_{i\alpha 33}(x', z')\, z_\beta \, d\sigma(z') = 0, \tag{3.27}$$

$$\sigma_{\alpha i,\alpha}^{(3)} + C_{i\alpha 33,\alpha} + \frac{\partial}{\partial x_\alpha} \int_\Sigma G_{i\alpha 33}(x', z')\, d\sigma(z') = 0, \tag{3.28}$$

$$\sigma_{\alpha i,\alpha}^{(4)} - \varepsilon_{3\lambda\beta} \Big\{ (C_{i\alpha\lambda 3}\, x_\beta)_{,\alpha} + \frac{\partial}{\partial x_\alpha} \int_\Sigma G_{i\alpha\lambda 3}(x', z')\, z_\beta \, d\sigma(z') \Big\} = 0, \quad \text{on } \Sigma, \tag{3.29}$$

and the boundary conditions

$$\sigma_{\alpha i}^{(\beta)} n_\alpha = - C_{i\alpha 33}\, x_\beta\, n_\alpha - n_\alpha \int_\Sigma G_{i\alpha 33}(x', z')\, z_\beta \, d\sigma(z'), \tag{3.30}$$

$$\sigma_{\alpha i}^{(3)} n_\alpha = - C_{i\alpha 33}\, n_\alpha - n_\alpha \int_\Sigma G_{i\alpha 33}(x', z')\, d\sigma(z'), \tag{3.31}$$

$$\sigma_{\alpha i}^{(4)} n_\alpha = \varepsilon_{3\lambda\beta}\, n_\alpha \Big[C_{i\alpha\lambda 3}\, x_\beta + \int_\Sigma G_{i\alpha\lambda 3}(x', z')\, z_\beta \, d\sigma(z') \Big], \quad \text{on } L. \tag{3.32}$$

Obviously, the necessary and sufficient conditions (3.25) for the existence of the displacements $v^{(s)}$ in the problem $P^{(s)}$ are satisfied. Thus, we can consider that the functions $v_i^{(s)}$ and $\sigma_{i\alpha}^{(s)}$ are determined.

If we use the notations

$$(3.33)\quad \begin{aligned} t_{ij}^{(\beta)} &= C_{ij33}\, x_\beta + \int_\Sigma G_{ij33}(x', z')\, z_\beta\, d\sigma(z') + \sigma_{ij}^{(\beta)}, \\ t_{ij}^{(3)} &= C_{ij33} + \int_\Sigma G_{ij33}(x', z')\, d\sigma(z') + \sigma_{ij}^{(3)}, \\ t_{ij}^{(4)} &= -\varepsilon_{3\lambda\beta}\left\{C_{ij\lambda3}\, x_\beta + \int_\Sigma G_{ij\lambda3}(x', z')\, z_\beta\, d\sigma(z')\right\} + \sigma_{ij}^{(4)}, \end{aligned}$$

then, in view of the relations (3.26) — (3.32), we obtain

$$(3.34)\quad t_{\alpha i,\alpha}^{(s)} = 0, \text{ on } \Sigma,$$

$$(3.35)\quad t_{\alpha i}^{(s)}\, n_\alpha = 0, \text{ on } L, \ (s = 1, 2, 3, 4).$$

Using the relations (3.34), (3.35) we get

$$(3.36)\quad \int_\Sigma t_{3\alpha}^{(s)}\, d\sigma = 0.$$

4. Saint-Venant's problem. We suppose that body forces are absent and the lateral surface is free of applied tractions. Thus, the equilibrium equations become

$$(4.1)\quad t_{ji,j} = 0,$$

and the boundary conditions on B are

$$(4.2)\quad t_{\alpha i} n_\alpha = 0.$$

Let the loading applied on the end $x_3 = 0$, be statically equivalent to a force R (R_i) and a moment M (M_i). On the end $x_3 = l$ there are applied tractions in a way which fulfills the equilibrium conditions of a rigid body. Thus, for $x_3 = 0$, we have the following conditions

$$(4.3)\quad \int_\Sigma t_{3\alpha}\, d\sigma = -R_\alpha, \qquad (4.4)\quad \int_\Sigma t_{33}\, d\sigma = -R_3,$$

$$(4.5)\quad \int_\Sigma x_\alpha\, t_{33}\, d\sigma = \varepsilon_{3\alpha\beta}\, M_3, \qquad (4.6)\quad \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha\, t_{3\beta}\, d\sigma = -M_3.$$

Our aim is to find the solution of the problem consisting of the equations (2.1), (3.2), (4.1) and of the conditions (4.2) — (4.6).

We try to solve this problem assuming that

$$(4.7)\quad \begin{aligned} u_\alpha &= -\frac{1}{2} b_\alpha x_3^2 - \frac{1}{6} c_\alpha x_3^3 - \left(b_4 + \frac{1}{2} c_4 x_3\right) \varepsilon_{3\alpha\beta}\, x_\beta\, x_3 + \sum_{s=1}^{4} (b_s + c_s x_3) v_\alpha^{(s)} + v_\alpha(x_1, x_2), \\ u_3 &= (b_1 x_1 + b_2 x_2 + b_3)\, x_3 + \frac{1}{2}(c_1 x_1 + c_2 x_2 + c_3)\, x_3^2 + \\ &+ \sum_{s=1}^{4} (b_s + c_s x_3)\, v_3^{(s)} + v_3(x_1, x_2), \end{aligned}$$

where $v_i^{(s)}$ are the components of the displacement vectors from the auxiliary plane strain problems, b_s, c_s $(s = 1, 2, 3, 4)$ are unknown constants and $v_i(x_1, x_2)$ are unknown functions.

With the use of (2.1), (3.2) and (4.7) and recalling (3.33) we obtain:

$$(4.8)\quad t_{ij} = \tau_{ij} + \sum_{s=1}^{4} (b_s + c_s x_3)\, t_{ij}^{(s)} + h_{ij},$$

where

$$(4.9)\quad \tau_{ij} = C_{ijk\alpha}\, v_{k,\alpha} + \int_\Sigma G_{ijk\alpha}(x', z')\, \frac{\partial v_k}{\partial z_\alpha}\, d\sigma(z'),$$

$$(4.10)\quad h_{ij} = \sum_{s=1}^{4} c_s \left[C_{ijk3}\, v_k^{(s)} + \int_\Sigma G_{ijk3}(x', z')\, v_k^{(s)}(z')\, d\sigma(z')\right].$$

With the help of (3.34), the equilibrium equations reduce to

$$(4.11)\quad \tau_{\alpha i,\alpha} + F_i = 0, \text{ on } \Sigma,$$

where

$$(4.12)\quad F_i = h_{\alpha i,\alpha} + \sum_{s=1}^{4} c_s t_{3i}^{(s)}.$$

Taking into account the relation (3.35), the boundary conditions (4.2) become

$$(4.13)\quad \tau_{\alpha i}\, n_\alpha = p_i, \text{ on } L, \qquad (4.14)\quad p_i = -h_{\alpha i}\, n_\alpha.$$

Thus, the functions v_i are the components of the displacement vector in the plane strain problem (4.9), (4.11), (4.13). The necessary and sufficient conditions to solve the above problem are

$$(4.15)\quad \int_\Sigma F_i\, d\sigma + \int_L p_i\, ds = 0, \quad \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha\, F_\beta\, d\sigma + \int_L \varepsilon_{3\alpha\beta}\, x_\alpha\, p_\beta\, ds = 0.$$

Using the relations (3.36), (4.12) and (4.14), it follows that the first two conditions (4.15) are identically satisfied, while from the remaining conditions we get

$$\sum_{s=1}^{4} K_{rs}\, c_s = 0, \quad (r = 3,\ 4). \tag{4.16}$$

From (4.3) we deduce

$$\sum_{s=1}^{4} K_{\alpha s}\, c_s = -R_\alpha, \tag{4.17}$$

where

$$K_{\alpha s} = \int_\Sigma x_\alpha\, t_{33}^{(s)}\, d\sigma, \quad K_{3s} = \int_\Sigma t_{33}^{(s)}\, d\sigma, \quad K_{4s} = \int_\Sigma \varepsilon_{3\eta\beta}\, x_\eta\, t_{33}^{(s)}\, d\sigma. \tag{4.18}$$

As in [6] we can prove that $\det(K_{rs}) \neq 0$, so that the system (4.16), (4.17) determine uniquely the constants c_s $(s = 1,\ 2,\ 3,\ 4)$. Thus the conditions (4.15) are satisfied and in what follows we assume that the functions v_i are known.

From the conditions (4.4) — (4.6) and using the relation (4.8), we get

$$\sum_{s=1}^{4} K_{rs} b_s = g_r, \quad (r = 1,\ 2,\ 3,\ 4), \tag{4.19}$$

where

$$\begin{aligned} g_\alpha &= -\int_\Sigma x_\alpha\,(\tau_{33} + h_{33})\, d\sigma + \varepsilon_{3\alpha\beta}\, M_\beta, \\ g_3 &= -\int_\Sigma (\tau_{33} + h_{33})\, d\sigma - R_3, \qquad g_4 = -\int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha\,(\tau_{33} + h_{33})\, d\sigma - M_3. \end{aligned} \tag{4.20}$$

The system (4.19) determine uniquely the constants b_s. Thus, the problem is solved.

As an illustration of these results, we consider some special problems.

a. Extension and bending. Let the loading applied on $x_3 = 0$, be statically equivalent to a force $\boldsymbol{R}\,(0,\ 0,\ R_3)$ and a moment $\boldsymbol{M}\,(M_1,\ M_2,\ 0)$. Thus, the conditions (4.3), (4.6) become

$$\int_\Sigma t_{3\alpha}\, d\sigma = 0, \qquad \int_\Sigma \varepsilon_{3\alpha\beta}\, x_\alpha\, t_{3\beta}\, d\sigma = 0. \tag{4.21}$$

Let us suppose that the material occupying the cylinder is isotropic and homogeneous. In this case the constitutive equations (3.2) can be written in the form

$$\begin{aligned} t_{ij} = {} & \lambda e_{rr}\, \delta_{ij} + 2\mu e_{ij} + \int_\Sigma \{\lambda'(|x' - z'|)\, e_{rr}\,(z', x_3)\, \delta_{ij} + \\ & + 2\mu'(|x' - z'|)\, e_{ij}\,(z', x_3)\}\, d\sigma\,(z'). \end{aligned} \tag{4.22}$$

On account of the hypotheses just made, our problem consists in determining a solution of the equations (2.1), (4.22), (4.1) with the conditions (4.2), (4.21), (4.4), (4.5). We study the case when the nonlocal elastic coefficients have the form (E d e l e n [4])

$$\lambda'(|x' - z'|) = \lambda K(|x' - z'|), \quad \mu'(|x' - z'|) = \mu K(|x' - z'|). \tag{4.23}$$

In this case we can determine the displacements $v_i^{(s)}$ $(s = 1,\ 2,\ 3,\ 4)$ in the auxiliary plane strain problems. It is not difficult to see that the solutions of the problems $P^{(s)}$ are

$$\begin{aligned} & v_1^{(1)} = -\frac{\lambda}{4(\lambda + \mu)}(x_1^2 - x_2^2), \quad v_2^{(1)} = -\frac{\lambda}{2(\lambda + \mu)}\, x_1 x_2, \quad v_3^{(1)} = 0, \\ & v_1^{(2)} = -\frac{\lambda}{2(\lambda + \mu)}\, x_1 x_2, \quad v_2^{(2)} = \frac{\lambda}{4(\lambda + \mu)}(x_1^2 - x_2^2), \quad v_3^{(2)} = 0, \\ & v_\alpha^{(3)} = -\frac{\lambda}{2(\lambda + \mu)}\, x_\alpha, \quad v_3^{(3)} = 0, \quad v_\alpha^{(4)} = 0, \quad v_3^{(4)} = \varphi\,(x_1,\ x_2), \end{aligned} \tag{4.24}$$

where φ is the solution of the boundary value problem

$$\begin{aligned} & \mu \Delta\varphi + \frac{\partial}{\partial x_1}\int_\Sigma \mu'(|x' - z'|)\, \frac{\partial\varphi}{\partial z_1}\, d\sigma(z') + \frac{\partial}{\partial x_2}\int_\Sigma \mu'(|x' - z'|)\, \frac{\partial\varphi}{\partial z_2}\, d\sigma(z') = \\ & = \int_\Sigma \left[\frac{\partial}{\partial x_1}\, \mu'(|x' - z'|)\, z_2 - \frac{\partial}{\partial x_2}\, \mu'(|x' - z'|)\, z_1\right] d\sigma\,(z'), \quad \text{on } \Sigma, \end{aligned} \tag{4.25}$$

$$\begin{aligned} & \mu\, \frac{\partial\varphi}{\partial n} + \int_\Sigma \mu'(|x' - z'|)\left[n_1(x')\,\frac{\partial\varphi}{\partial z_1} + n_2(x')\,\frac{\partial\varphi}{\partial z_2}\right] d\sigma\,(z') = x_2 n_1 - x_1 n_2 + \\ & + \int_\Sigma \mu'(|x' - z'|)\,[n_1(x')\, z_2 - n_2\,(x')\, z_1]\, d\sigma\,(z'), \quad \text{on } L. \end{aligned} \tag{4.26}$$

Making use of these results in the solution of Saint-Venant's problem, we easily find that

$$c_s = 0, \ (s = 1,2,3,4), \ v_i = 0, \ b_4 = 0, \ K_{4i} = 0, \ \tau_{ij} = 0, \ t_{\alpha\beta} = t_{3\alpha} = 0,$$

$$t_{33} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\left[b_1 x_1 + b_2 x_2 + b_3 + \int_\Sigma K(|x' - z'|)(b_1 z_1 + b_2 z_2 + b_3)\, d\sigma\,(z')\right], \tag{4.27}$$

so that the solution takes the form

$$u_\alpha = -\frac{1}{2}\, b_\alpha\, x_3^2 + b_i v_\alpha^{(i)}, \quad u_3 = b_1 x_1 + b_2 x_2 + b_3, \tag{4.28}$$

where $v_\alpha^{(i)}$ are given by (4.24). The constants b_i are determined from the system

$$K_{\alpha i} b_i = \varepsilon_{3\alpha\beta}\, M_\beta, \quad K_{3i} b_i = -R_3, \tag{4.29}$$

where, in this case, from (4.18) we deduce

$$\frac{1}{E} K_{\alpha 3} = \int_{\Sigma} x_\alpha x_3 \, d\sigma + \int_{\Sigma}\int_{\Sigma} K(|x' - z'|) \, x_\alpha z_3 \, d\sigma(x') \, d\sigma(z'),$$

$$\frac{1}{E} K_{\alpha 3} = \int_{\Sigma} x_\alpha \, d\sigma + \int_{\Sigma}\int_{\Sigma} K(|x' - z'|) \, x_\alpha \, d\sigma(x') \, d\sigma(z'), \tag{4.30}$$

$$\frac{1}{E} K_{33} = \int_{\Sigma} d\sigma + \int_{\Sigma}\int_{\Sigma} K(|x' - z'|) \, d\sigma(x') \, d\sigma(z'); \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

We remark in conclusion that the solution of the problem of extension and bending has the same form as the corresponding solution in classical elasticity.

b. Torsion. Making $R_i = M_\alpha = 0$ in the Saint-Venant's problem we obtain the solution of the problem of torsion. Thus, the solution (4.7) for an isotropic and homogeneous material, takes the form

$$u_1 = -b_4 x_2 x_3, \quad u_2 = b_4 x_1 x_3, \quad u_3 = b_4 \varphi(x_1, x_2), \tag{4.31}$$

where the torsion function φ is the solution of the boundary value problem (4.25), (4.26). For the constant b_4 we get

$$D_t b_4 = -M_3,$$

where

$$D_t = K_{44} = \mu \int_{\Sigma} [x_1(\varphi_{,2} + x_1) - x_2(\varphi_{,1} - x_2)] \, d\sigma +$$

$$+ \int_{\Sigma}\int_{\Sigma} \mu'(|x' - z'|) \left[x_1 \left(\frac{\partial \varphi}{\partial z_2} + z_1 \right) - x_2 \left(\frac{\partial \varphi}{\partial z_1} - z_2 \right) \right] d\sigma(z') \, d\sigma(x'),$$

is the torsional rigidity of the cylinder.

REFERENCES

1. Kröner E. — *Elasticity theory of materials with long range cohesive forces*, Int. J. Solids Structures, 3(1967) 731—742.
2. Eringen A. C. — *Theory of nonlocal thermoelasticity*, Int. J. Engng Sci., 12 (1974) 1063—1077.
3. Eringen A. C. — *On Rayleigh surface waves with small wave lengths*, Lett. Appl. Engng. Sci., 1 (1973) 11—17.
4. Edelen D. G. B. — *On compatibility conditions and stress boundary value problems in linear nonlocal elasticity*, Int. J. Engng Sci., 13 (1975) 971—977.
5. Chiriță S. — *On some boundary value problems in nonlocal elasticity*, An. şt. Univ. Iaşi, Matematică, 22 (1976) 231—237.
6. Ieşan D. — *Saint-Venant's problem for inhomogeneous and anisotropic elastic solids*, J. Eng. Math., 9 (1975) 281—290.
7. Fichera G. — *Existence theorems in elasticity*. In vol. VI a/2 of the Handbuch der Physik, edited by S. Flügge, Berlin-Heidelberg-New York, Springer (1972).

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UNIQUENESS OF WEAK SOLUTIONS OF THE MICROPOLAR FLUID EQUATIONS

BY

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1. Notation. We begin with some notational conventions which will be in effect throughout this paper. The simbol Ω denotes a domain (an open, connected set), in R^3, with sufficiently smooth boundary $\partial\Omega$. We use $L^p = L^p(\Omega)$, $1 \leq p < \infty$, to denote the usual Banach space of „p-integrable" functions from Ω into R^3, endowed with the norm

$$\|u\|_p = \left(\int_{\Omega} |u(x)|^p \, dx \right)^{1/p}, \tag{1.1}$$

where $|\cdot|$ is the norm in R^3. C_0^∞ is the set of functions from Ω into R^3, having continuous derivatives of all orders, and vanishing in a neighbourhood of $\partial\Omega$. The set of functions in C_0^∞ that are free divergence will be denoted by $\mathcal{C}_0^\infty$. The spaces $H_0^{1,2}$ and $\mathcal{H}_0^{1,2}$ are the completions of C_0^∞ and $\mathcal{C}_0^\infty$, respectively, in the Dirichlet norm

$${}_H\|u\|_2 = \left(\int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}. \tag{1.2}$$

We denote the space of all measurable functions $u : (0, T) \to B$, with p-integrable norm by $L^p(0, T; B)$. This space is endowed with the norm

$$\|u\|_{p,B} = \left(\int_0^T \|u(t)\|_B^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty. \tag{1.3}$$

This paper will be concerned with the uniqueness of weak solution of the equations which govern the flow of micropolar fluids.

2. Preliminary results. As it is well known, the micropolar fluid equations have been introduced for the first time by Eringen [1]. Such equations, in the incompressible case, are given by [1],

$$(\mu + k) \nabla^2 v + k \nabla \times \nu - \nabla p + \rho f = \rho \dot{v},$$