For the differetial 1-form ω on M we may define the exterior differentiation d induced by the linear connection $\nabla^{\perp}$ on the normal bundle

$$d\omega(X, Y) = \nabla^{\perp}_{X}\,(\omega(Y)) - \nabla^{\perp}_{Y}\,(\omega(X)) - \omega([X, Y]) \tag{3.9}$$

for any vector fields X, Y on M.

If we take vector fields X, Y from D, then (3,9) becomes

$$d\omega(X, Y) = -JQ([X, Y]) \tag{3.10}$$

Thus we have

Theorem 3.4. *A necessary and sufficient condition for the horizontal distribution to be integrable is that*

$$d\omega(X, Y) = 0 \tag{3.11}$$

for each vector fields X and Y from D.

Now, from (2.2) we can see that the natural F-structure on the CR submanifold M verifies $F^3 + F = 0$. This means that F has properties of the structure introduced and studied by K. Yano [6]. We know [3] that F is called partially integrable, if the horizontal distribution is integrable and the almost complex structure $\tilde{J}X = FX$ induced by F on each integral manifold of D is also integrable. Thus we can state

Theorem 3.5. *For a CR submanifold of an Hermitian manifold, the following four conditions are equivalent to each other*

1. *The natural F-structure is partially integrable;*
2. $N_F(X, Y) = 0$ for all X, $Y \in D$;
3. $d\omega(X, Y) = 0$ for all X, $Y \in D$;
4. *The horizontal distribution is integrable.*

Proof. The conditions 3 and 4 are equivalent by theorem 3.4. We note that the almost complex structure $\tilde{J}$ induced by F on each integral manifold of D is just the complex structure J on N. This implies the conditions 4. and 1. are equivalent. The equivalence of conditions 1 and 2 has been proved by K. Yano and S. Ishihara [3]. Thus the proof is complete.

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A GEOMETRIC INTERPRETATION FOR THE PROLONGATION OF A *G*-SPACE

BY

ALEXANDRU NEAGU

In this paper we give a geometric interpretation as a frame space for the prolongation of order r of a differentiable G-space with all orbits of the same type and G a compact Lie group.

1. Let X be a differentiable G-space with all orbits of the same type $H\backslash G$, where G is a compact Lie group, H is a closed Lie subgroup of G and $H\backslash G$ is the homogeneous quotient space. It is well known that canonical projection of X on the orbit space $G\backslash X = X^*$ defines a fibre bundle structure with the fibre $H\backslash G$ and the structure group $H\backslash N = K$, where N denotes the normalizer of H in G (see [1], p. 309).

Let $X^{(H)}$ be the fixed point space of H (where H acts on X by restriction), thus X may be identified with the twisted products $X^{(H)} \times_H G$ and $X^{(H)} \times_K (H\backslash G)$ respectively. Also $X^{(H)}$ is a principal fibre bundle over X^* with the structure group K (see [1], p. 182).

In [4] we constructed the prolongation of order r of X. This prolongation is denoted by $\tau^r X$ and its points are the pairs $(j^r_0\varphi, j^r_{\varphi(0)}\sigma)$ where $\varphi : U \subset R^m \to X^*$ is a local diffeomorphism around $0 \in R^m$, $\sigma : \varphi(U) \subset X^* \to X$ is a local cross section over $\varphi(U)$ of the fibre bundle $X \to X^*$ and $j^r_0\varphi$ (resp. $j^r_{\varphi(0)}\sigma$) denotes the r-jet of φ (resp. σ) at the point $0 \in R^m$ (resp. $\varphi(0) \in X^*$). $\tau^r X$ is canonically identified with the fibre product $H^r(X^*) \times_{X^*} J^r X$, where $H^r(X^*)$ denotes the principal fibre bundle of all tangent r-frames of X^* and $J^r X$ denotes the fibre bundle of the r-jets of all local cross section of the fibre bundle $X \to X^*$. It follows that $\tau^r X$ is a differentiable fibre bundle over X^*.

Let G^r_m be the group defined by $G^r_m = \{(j^r_0\bar{\varphi}, j^r_0\bar{\sigma})\}$, where $\bar{\varphi} : U \subset R^m \to R^m$ is a local diffeomorphism such that $\bar{\varphi}(0) = 0$ and $\bar{\sigma} : \bar{\varphi}(U) \subset R^m \to G$ is a differentiable map. Thus G^r_m may be identified with the semi-direct product $L^r_m \times T^r_m G$, where L^r_m is the structure group of $H^r(X^*)$ and $T^r_m G$ is the group of m^r-velocity on G (see [2] and [3]). The product of the elements of G is:

$$\bar{h}_2 \cdot \bar{h}_1 = (\bar{y}_2, \bar{S}_2) \cdot (\bar{y}_1, \bar{S}_1) = (\bar{y}_2 \cdot \bar{y}_1, (\bar{S}_2 \cdot \bar{y}_1) \cdot \bar{S}_2) \tag{1}$$

In [4] we proved the following:

Theorem. *$\tau^r X$ is a G^r_m-space with all orbits of the same type $T^r_m H \diagdown G^r_m$, where $T^r_m H$ is the group of m^r-velocity on H. The action of G^r_m on $\tau^r X$ is:*

$$h \,.\, \bar{h} = (y \,.\, \bar{y},\ S \,.\, (\bar{S} \,.\, \bar{y}^{-1} \,.\, y^{-1})), \tag{2}$$

where $h = (y, S) \in \sigma^r X$ *and* $\bar{h} = (\bar{y}, \bar{S}) \in G^r_m$.

The isotropy group of every point of $\tau^r X$ is conjugate to $T^r_m H$ and the orbit space $G^r_m \diagdown \tau^r X$ may be identified with X^*. The orbits of G^r_m coincide with the local fibres of the fibre bundle $\tau^r X \to X^*$ (see [4]). Thus $\tau^r X \to X^*$ has a fibre bundle structure with the structure group $K^r_m = T^r_m H \diagdown N^r_m$, where $N^r_m = L^r_m \bar{\times} T^r_m N$ (see [4]).

2. The fixed point space of $T^r_m H$, denoted by $(\tau^r X)^{(T^r_m H)}$, may be identified with the prolongation $\tau^r X^{(H)}$, of order r of the principal fibre bundle $X^{(H)} \to X^*$. Hence $\tau^r X$ can be identified with the twisted product $\tau^r X^{(H)} \times_{N^r_m} G^r_m$. Analogously, $\tau^r X$ can be identified with the twisted product $\tau^r X^{(H)} \times_{K^r_m} (T^r_m H \diagdown G^r_m)$. It follows that $(\tau^r X)^{(T^r_m H)}$ is a principal fibre bundle over X^* with the structure group K^r_m (see [4]). The map $\tau^r X^{(H)} \to X^{(H)}$ defined by $h = (j^r_0 \varphi,\ j^r_{\varphi(0)} \sigma) \to \sigma(\varphi(0))$ induces on $\tau^r X^{(H)}$ a fibre bundle structure with the structure group $L^r_m \bar{\times} T^r_{m,e} K$, where $T^r_{m,e} K = \{j^r_0 \sigma,\ \sigma : R^m \to K,\ \bar{\sigma}(0) = e = H \,.\, e\}$. Let us show that $j^r_1 : h \in \tau^r X \to j^r_1(h) = \sigma(\varphi(0)) \in X$; where $h = (j^r_0 \varphi,\ j^r_{\varphi(0)} \sigma)$ is well defined. Indeed, $\sigma^r X$ may be identified with the twisted product $\tau^r X^{(H)} \times_{N^r_m} G^r_m$ by means $[h, \bar{h}] \to h \,.\, \bar{h}$, where $[h, \bar{h}]$ is the orbit of $(h, \bar{h})$ in the N^r_m-space $\tau^r X^{(H)} \times G^r_m$ with the action:

$$(h, \bar{h}) \,.\, \bar{h}_1 = (h \,.\, \bar{h}_1,\ \bar{h}_1^{-1} \,.\, \bar{h}),\ \bar{h}_1 \in N^r_m.$$

Let us consider $(h_0, \bar{h}_0) \in [\bar{h}, \bar{h}]$; then there exists $\bar{h}_1 \in N^r_m$ such that $(h_0, \bar{h}_0) = (h \,.\, \bar{h}_1,\ \bar{h}_1^{-1} \,.\, \bar{h})$ and we have:

$$j^r_1[h_0, \bar{h}_0] = j^r_1\ (h \,.\, \bar{h}_1, \bar{h}_1 \,.\, \bar{h}) = \sigma(\varphi(0) \,.\, \bar{\sigma}_1(0) \,.\, \bar{\sigma}^{-1}(0)\, \sigma(0) =$$

$$= \sigma(\varphi(0)) \,.\, \bar{\sigma}(0) = j^r_1[h, \bar{h}]$$

Also, we have (see in [4]) that the map $j^0_r : \bar{h} = (j^r_0 \bar{\varphi}.\ \ j^r_0\ \bar{\sigma}) \in G^r_m \to j^0_r(\bar{h}) = \bar{\sigma}(0) \in G$ is a morphism of Lie groups.

Theorem 1. *The G-space mapping:*

$$(j^r_1,\ j^0_r) : (\tau^r X, G^r_m) \to (X, G)$$

is equivariant.

Proof. Using (2), we obtain:

$$j^r_1(h \,.\, \bar{h}) = j^r_1(j^r_0 \varphi \circ \bar{\varphi},\ j^r_{\varphi(0)} \sigma(\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1})) = (\sigma(\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1}).$$

$$(\varphi \circ \bar{\varphi}(0)) = \sigma(\varphi(0)) \,.\, \bar{\sigma}(0) = j^r_1(h) \,.\, j^0_r(\bar{h}).$$

Theorem 2. *The group $(j^0_r)^{-1}(H)$ coincides with ${}^H G^r_m = L^r_m \bar{\times} (H \,.\, T^r_{m,e} G)$.*

Proof. If $\bar{h} \in (j^0_r)^{-1}\ (H)$, it follows that $j^0_r(\bar{h}) = \bar{\sigma}(0) \in H$. We suppose that $\bar{h} = (j^r_0 \bar{\varphi},\ a \,.\, j^r_0 \bar{\sigma}_1)$, where $a \in G$ and $j^r_0 \bar{\sigma}_1 \in T^r_{m,e}\ G$. Then $j^0_r(\bar{h}) = a \,.\, \bar{\sigma}_1(0) = a \in H$ and consequently $h \in {}^H G^r_m$. Hence $(j^0_r)^{-1}(H) \subset {}^H G^r_m$. Conversely, if $\bar{h} = (j^r_0 \bar{\varphi},\ a \,.\, j^r_0 \bar{\sigma}_1) \in {}^H G^r_m$, then we have $j^0_r(\bar{h}) = (a \,.\, \bar{\sigma}_1)(0) = a.\, \bar{\sigma}_1(0) = a \in H$ and hence $\bar{h} \in (j^0_r)^{-1}(H)$ from which ${}^H G^r_m \subset (j^0_r)^{-1}(H)$.

Theorem 3. *There exists a right action of ${}^H G^r_m$ on $\tau^r X$, denoted by „$*$" with respect to which all orbits have the same type and the orbit space is X. The canonical projection of $\tau^r X$ in the orbit space X is the map j^r_1.*

Proof. Let $[h, \bar{h}] = h \,.\, \bar{h}$ be a point of $\tau^r X^{(H)} \times_{N^r_m} G^r_m$, where $\bar{h} = (j^r_0 \bar{\varphi}.\ a \,.\, j^r_0\, \bar{\sigma}),\ j^r_0\, \bar{\sigma} \in T^r_{m,e} G \,.\, a \in G$. We define the action of ${}^H G^r_m$ on $\tau^r X$ by:

$$([h \,.\, \bar{h}], \bar{h}_1) \in \tau^r X \times {}^H G^r_m \to [h \,.\, \bar{h}] * \bar{h}_1 = h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a \in \tau^r X. \tag{3}$$

We have:

$$[h, \bar{h}] * (\bar{h}_1 \,.\, \bar{h}_2) = h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, \bar{h}_2 \,.\, a,$$

$$(([h \,.\, \bar{h}] * \bar{h}_1) * \bar{h}_2) = (h \,.\, \bar{h} \,.\, a^{-1} \,.\, h_1 \,.\, a) * \bar{h}_2 \,.$$

But:

$$\bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a = (j^r_0\, \bar{\varphi} \,.\, a \,.\, j^r_0\, \bar{\sigma}) \,.\, a^{-1} \,.\, (j^r_0\, \bar{\varphi}_1 \,.\, b \,.\, j^r_0\, \bar{\sigma}_1) =$$

$$= (j^r_0\, \bar{\varphi} \,.\, a \,.\, j^r_0\, \bar{\sigma} \,.\, a^{-1}) \,.\, (j^r_0\, \bar{\varphi}_1 \,.\, b \,.\, j^r_0\, \bar{\sigma}_1 \,.\, a) =$$

$$= (j^r_0\, \bar{\varphi} \circ \bar{\varphi}_1 \,.\, a \,.\, a^{-1} \,.\, j^r_0\, (\bar{\varphi}' \circ \bar{\varphi}_1) \,.\, b \,.\, j^r_0\, \bar{\sigma}_1 \,.\, a) =$$

$$= (j^r_0\, \bar{\varphi} \circ \bar{\varphi}_1 \,.\, b \,.\, j^r_0\, \bar{\sigma}_0 \,.\, j^r_0\, \bar{\sigma}_1 \,.\, a) = (j^r_0\, \bar{\varphi} \circ \bar{\varphi}_1 \,.\, b \,.\, a \,.\, j^r_0\, \bar{\sigma}_0') = \bar{h}' =$$

$$= (\bar{y}_0 \,.\, b \,.\, a \,.\, \bar{S}_0') = b \,.\, \bar{h}_0' = b \,.\, (\bar{y}_0 \,.\, a \,.\, \bar{S}_0').$$

(since $T^r_{m,e} G$ is a normal subgroup of $T^r_m G$). It follows that:

$$(h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a) * \bar{h}_2 = [h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a] * \bar{h}_2 =$$

$$= h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a \,.\, a^{-1} \,.\, \bar{h}_2 \,.\, a = h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, \bar{h}_2 \,.\, a.$$

But:

$[h \,.\, \bar{h}] * \bar{h}_1 = [h, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a] = [h, \bar{h}'] = h \,.\, \bar{h}' = h \,.\, b \,.\, \bar{h}_0' = h \,.\, \bar{h}_0' = [h \,.\, \bar{h}_0'],$

where $h_0' = (j^r_0\, \bar{\varphi} \circ \bar{\varphi}_1 \,.\, a \,.\, \bar{S}_0') \,.\, b \in H \subset T^r_m H$ and $T^r_m H$ is the isotropy group of $h \in \tau^r X^{(H)}$.

On the other hand, we have:

$$j^r_1\, ([h \,.\, \bar{h}] * \bar{h}_1) = j^r_1\, (h \,.\, \bar{h} \,.\, a^{-1} \,.\, \bar{h}_1 \,.\, a) =$$

$$= \sigma\, (\varphi\, (0)) \,.\, a \,.\, \bar{\sigma}\, (0) \,.\, a^{-1} \,.\, \bar{\sigma}_1\, (0) \,.\, a = \sigma\, (\varphi\, (0)) \,.\, a \,.\, a^{-1} \,.\, b \,.\, a =$$

$$= \sigma(\varphi(0)) . b . a = \sigma(\varphi(0)) . a$$

since $j^0_r(j^r_0\bar{\sigma}_1) = \bar{\sigma}_1(0) = b \in H$ and $\sigma(\varphi(0)) \in X^{(H)}$. Hence

$$\sigma(\varphi(0)) . b = \sigma(\varphi(0)).$$

Let h_1, h_2 be two points of $\tau^r X$ such that $j^r_1(h_1) = j^r_1(h_2)$, then $(\pi \circ j^r_1)(h_1) = j^r(h_1) = \varphi_1(0) = (\pi \circ j^r_1)(h_2) = j^r(h_2) = \varphi_2(0)$, since $\pi \circ j^r_1 = j^r : \tau^r X \to X^*$, where $\pi : X \to X^*$ is the canonical projection. It results that h_1 and h_2 belong to the same orbit, and hence there exist $\bar{h}_i \in G^r_m (i = 1, 2)$ and $h \in \tau^r X^{(H)}$ such that $h_1 = h . \bar{h}_1, h_2 = h . \bar{h}_2$ from, which $h_2 = h_1 . \bar{h}_1^{-1} . \bar{h}_2$. We have:

$$j^r_1(h_1) = \sigma(\varphi(0)) . \bar{\sigma}_1(0) \text{ and } j^r_1(h_2) = \sigma(\varphi(0)) . \bar{\sigma}_2(0).$$

Then $\bar{\sigma}_1^{-1}(0) . \bar{\sigma}_2(0) \in H$. Hence $j^0_r(\bar{h}_1^{-1} . h_2) = \bar{\sigma}_1^{-1}(0) . \bar{\sigma}_2(0) \in H$. It follows that $\bar{h} = \bar{h}_1^{-1} . \bar{h}_2 \in {}^H G^r_m$ and hence the orbit space of the action „ $*$ " on $\tau^r X$ is X.

Now, we prove that all orbits have the same type $T^r_m H \setminus {}^H G^r_m$. Let $\bar{h}_1$ be an element of ${}^H G^r_m$ such that $[h . \bar{h}] * \bar{h}_1 = [h . \bar{h}]$; hence $h . \bar{h} . a^{-1} . \bar{h}_1 . a = h . \bar{h}$. Since $h \in \tau^r X^{(H)}$ it follows that $\bar{h} . a^{-1} . \bar{h}_1 . \bar{h}^{-1} \in$ $\in T^r_m H$.

$$(j^r_0\bar{\varphi} . a . j^r_0\bar{\sigma}) . a^{-1} . (j^r_0\bar{\varphi}_1 . j^r_0\bar{\sigma}_1) . a . (j^r_0\bar{\varphi}^{-1} . j^r_0\bar{\sigma}^{-1} . a^{-1} . \bar{\varphi}^{-1}) \in T^r_m H,$$

$$(j^r_0\bar{\varphi} \circ \bar{\varphi}_1 , a . j^r_0\bar{\sigma} \circ \bar{\varphi}_1 . a^{-1} . j^r_0\bar{\sigma}_1) . (j^r_0\bar{\varphi}^{-1} . a . j^r_0\bar{\sigma}^{-1} \circ \bar{\varphi}^{-1} . a^{-1}) \in T^r_m H,$$

$$(j^r_0\bar{\varphi} \circ \bar{\varphi}_1 \circ \bar{\varphi}^{-1} . a . j^r_0\bar{\sigma} \circ \bar{\varphi}_1 \circ \bar{\varphi}^{-1} . a . j^r_0\bar{\sigma}_1 \circ \bar{\varphi}^{-1} . a . j^r_0\bar{\sigma}^{-1} \circ \bar{\varphi}^{-1} . a^{-1}) \in T^r_m H.$$

It results that $j^r_0\bar{\varphi} \circ \bar{\varphi}_1 \circ \bar{\varphi}^{-1} = j^r_0 \, \mathrm{id}_{R^m}$, hence $j^r_0\bar{\varphi}_1 = j^r_0 \, \mathrm{id}_{R^m}$ and

$$a . j^r_0\bar{\sigma} \circ \bar{\varphi}^{-1} . a^{-1} . j^r_0\bar{\sigma}_1 \circ \bar{\varphi}^{-1} . a . j^r_0\bar{\sigma}^{-1} \circ \bar{\varphi}^{-1} . a^{-1} \in T^r_m H.$$

from which:

$$(a . j^r_0\bar{\sigma} . a^{-1} . j^r_0\bar{\sigma}_1 . a . j^r_0\bar{\sigma}^{-1} . a^{-1}) . j^r_0\bar{\varphi}^{-1} \in T^r_m H,$$

$$(a . j^r_0\bar{\sigma} . a^{-1}) . j^r_0\bar{\sigma}_1 . (a . j^r_0\bar{\sigma} . a^{-1})^{-1} \in T^r_m H,$$

and we obtain: $j^r_0\bar{\sigma}_1 \in (a . j^r_0\bar{\sigma} . a^{-1})^{-1} . T^r_m H . (a . j^r_0\bar{\sigma} . a^{-1})$. But $j^0_r(a . j^r_0\bar{\sigma} . a^{-1}) =$ $= (a . \bar{\sigma} . a^{-1})(0) = a . \bar{\sigma}(0) . a^{-1} = a . a^{-1} = e$ hence $a . j^r_0\bar{\sigma} . a^{-1} \in \operatorname{Ker} j^0_r =$ $= L^r_m \overline{\times} T^r_{m,e} G \subset {}^H G^r_m$. And the proof is completed.

Remark. $\tau^r X$ is canonically identified with the twisted product ${}^H\tau^r X \times_{\tilde{N}^r_m} {}^H G^r_m$, where ${}^H\tau^r X$ is the fixed point space of $T^r_m H$ with respect to „ $*$ " and $\tilde{N}^r_m$ is the normalizer of $T^r_m H$ in ${}^H G^r_m$. ${}^H\tau^r X$ is a principal fibre bundle over X with the structure group $\tilde{K}^r_m = T^r_m H \setminus \tilde{N}^r_m$.

Lemma 1. *The group* $T^r_{m,e}(H \setminus N)$ *is canonically identified with* $T^r_{m,e}H \setminus T^r_{m,e}N$.

Proof. Let μ be the map:

$$\mu : T^r_{m,e}H . j^r_0\bar{\sigma} \in T^r_{m,e}H \setminus T^r_{m,e}N \to j^r_0(\pi \circ \bar{\sigma}) \in T^r_{m,e}(H \setminus N),$$

where π denotes the canonical projection $N \to H \setminus N$ and $j^r_0\bar{\sigma} \in T^r_{m,e}N$; hence $\bar{\sigma} : R^m \to N, \bar{\sigma}(0) = e$. Next, let $j^r_0\bar{\sigma}_1 \in T^r_{m,e}H . j^r_0\bar{\sigma}$ then $j^r_0\bar{\sigma}_1 . \bar{\sigma}^{-1} \in$ $\in T^r_{m,e}H$ and hence $\bar{\sigma}_1 . \bar{\sigma}^{-1} : R^m \to H, (\bar{\sigma}_1 . \bar{\sigma}^{-1})(0) = e$. It follows that $\pi \circ \bar{\sigma}_1(u) = H . \bar{\sigma}_1(u) = (\pi \circ \bar{\sigma})(u) = H . \bar{\sigma}(u)$ for every $u \in R^m$. Then $j^r_0(\pi \circ \bar{\sigma}_1) = j^r_0(\pi \circ \bar{\sigma})$ and hence it is well defined.

$$\mu((T^r_{m,e}H . j^r_0\bar{\sigma}_1) . (T^r_{m,e}H . j^r_0\bar{\sigma}_2)) = \mu(T^r_{m,e}H . j^r_0\bar{\sigma}_1 . \bar{\sigma}_2) =$$

$$= j^r_0(\pi \circ (\bar{\sigma}_1 . \bar{\sigma}_2)) = j^r_0((\pi \circ \bar{\sigma}_1) . (\pi \circ \bar{\sigma}_2)) = j^r_0(\pi \circ \bar{\sigma}_1) . j^r_0(\pi \circ \bar{\sigma}_2) =$$

$$= \mu(T^r_{m,e}H . j^r_0\bar{\sigma}_1) . \mu(T^r_{m,e}H . j^r_0\bar{\sigma}_2).$$

Thus μ is a morphism of Lie groups. We suppose that $\mu(T^r_{m,e}H . j^r_0\bar{\sigma}_1)$ is the unit of the group $T^r_{m,e}(H \setminus N)$. Then $\pi \circ \bar{\sigma} : R^m \to H \setminus N$ satisfies the relation $(\pi \circ \bar{\sigma})(u) = H . e$ for all $u \in U \subset R^m$ and hence $\bar{\sigma} : R^m \to H \subset N$. Since $j^r_0\bar{\sigma} \in T^r_{m,e}N$, it results that $\bar{\sigma}(0) = e$ and thus $j^r_0\bar{\sigma} \in T^r_{m,e}H$. It follows that $T^r_{m,e}H . j^r_0\bar{\sigma} = T^r_{m,e}H$ and hence μ is a isomorphism.

Lemma 2. $\tilde{K}^r_m$ *is canonically identified with the semi-direct product* ${}^\circ K^r_m = L^r_m \overline{\times} T^r_{m,e}K$.

Proof. First, we observe that:

$$\tilde{K}^r_m = T^r_m H \setminus (L^r_m \overline{\times} H . T^r_m N) = L^r_m \overline{\times} (T^r_m H \setminus (H . T^r_{m,e}N)) =$$

$$L^r_m \overline{\times} ((H . T^r_{m,e}H) \setminus (H . T^r_{m,e}N))$$

and ${}^\circ K^r_m = L^r_m \overline{\times} T^r_{m,e}(H \setminus N)$.

Let λ be the map:

$$\lambda : H . T^r_{m,e}H . a . j^r_0\bar{\sigma} \in H . T^r_{m,e}H \setminus H . T^r_{m,e}N \to T^r_{m,e}H . j^r_0\bar{\sigma} \in$$

$$T^r_{m,e}H \setminus T^r_{m,e}N.$$

where $a \in H$ and $j^r_0\bar{\sigma} \in T^r_{m,e}N$. If $a_1 . j^r_0\bar{\sigma}^{-1} \in T^r_m H . a . j^r_0\bar{\sigma}$, then: $a_1 . j^r_0\bar{\sigma}_1 . (a^{-1} . j^r_0\bar{\sigma}^{1}) \in T^r_m H$ and hence $j^r_0\bar{\sigma}_1 = M . j^r_0\bar{\sigma}$, where $M \in T^r_{m,e}H$. It results that $\lambda(T^r_m H . a_1 . j^r_0\bar{\sigma}_1) = T^r_{m,e}H . j^r_0\bar{\sigma}_1 = T^r_{m,e}H . j^r_0\bar{\sigma} = \lambda(T^r_{m,e} H . a . j^r_0\bar{\sigma})$, hence λ is well defined. If we consider $T^r_m H . a_i . j^r_0\bar{\sigma}_i \in T^r_m H \setminus$ $\setminus H . T^r_{m,e}N$ $(i = 1,2)$ then:

$$\lambda((T^r_m H . a_1 . j^r_0\bar{\sigma}_1) . (T^r_m H . a_2 . j^r_0\bar{\sigma}_2)) = \lambda(T^r_m H . a_1 . j^r_0\bar{\sigma}_1 . a_2 . j^r_0\bar{\sigma}_2) =$$

$$= \lambda(T^r_m H . j^r_0\bar{\sigma}_1 . a_2 . j^r_0\bar{\sigma}_2) = \lambda(j^r_0\bar{\sigma}_1 . T^r_m H . j^r_0\bar{\sigma}_2) =$$

$$= \lambda(T^r_m H . j^r_0\bar{\sigma}_1 . j^r_0\bar{\sigma}_2) = T^r_{m,e}H . j^r_0\bar{\sigma}_1 . j^r_0\bar{\sigma}_2 =$$

$$= (T^r_m H . j^r_0\bar{\sigma}_1) . (T^r_m H . j^r_0\bar{\sigma}_2) = \lambda(T^r_m H . a_1 . j^r_0\bar{\sigma}_1) . \lambda(T^r_m H . a_2 . j^r_0\bar{\sigma}_2).$$

It follows that λ is a morphism of Lie groups.

Let $T^r_m H . a . j^r_0\bar{\sigma} \in T^r_m H \,/\, (H . T^r_{m,e}N)$ such that $\lambda(T^r_m H . a . j^r_0\bar{\sigma}) =$ $= T^r_m H$. Then $T^r_{m,e}H . j^r_0\bar{\sigma} = T^r_{m,e}H$, hence $j^r_0\bar{\sigma} \in T^r_{m,e}H$. It results that $T^r_m H . a . j^r_0\bar{\sigma} = T^r_m H$ and hence λ is a isomorphism.

3. Theorem 3. ${}^H\tau^r X \to X$ *is an r-frame bundle over* X *(a G-structure of order r over* X*) with the structure group* $K^r_m = L^r_m \overline{\times} T^r_{m,e}\tilde{K}$.

Proof. Let $h = [h_0 . \bar{h}] = h_0 . \bar{h}$ be a point of ${}^{H}\tau^{r}X$, where $h_0 = (y_0 . S_0)$ and $\bar{h} = (\bar{y} . a . \bar{S})$. We suppose that $\bar{h}_0 = (\bar{y}_0 . \bar{S}_0) \in T^r_m H$, hence $\bar{y}_0 = j^r_0 \operatorname{id}_{R^m}$ and $j^r_0 \bar{\sigma}_0 = \bar{S}_0 . \bar{\sigma}_0 : R^m \to H$. Then $h * \bar{h}_0 = h$ hence $h * \bar{h}_0 = (h . \bar{h}) * \bar{h}_0 = \bar{h}_0 . \bar{h} . a^{-1} . \bar{h}_0 . a = (y_0 . \bar{y} . S_0 . (a . \bar{S} . a^{-1} . \bar{S}_0 . a . \bar{y}^{-1} . y_0^{-1})) = (y_0 . \bar{y} . S_0 . (a . \bar{S} . \bar{y}^{-1} . y_0^{-1}))$.

It follows that:

$$((a . \bar{S} . a^{-1}) . \bar{S}_0 . (a . \bar{S}^{-1} . a^{-1})) . \bar{y}^{-1} . y_0^{-1} \in T^r_m H, \text{ hence}$$
$$(a . \bar{S} . a^{-1}) . \bar{S}_0 . (a . \bar{S} . a^{-1})^{-1} \in T^r_m H.$$

Since $\bar{S}_0 \in T^r_m H$ is arbitrary, it results that $a . \bar{S} . a^{-1} \in T^r_{m,e} N$; hence $a . \bar{S} \in T^r_{m,e} N . a$. We obtain:

$$\bar{h} = (\bar{y} . a . \bar{S}) \in L^r_m \overline{\times} T^r_{m,e} N . a,$$

$$\bar{h} = a . h^{-1} = a . (\bar{y} . \bar{S}) \in (L^r_m \overline{\times} T^r_{m,e} N) . a;$$

hence

$$h^{-1} \in \{a^{-1} . {}^{0}N^r_m . \setminus a \in G\} = \{(a^{-1} . S_0^{-1} . a) \setminus a \in G,\ S_0^{-1} \in L^r_m \overline{\times} T^r_{m,e} N\}.$$

On the other side:

$$\bar{h} = h_0 . \bar{h} = (y_0 . \bar{y}, S_0 . (S_0^{-1} . a . \bar{y}^{-1} . y_0^{-1})) =$$
$$= (j^r_0 \varphi \circ \bar{\varphi}, j^r_0 \sigma_0 . (\bar{\sigma}^1_0 . a . \bar{\varphi}^{-1} . \varphi_0^{-1})).$$

Then the mapping:

$$(u, H . g) \in R^m \times H \setminus G \xrightarrow{\psi} (\sigma_0 \circ \varphi_0 \circ \bar{\varphi})(u) . \sigma_0^{-1}(u) . g \in X$$

is an equivariant diffeomorphism.

Let us consider a neighbourhood U of $0 \in R^m$ and a neighbourhood coordinate V around $H . g \in H \setminus G$ (denoting dim $H \setminus G$ by p, V may be identified to an open set W of R^p with $0 \in W$). It follows that $\varphi : U \times V \subset R^m \times R^p \to X$ is a diffeomorphism. Taking the r-jets, we deduce that h is a r-frame on X. In the rest, the proof is similar to that of a previous paper [5].

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GROUPES DE TRANSFORMATIONS DANS LA THÉORIE DES QUASIGROUPES

PAR

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On sait qu'un quasigroupe $(Q;\ .\ ,\ /\ ,\ \setminus)$ s'appelle quasigroupe topologique si Q est un espace topologique et ses applications „.", „/" et „\\" sont des applications continues. Puisque l'application $s : Q \times Q \to Q \times Q$ définie par $s(x, y) = (y, x)$ est un homéomorphisme, il résulte immédiatement que les parastrophiques d'un quasigroupe topologique sont aussi des quasigroupes topologiques. Ce fait nous assure que toutes les translations à gauche et à droite associées à chacune des opérations binaires définissant la structure du quasigroupe sont des homéomorphismes. Si $A \subset Q$ est ouvert et $B \subset Q$ quelconque, AB, BA, A/B, B/A, $A \setminus B$ et $B \setminus A$ sont des ouverts de Q comme réunion d'ensembles ouverts. Il résulte immédiatement que les applications qui définissent la structure du quasigroupe sont des applications ouvertes. On peut prouver:

Lemme 1. *Pour tout quasigroupe topologique les applications* $(x, y) \xrightarrow{\alpha} (xy, y)$, $(x, y) \xrightarrow{\beta} (x / y, y)$, $(x, y) \xrightarrow{\gamma} (x, xy)$ *et* $(x,y) \xrightarrow{\delta} (x, x \setminus y)$ *de* $Q \times Q$ *dans lui-même sont des homéomorphismes.*

Démonstration. L'assertion du lemme résulte immédiatement compte tenu du fait que ces applications sont continues et satisfont aux conditions: $\alpha \circ \beta = \beta \circ \alpha = \gamma \circ \delta = \delta \circ \gamma = \operatorname{id}_{Q \times Q}$.

En utilisant la lemme 1 nous pouvons prouver:

Proposition 1. *Soit Q un quasigroupe topologique qui est T_1-espace. Si $A \subset Q$ est fermé (resp. compact) et $K \subset Q$ est compact, alors KA est fermé (resp. compact) de Q.*

Démonstration. Considérons $K \times Q \xrightarrow{pr_2} Q$. K étant compact, d'après la conséquence 5 du théorème 1 de [5], p. 159, pr_2 est propre. Comme chaque application propre est fermée (voir prop. 1 de [5], p. 152) et γ est un homeomorphisme, il résulte que $pr_2 \circ \gamma$ est fermée. Donc $KA = (pr_2 \circ \gamma)(K \times A)$ est fermé.

Remarquons le fait que cette propriété reste vraie aussi quand les quasigroupes topologiques ne sont pas des T_1-espaces, si nous changeons la compacité avec la quasicompacité.

En utilisant ce résultat et la conséquence 2 du théorème 1 de [5], p. 158 nous obtenons: