Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXIV, S. I a, f. 1, 1978

ON INTEGRABILITY CONDITIONS ON A *CR* SUBMANIFOLD

BY

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0. Introduction. The differential geometry of submanifolds in almost Hermitian spaces has been intensively studied in the last decade. There exist two classes of submanifolds with wery interesting properties: complex submanifolds (see [4] for a survey of results and references) and totally real submanifolds which have been investigated by B.Y. Chen [2], M. Kon [7], K. Ogiue [2], K. Yano [7] and others.

Both these classes are included as particular cases of *CR* submanifolds. A *CR* submanifold is endowed with two distributions; the first one is complex (i.e. invariant to the almost complex structure on the ambient space) and the second one, is totally real (i.e. anti-invariant to the almost complex structure). If in particular, one of these distributions contains only the null vector field, then one of the above classes of submanifolds is obtained. We note that every real hypersurface of an almost Hermitian manifold is a *CR* submanifold.

CR submanifolds have been mostly investigated from the complex analysis' viewpoint. We initiated in [1] a study of *CR* submanifold from the viewpoint of differential geometry.

The purpose of the present paper is to study further *CR* submanifolds of an almost Hermitian manifold. The notion of (F, ω, π) - structure on a manifold is introduced and its existence on a *CR* submanifold is proved. Also, necessary and sufficient conditions for the integrability of both distributions are obtained.

1. (F, ω, π)-structures on a manifold. Let M be a differentiable manifold and $\pi : S \to M$ be a vector bundle over M. Throughout the paper, the class of differentiability will be C^∞.

Suppose on M a non-null tensor field of type (1, 1) and a π-valued differential 1-form are given. The mapping $x \to \operatorname{Ker} F_x$ (resp. $x \to \operatorname{Ker} \omega_x$) is supposed to be a regular differentiable distribution (i.e. dim Ker F_x is constant for each $x \in M$) denoted by Ker F (resp. Ker ω). The projectors which define the distributions Ker F and Ker ω are denoted by Q and P.

Definition. *An (F, ω, π)-structure on a manifold M is determined by a non-null tensor field F of type (1, 1) on M and a π - valued differential 1-form on M which satisfy the following conditions*

$$F^2 = -I + Q, \tag{1.1}$$

$$\omega \circ F = 0, \tag{1.2}$$

where I is the identity transformation on TM.

If in particular, the vector bundle π is the trivial vector bundle $\pi : M \times R \to M$, the distributions $\operatorname{Ker} F$ and $\operatorname{Ker} \omega$ are complementary to each other, dim. $\operatorname{Ker} F = 1$ and there exists a vector field X on M such that $\omega(X) = 1$, then the (F, ω, π)-structure becomes the well-known almost contact structure [5].

Of course, special investigations should be made on (F, ω, π)-structures on a manifold. But our purpose in this paper is only to show that an (F, ω, π)-structure appears in a natural way on every CR submanifold, and to study its properties.

2. CR submanifolds in an almost Hermitian manifold. Let N be a $2n$-dimensional differentiable manifold endowed with an almost Hermitian structure (J, g) and M be an m-dimensional differentiable manifold immersed in N.

The immersion of M into N is denoted by φ. We identify for each $x \in M$, the tangent space T_xM with $\varphi_*(T_xM) \subset T_{\varphi(x)}N$ by means of φ.

Now, suppose on M two complementary distributions D and $\tilde{D}$ are given. The distribution D is assumed to be of dimension $2p$ and invariant under the almost complex structure (i.e. $J(D_x) = D_x$ for each $x \in M$). The distribution $\tilde{D}$ is of dimension q and anti-invariant under the almost omplex structure (i.e. $J(\tilde{D}_x) \subset \nu_x$ for each $x \in M$), where ν_x is the normal space to M at x. This implies $q \leqslant 2n - m$. The distribution D (resp. $\tilde{D}$) may be defined by a projector P (resp. Q) which satisfy the conditions

$$P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0. \tag{2.1}$$

We shall call distribution D (resp. $\tilde{D}$) the horizontal (resp. vertical) distribution on M.

Definition. *The submanifold M endowed with the above pair of distributions $(D, \tilde{D})$ is called a CR submanifold of N.*

Theorem 2.1. *On every CR-submanifold of an almost Hermitian manifold there exists an (F, ω, π) - structure.*

Proof. The normal bundle to M is taken as the vector bundle π. Define the tensor field F and the π-valued differential 1-form ω by

(2.2) $\quad FX = JPX$ and $\qquad$ (2.3) $\quad \omega X = JQX$

for any vector field X on M. It is easy to see that $\operatorname{Ker} F = \tilde{D}$ and $\operatorname{Ker} \omega = D$. The relations (1. 1) — (1.3) follow by direct computations from (2. 2) and (2. 3).

Since hereafter we consider only the (F, ω, π) - structure whose existence has been proved in theorem 2.1 we shall call it the natural F-structure.

3. Integrability conditions on a CR submanifold. Let N be an almost Hermitian manifold and M be a CR submanifold in N. The integrability of the natural structure F and the integrability of both distributions D and $\tilde{D}$ will be studied by means of the Nijenhuis tensors corresponding to F and J, denoted by N_F and N_J.

For each vector field X on M we have

$$JX = FX + \omega X. \tag{3.1}$$

If we make a computation for N_F and N_J we obtain

$$N_J(X, Y) = N_F(X, Y) - Q([X, Y]) - \omega([JX, Y] + [X, JY]) \tag{3.2}$$

for all $X, Y \in D$.

For each vector field Z on N denote by tZ the tangent part of Z and by $t\perp Z$ its normal part. Then from (3. 2) we have

Theorem 3.1. *A necessary and sufficient condition for the horizontal distribution to be integrable is that*

$$tN_J(X, Y) = N_F(X, Y) \textit{ for all } X, \; Y \in D. \tag{3.3}$$

Also, from (3.2) we have

$$t\perp N_J(X, Y) = -\omega([JX, Y] + [X, JY]) \tag{3.4}$$

for all vector fields $X, Y \in D$.

Theorem 3.2. *A necessary and sufficient condition for the horizontal distribution to be integrable is that the following conditions be satisfied*

1. $t\perp N_J(X, Y) = 0 \quad$ 2. $QN_F(X, Y) = 0$

for all vector fields X, Y from D.

Proof. If the horizontal distribution is integrable, then the condition 1. follows from (3.4). For N_F we have

$$N_F(X, Y) = [JX, JY] - JP([JX, Y] + [X, JY]) - P([X, Y])$$

for all X, $Y \in D$. Thus $N_F(X, Y) \in D$ and the condition 2. is accomplished. Conversely, from the condition 1. we have

$$Q([JX, Y] + [X, JY]) = 0 \text{ for all } X, \; Y \in D.$$

Hence $QtN_J(X, Y) = 0$. From (3.2) we obtain

$$QtN_J(X, Y) = QN_F(X, Y) - Q([X, Y]), \text{ for all } X, Y \in D.$$

Thus, the condition 2 implies $Q([X, Y]) = 0$, that is, D is integrable.

Now, if we take X, Y from the vertical distribution, then we obtain

$$N_F(X, Y) = -P([X, Y]). \tag{3.5}$$

Thus we have

Theorem 3.3. *A necessary and sufficient condition for the vertical distribution to be integrable is that*

$$N_F(X, Y) = 0 \text{ for all } X, \; Y \in \tilde{D}. \tag{3.6}$$

Denote by $\tilde{\nabla}$ the Hermitian connection on N. The Gauss equations and those of Weingarten are given by

(3.7) $\quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\qquad$ (3.8) $\quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$

where ∇ is the Riemannian connection on M, $\nabla^\perp$ is the linear connection induced on the normal bundle by $\tilde{\nabla}$ and h is the second fundamental form of the immersion.

For the differential 1-form ω on M we may define the exterior differentiation d induced by the linear connection $\nabla^{\perp}$ on the normal bundle

$$d\omega(X, Y) = \nabla^{\perp}_X\,(\omega(Y)) - \nabla^{\perp}_Y\,(\omega(X)) - \omega([X, Y]) \tag{3.9}$$

for any vector fields X, Y on M.

If we take vector fields X, Y from D, then (3,9) becomes

$$d\omega(X, Y) = -JQ([X, Y]) \tag{3.10}$$

Thus we have

Theorem 3.4. *A necessary and sufficient condition for the horizontal distribution to be integrable is that*

$$d\omega(X, Y) = 0 \tag{3.11}$$

for each vector fields X and Y from D.

Now, from (2.2) we can see that the natural F-structure on the CR submanifold M verifies $F^3 + F = 0$. This means that F has properties of the structure introduced and studied by K. Yano [6]. We know [3] that F is called partially integrable, if the horizontal distribution is integrable and the almost complex structure $\tilde{J}X = FX$ induced by F on each integral manifold of D is also integrable. Thus we can state

Theorem 3.5. *For a CR submanifold of an Hermitian manifold, the following four conditions are equivalent to each other*

1. *The natural F-structure is partially integrable;*
2. $N_F(X, Y) = 0$ for all X, $Y \in D$;
3. $d\omega(X, Y) = 0$ for all X, $Y \in D$;
4. *The horizontal distribution is integrable.*

Proof. The conditions 3 and 4 are equivalent by theorem 3.4. We note that the almost complex structure $\tilde{J}$ induced by F on each integral manifold of D is just the complex structure J on N. This implies the conditions 4. and 1. are equivalent. The equivalence of conditions 1 and 2 has been proved by K. Yano and S. Ishihara [3]. Thus the proof is complete.

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Received 26.X. 1977

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXIV, S. I a, f. 1, 1978

A GEOMETRIC INTERPRETATION FOR THE PROLONGATION OF A G-SPACE

BY

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In this paper we give a geometric interpretation as a frame space for the prolongation of order r of a differentiable G-space with all orbits of the same type and G a compact Lie group.

1. Let X be a differentiable G-space with all orbits of the same type $H\backslash G$, where G is a compact Lie group, H is a closed Lie subgroup of G and $H\backslash G$ is the homogeneous quotient space. It is well known that canonical projection of X on the orbit space $G\backslash X = X^*$ defines a fibre bundle structure with the fibre $H\backslash G$ and the structure group $H\backslash N = K$, where N denotes the normalizer of H in G (see [1], p. 309).

Let $X^{(H)}$ be the fixed point space of H (where H acts on X by restriction), thus X may be identified with the twisted products $X^{(H)} \times_H G$ and $X^{(H)} \times_K (H\backslash G)$ respectively. Also $X^{(H)}$ is a principal fibre bundle over X^* with the structure group K (see [1], p. 182).

In [4] we constructed the prolongation of order r of X. This prolongation is denoted by $\tau^r X$ and its points are the pairs $(j_0^r\varphi, j_{\varphi(0)}^r\sigma)$ where $\varphi : U \subset R^m \to X^*$ is a local diffeomorphism around $0 \in R^m$, $\sigma : \varphi(U) \subset X^* \to X$ is a local cross section over $\varphi(U)$ of the fibre bundle $X \to X^*$ and $j_0^r\varphi$ (resp. $j_{\varphi(0)}^r\sigma$) denotes the r-jet of φ (resp. σ) at the point $0 \in R^m$ (resp. $\varphi(0) \in X^*$). $\tau^r X$ is canonically identified with the fibre product $H^r(X^*) \times_{X^*} J^r X$, where $H^r(X^*)$ denotes the principal fibre bundle of all tangent r-frames of X^* and $J^r X$ denotes the fibre bundle of the r-jets of all local cross section of the fibre bundle $X \to X^*$. It follows that $\tau^r X$ is a differentiable fibre bundle over X^*.

Let G_m^r be the group defined by $G_m^r = \{(j_0^r\bar{\varphi}, j_0^r\bar{\sigma})\}$, where $\bar{\varphi} : U \subset R^m \to R^m$ is a local diffeomorphism such that $\bar{\varphi}(0) = 0$ and $\bar{\sigma} : \bar{\varphi}(U) \subset R^m \to G$ is a differentiable map. Thus G_m^r may be identified with the semi-direct product $L_m^r \bar{\times} T_m^r G$, where L_m^r is the structure group of $H^r(X^*)$ and $T_m^r G$ is the group of m^r-velocity on G (see [2] and [3]). The product of the elements of G is:

$$\bar{h}_2 \,.\, \bar{h}_1 = (\bar{y}_2, \bar{S}_2) \,.\, (\bar{y}_1, \bar{S}_1) = (\bar{y}_2 \,.\, \bar{y}_1, (\bar{S}_2 \,.\, \bar{y}_1) \,.\, \bar{S}_2) \tag{1}$$

In [4] we proved the following: