Corollaire 3. *Soit Q une boucle topologique localement compacte. Le groupe L_Q muni de l'une de topologies : c.o.-topologie, g-topologie ou bien l.i.s.g.-topologie, est localement compact si et seulement si il est un espace de Baire.*

Démontrons maintenant :

Proposition 12. *Soit Q une boucle topologique dont l'espace sousjacent est une variété connexe. Le groupe L_Q^* est localement compact si et seulement si* $\dim L_Q < +\infty$ *ou bien, la suite des dimensions des ensembles $A_{[\varepsilon_1,\dots,\varepsilon_n]}$ est bornée.*

Démonstration. Supposons que L_Q^* est localement compact. D'après le corollaire 3 du théorème 9 de [9], p. 69, il résulte qu'il est un groupe de Lie, donc $\dim L_Q$ est finie. Mais alors la suite de dimensions des ensembles $A_{[\varepsilon_1,\dots,\varepsilon_n]}$ est bornée. Réciproquement, supposons que $\dim L_Q$ est finie. Ceci signifie que la suite des dimensions des ensembles $A_{[\varepsilon_1,\dots,\varepsilon_n]}$ est bornée. Alors, chaque sous-ensemble compact, metrisable de L_Q^* est de dimension finie. Conformément au corollaire 7. 3 de [7], p. 645, L_Q^* est un groupe de Lie, donc L_Q^* est un groupe topologique localement compacte.

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CHERN FORMS AND *H*-PROJECTIVE CURVATURE OF COMPLEX MANIFOLDS

BY

A. NEAGU and V. OPROIU

1. Introduction. It is well known that the Chern classes of a complex vector bundle E are real cohomology classes (in fact they are integer cohomology classes).

Using the real representation of a complex vector bundle we get some real closed differential forms such that their de Rham cohomology classes generate the Chern ring of the given complex vector bundle E. These forms are obtained using the complex structure operator F on E and the curvature forms of a real connection ∇ on E such that $\nabla F = 0$. We call these forms the Chern forms of ∇.

Now, let M be a complex manifold and let E be the tangent bundle of M. Let ∇ be a real torsion free connection on M such that $\nabla F = 0$. Our main result is a formula relating the Chern forms of ∇ to the H-projective curvature tensor field of ∇. Then, if on M there exists a H-projectively flat connection, the Chern forms of this connection are powers of the first Chern form. Particularly, if M is a Kaehler manifold of constant holomorphic curvature, the Chern classes of M are powers of the first Chern class. Thus, this is an extension to Kaehler manifolds, of Chern's result [3] concerning the vanishing of the Pontrjagin classes of a Riemannian manifold of constant curvature.

2. The Chern forms of a complex vector bundle. Let $\pi : E \to M$ be a differentiable complex vector bundle over the (real or complex) manifold M. Denote by m the complex dimension and by $n = 2m$ the real dimension of the fibre of E. The multiplication by the complex unity i on E (as a complex vector bundle) induces an operator F on E (as a real vector bundle) such that $F^2 = -I$ where I is the identity operator. Let $E^c = E \otimes C$ be the complexification of E. Then E can be identified with the vector subbundle of E^c consisting of all vectors of type (1, 0). Namely, $E \subset E^c$ is the vector subbundle of all vectors of the form $u - \mathrm{i}.\, Fu$, where $u \in E$ (real).

Let ∇ be a real linear connection on E, preserving the complex structure of E, i.e. $\nabla F = 0$. Then ∇ can be extended usually to a connection ∇^c on E^c and the property $\nabla F = 0$ of ∇ implies that the covariant derivatives of the sections in $E \subset E^c$ are also sections in E. Let $(u_1, u_2, \dots, u_m, Fu_1, Fu_2, \dots, Fu_m)$ be a local frame of E over $U \subset M$. From now on we shall denote $Fu_\alpha = u_{\alpha^*}$; $\alpha = 1, 2, \dots, m$. Then :

$$w_\alpha = u_\alpha - \mathrm{i}\,.\,u_{\alpha^*}\,;\ \alpha = 1,\ 2,\ \ldots,\ m$$

is a local frame of $E \subset E^c$.

Let R be the curvature tensor field of ∇ and let R^c be the curvature tensor field of ∇^c. Then:

$$R^c(X, Y)w_\alpha = (\nabla^c_X \nabla^c_Y - \nabla^c_Y \nabla^c_X - \nabla^c_{[X,Y]})(u_\alpha - \mathrm{i}\,.\,u_{\alpha^*}) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X -$$
$$- \nabla_{[X,Y]})\,u_\alpha - \mathrm{i}\,.\,(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\,u_{\alpha^*} = R(X, Y)u_\alpha - \mathrm{i}\,.\,R(X, Y)u_{\alpha^*},$$

for any real vector fields X, Y over M.

Since $\nabla F = 0$ it follows easily that $FR(X, Y) = R(X, Y)F$. Let:

$$\begin{bmatrix} \Omega^\alpha_\beta & -\Omega^{\alpha^*}_\beta \\ \Omega^{\alpha^*}_\beta & \Omega^\alpha_\beta \end{bmatrix} \quad (\alpha,\ \beta = 1,\ 2,\ \ldots,\ m)$$

be the matrix of FR with respect to the local frame $(u_1, u_2, \ldots, u_m, u_{1^*}, u_{2^*}, \ldots, u_{m^*})$ i.e.:

$$FR(X, Y)u_\alpha = \sum_{\beta=1}^{m} (\Omega^\beta_\alpha(X, Y)u_\beta + \Omega^{\beta^*}_\alpha(X, Y)u_{\beta^*})$$

$$(FR)(X, Y)\,u_{\alpha^*} = (FR)(X, Y)\,Fu_\alpha = F\,(FR)(X, Y)\,u_\alpha =$$

$$= \sum_{\beta=1}^{m} (-\Omega^{\beta^*}_\alpha(X, Y)u_\beta + \Omega^\beta_\alpha(X, Y)u_{\beta^*}).$$

The curvature matrix Φ of the restriction of ∇^c to $E \subset E^c$, with respect to the local frame $(w_1, \ldots, w_m)$ is given by:

$$\sum_{\beta=1}^{m} \Phi^\beta_\alpha(X, Y)\,w_\beta = R^c(X, Y)\,w_\alpha = R(X, Y)\,u_\alpha - \mathrm{i}R(X, Y)\,u_{\alpha^*} =$$

$$= -(FR)(X, Y)\,u_{\alpha^*} - \mathrm{i}\,(FR)(X, Y)u_\alpha = \sum_{\beta=1}^{m} (\Omega^{\beta^*}_\alpha(X, Y)\,u_\beta - \Omega^\beta_\alpha(X, Y)\,u_{\beta^*} -$$

$$- \mathrm{i}\Omega^\beta_\alpha(X, Y)\,u_\beta - \mathrm{i}\Omega^{\beta^*}_\alpha(X, Y)\,u_{\beta^*}) = \sum_{\beta=1}^{m} (\Omega^{\beta^*}_\alpha(X, Y)\,(u_\beta - \mathrm{i}u_{\beta^*}) -$$

$$- \mathrm{i}\Omega^\beta_\alpha(X, Y)\,(u_\beta - \mathrm{i}u_{\beta^*})) = \sum_{\beta=1}^{m} (\Omega^{\beta^*}_\alpha(X, Y) - \mathrm{i}\Omega^\beta_\alpha(X, Y))w_\beta$$

Thus:

$$\Phi^\beta_\alpha = \Omega^{\beta^*}_\alpha - \mathrm{i}\Omega^\beta_\alpha = \frac{1}{\mathrm{i}}(\Omega^\beta_\alpha + \mathrm{i}\Omega^{\beta^*}_\alpha)$$

On the other hand, the Chern forms of the of the complex vector bundle E are determined as the homogeneous forms in the expression:

$$\det\left(I + \frac{\mathrm{i}}{2\pi}\,\Phi\right).$$

Suppose that ∇ has the property that a Hermitian metric given on E is paralel with respect to ∇. If $(u_1, \ldots, u_m, u_{1^*}, \ldots, u_{m^*})$ is a local frame orthonormal with respect to this metric then the matrix Φ has the property:

$${}^t\Phi = -\overline{\Phi}$$

and then Chern forms defined above are real forms. Thus the Chern forms are essentially determined by the matrix:

$$\mathrm{i}\Phi = \Omega + \mathrm{i}\Omega^*$$

and the invariant polynomials obtained from:

$$\det\,(I + tA)\,;\ A \in gl(m, \mathbf{C})$$

generating the algebra of the invariant polynomials. We denoted by Ω the $m \times m$ matrix with entries the two-forms Ω^β_α and by Ω^* the $m \times m$ matrix with entries the two-forms $\Omega^{\beta^*}_\alpha$. On the other hand the algebra of the invariant polynomials is also generated by the polynomials:

$$\mathrm{trace}\ A^p\,;\ A \in gl(m, \mathbf{C})\,;\ p = 0,\ 1,\ \ldots,\ m\,.$$

These polynomials can be used to construct some closed forms which are also called the Chern forms of ∇.

Theorem 1. *The Chern forms trace $(\mathrm{i}\Phi)^p$ are, up to a constant factor, the forms:*

$$\mathrm{trace}\begin{bmatrix} \Omega & -\Omega^* \\ \Omega^* & \Omega \end{bmatrix}^p,$$

i.e. they can be constructed using the matrix of FR, where F is the complex structure operator on E and R is the curvature of ∇.

Proof. We use an orthonormal local frame $(u_1, u_2, \ldots, u_m, u_{1^*}, u_{2^*}, \ldots, u_{m^*})$ with respect to a Hermitian metric on E. Then:

$${}^t\Omega + \mathrm{i}\,{}^t\Omega^* = {}^t(\mathrm{i}\Phi) = \overline{\mathrm{i}\Phi} = \Omega - \mathrm{i}\Omega^*.$$

Thus Ω is a symmetric matrix and Ω^* is a skew-symmetric matrix. The form trace $(\mathrm{i}\,\Phi)^p$ is real since:

$$\overline{\mathrm{trace}\,(\mathrm{i}\Phi)^p} = \mathrm{trace}\,(\overline{\mathrm{i}\Phi})^p = \mathrm{trace}\,(\overline{\mathrm{i}\Phi})^p = \mathrm{trace}({}^t(\mathrm{i}\Phi))^p =$$

$$\mathrm{trace}\ {}^t((\mathrm{i}\Phi)^p) = \mathrm{trace}\,(\mathrm{i}\Phi)^p.$$

It is well known that to the complex matrix $i\Phi = \Omega + i\Omega^*$ it corresponds its real representation:

$$\begin{bmatrix} \Omega & -\Omega^* \\ \Omega^* & \Omega \end{bmatrix},$$

and to $(\mathrm{i}\,\Phi)^p = \Pi + i\,\Pi^*$:

$$\begin{bmatrix} \Pi & -\Pi^* \\ \Pi^* & \Pi \end{bmatrix} = \begin{bmatrix} \Omega & -\Omega \\ \Omega^* & \Omega \end{bmatrix}^p.$$

Then:

$$\operatorname{trace}(i\,\Phi)^p = \operatorname{trace}(\Pi + i\Pi^*) = \operatorname{trace}\Pi =$$
$$= \frac{1}{2}\operatorname{trace}\begin{vmatrix} \Pi & -\Pi^* \\ \Pi^* & \Pi \end{vmatrix} = \frac{1}{2}\operatorname{trace}\begin{vmatrix} \Omega & -\Omega^* \\ \Omega^* & \Omega \end{vmatrix}^p.$$

Since the Chern forms do not depend on the local frame used for their local representation, it follows that we can use the matrices associated to FR to get them.

The result obtained above can be extended to the case of a linear connection on E which does not necessarilly satisfy the condition of paralelism for a given Hermitian metric on E. In this case the imaginary part of the form trace $(i\ \Phi)^p$ is an exact form on M and the corresponding p-Chern form is given by the real part of $(i\ \Phi)^p$. Then the p-Chern form is given by the trace of the p-power of the matrix associated with FR in an arbitrary local frame of E.

3. The Chern forms and the H-projective curvature tensor field. Let M be a complex manifold, i.e. M is a real, $n = 2m$ - dimensional manifold with an integrable almost complex structure F. Let ∇ be a torsion free linear connection on M, such that $\nabla F = 0$. If $R_{kji}^{\ \ \ h}$, $R_{ji} = R_{kji}^{\ \ \ k}$ are the local expressions of the curvature tensor field and the Ricci tensor field of ∇, then the H-projective curvature tensor field $P_{kji}^{\ \ \ h}$ of ∇ is defined by:

$$R_{kji}^{\ \ \ h} = P_{kji}^{\ \ \ h} + S_{kji}^{\ \ \ h}. \tag{1}$$

where:

$$S_{kji}^{\ \ \ h} = -\delta_k^h P_{ji} + \delta_j^h P_{ki} + \delta_i^h (P_{kj} - P_{jk}) + F_k^h P_{js} F_i^s - F_j^h P_{ks} F_i^s -$$
$$- P_{kt} F_j^t F_i^h + P_{jt} F_k^t F_i^h$$

and:

$$P_{ji} = -\frac{1}{n+2}\left\{R_{ji} + \frac{2}{n-2} O_{ji}^{ts}(R_{ts} + R_{st})\right\}; \quad O_{ji}^{ts} = \frac{1}{2}(\delta_j^t \delta_i^s - F_j^t F_i^s)$$

(see [4]).

It is known that the tensor field P is invariant under a H-projective change of torsion free connections, given by:

$$\bar{\Gamma}_{ji}^h = \Gamma_{ji}^h + \delta_j^h p_i + \delta_i^h p_j + F_j^h q_i + F_i^h q_j. \tag{2}$$

where $q_j = -p_t F_j^t$.

We intend to expres the Chern forms of the connexion ∇ with the help of the H-projective curvature tensor field of ∇. For this we need some auxiliary results.

Proposition 2.

$$R_{j_1 j_2 i_2}^{\ \ \ \ \ \ \ i_1} \cdot S_{j_3 j_4 i_1}^{\ \ \ \ \ \ \ i_2} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4} =$$
$$= \frac{1}{n+2} R_{j_1 j_2 i_1}^{\ \ \ \ \ \ \ i_1}(R_{j_3 j_4} - R_{j_4 j_3}) \cdot dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4} + \tag{3}$$
$$+ \frac{1}{n+2} F_{i_2}^{i_1} R_{j_1 j_2 i_1}^{\ \ \ \ \ \ \ i_2}(R_{j_3 t} F_{j_4}^t - R_{j_4 t} F_{j_3}^t)\, dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4}.$$

Similar formulas are valid if R is replaced by P or S.

The proof consists in a direct computation, where it is used the Bianchi identity:

$$R_{kji}^{\ \ \ h} + R_{jik}^{\ \ \ h} + R_{ikj}^{\ \ \ h} = 0$$

and similar Bianchi identities satisfied by P and S.

On the other hand, using of the same Bianchi identity gives the following result:

$$R_{kj} - R_{jk} = -R_{kji}^{\ \ \ i}. \tag{4}$$

But, the two form $\frac{1}{2} R_{jik}^{\ \ \ k}\, dx^j \wedge dx^i$ is the local expression of the Pontrjagin form of degree two and this form is exact. Thus, we can write:

$$R_{jk} - R_{kj} = \nabla_j r_k - \nabla_k r_j. \tag{5}$$

where r is a globally defined one form, and we use the fact that ∇ is torsion free.

Proposition 3. *The connection ∇ can be transformed by a H-projective change of connections into a torsion free connection $\bar{\nabla}$ with symmetric Ricci tensor field.*

Proof. The relation between the curvature tensor fields of the connection $\nabla(\Gamma_{ji}^h)$ and $\bar{\nabla}(\bar{\Gamma}_{ji}^h)$ is (see [5]):

$$\bar{R}_{kji}^{\ \ \ h} = R_{kji}^{\ \ \ h} + \delta_j^h(\nabla_k p_i - p_k p_i + q_k q_i) - \delta_k^h(\nabla_j p_i - p_j p_i + q_j q_i) +$$
$$+ \delta_i^h(\nabla_k p_j - \nabla_j p_k) + F_i^h(\nabla_k q_j - \nabla_j q_k) + F_j^h(\nabla_k q_i - p_k q_i - q_k p_i) - \tag{6}$$
$$- F_k^h(\nabla_j q_i - p_j q_i - q_j p_i).$$

Then:

$$\bar{R}_{kji}^{\ \ \ i} = R_{kji}^{\ \ \ i} + (n+2)(\nabla_k p_j - \nabla_j p_k) \tag{7}$$

and:

$$\bar{R}_{jk} - \bar{R}_{kj} = R_{jk} - R_{kj} + (n+2)(\nabla_k p_j - \nabla_j p_k) = \nabla_j r_k - \nabla_k r_j +$$
$$+ (n+2)(\nabla_k p_j - \nabla_j p_k) = \nabla_j(r_k - (n+2)p_k) - \nabla_k(r_j - (n+2)p_j).$$

If we take: $p_k = [1/(n+2)]\, r_k$ then $\bar{R}_{jk} - \bar{R}_{kj} = 0$.

From now on we suppose that the connection ∇ has its Ricci tensor field symmetric. Thus the formula (3) becomes:

$$R_{j_1 j_2 i_2}^{\ \ \ \ \ \ \ i_1} S_{j_3 j_4 i_1}^{\ \ \ \ \ \ \ i_2} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4} =$$
$$= \frac{1}{n+2} F_{i_2}^{i_1} R_{j_1 j_2 i_1}^{\ \ \ \ \ \ \ i_2}(R_{j_3 t} F_{j_4}^t - R_{j_4 t} F_{j_3}^t)\, dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge dx^{j_4}. \tag{8'}$$

Similar formulas are obtained when the curvature tensor field is replaced by P or S.

Let Ω, Π and Θ be the matrices associated to the tensor fields FR, FP and FS respectively, i.e. :

$$\Omega^{i_1}_{i_2} = \frac{1}{2!} F^{i_1}_k R_{j_1 j_2 i_2}{}^k \, dx^{j_1} \wedge dx^{j_2},$$

$$\Pi^{i_1}_{i_2} = \frac{1}{2!} F^{i_1}_k P_{j_1 j_2 i_2}{}^k \, dx^{j_1} \wedge dx^{j_2},$$

$$\Theta^{i_1}_{i_2} = \frac{1}{2!} F^{i_1}_k S_{j_1 j_2 i_2}{}^k \, dx^{j_1} \wedge dx^{j_2}.$$

We shall denote by c_p the Chern forms of ∇, and by $\tilde{c}_p$ the forms constructed in the similar way, using the matrix Π, i.e. :

$$c_p = \operatorname{trace} \Omega^p, \ \tilde{c}_p = \operatorname{trace} \Pi^p. \tag{8}$$

Then, it can be verified by a straightforward computation, using the Bianchi identity and the commutation property of F and R; that :

$$c_1 = \frac{1}{2} F^{i_1}_k R_{j_1 j_2 i_1}{}^k \, dx^{j_1} \wedge dx^{j_2} = -\frac{1}{2} (R_{j_1 s} F^s_{j_2} - R_{j_2 s} F^s_{j_1}) \, dx^{j_1} \wedge dx^{j_2}; \ \tilde{c}_1 = 0. \tag{9}$$

Then, since F commutes also with S the formula (3) can by writen as follows :

$$\Omega \wedge \Theta = \frac{1}{n+2} \Omega \wedge c_1 = \frac{1}{n+2} c_1 \wedge \Omega.$$

where in the left hand side we have the matrix multiplication and in the right hand side we have the multiplication of the scalar form with a matrix form.

Similar formulae are obtained when Ω is replaced by Π and Θ :

$$\Pi \wedge \Theta = \frac{1}{n+2} c_1 \wedge \Pi, \quad \Theta \wedge \Theta = \frac{1}{n+2} c_1 \wedge \Theta.$$

For convenience, we shall denote $\dfrac{1}{n+2} \, c_1 = \theta$ such that :

$$\Omega \wedge \Theta = \theta \wedge \Omega, \ \Pi \wedge \Theta = \theta \wedge \Pi, \ \Theta \wedge \Theta = \theta \wedge \Theta.$$

Theorem 4. *The forms c_p and $\tilde{c}_p$ are related by the following formula :*

$$c_p = \tilde{c}_p + \binom{p}{1} \theta \wedge \tilde{c}_{p-1} + \binom{p}{2} \theta^2 \wedge \tilde{c}_{p-2} + \ldots + \theta^p. \tag{10}$$

where θ^p is the p-exterior power of θ.

Proof. We have :

$$\Omega = \Pi + \Theta,$$

and :

$$\operatorname{trace} \Pi^p = \tilde{c}_p, \ \operatorname{trace} \Omega \wedge \Pi^{p-1} = \tilde{c}_p + \theta \wedge \tilde{c}_{p-1}, \ \operatorname{trace} \Omega^2 \wedge \Pi^{p-2} = \tilde{c}_p + 2\theta \wedge \tilde{c}_{p-1} + \theta^2 \wedge \tilde{c}_{p-2}.$$

Then it can be proved by a usual induction the following formula :

$$\operatorname{trace} (\Omega^r \wedge \Pi^{p-r}) = \tilde{c}_p + \binom{r}{1} \theta \wedge \tilde{c}_{p-1} + \binom{r}{2} \theta^2 \wedge \tilde{c}_{p-2} + \ldots + \theta^r \wedge \tilde{c}_{p-r}.$$

If $r = p$ we obtain the desired result.

The obtained formula allows us to get some interesting results :

Corollary 5. *If the connection ∇ has vanishing first Chern form, then all its Chern forms are H-projectively invariant.* (compare with [1]).

Corollary 6. *If the manifold M is H-projectively flat, then its Chern classes are powers of the first Chern class.*

In fact, if M is projectively flat there exists a linear connection on M such that its H-projective curvature tensor field vanish. Then, from Theorem 4, we obtain :

$$c_p = \theta^p = \frac{1}{(n+2)^p} c_1^p.$$

Particularly, if M is a Kaehlerian manifold, then it is projectively flat iff it is of constant holomorphic curvature. Thus, for any Kaehlerian manifold of constant holomorphic curvature, the Chern classes are powers of the first Chern class (compare with [3]).

Generally, the first Chern class is not H-projectively invariant. In fact, from the formula (6) we obtain :

$$\bar{c}_1 = \frac{1}{2} F^i_h \bar{R}_{kji}{}^h \, dx^k \wedge dx^j = c_1 - \frac{1}{2} (n+2)(\nabla_k q_j - \nabla_j q_k) \, dx^k \wedge dx^j.$$

Then, c_1 remains invariant only under the H-projective transformations for which the one form $q = q_i \, dx^i = -F^j_i p_j \, dx^i$ is closed. On the other hand the condition that the connection ∇ with symmetric Ricci tensor field be H-projectively transformed into a connection with symmetric Ricci tensor field implies that the one form $p = p_k \, dx^k$ is also closed (see the formula (7)).

We should be interested in looking for the H-projective transformations on M such that the one forms $p = p_j \, dx^j$ and $q = q_j \, dx^j$ are closed.

If $q_j = -p_k F^k_j$ then the complex valued form $p + iq = p_j \, dx^j - iF^j_k p_j \, dx^k = p_j (dx^j - iF^j_k \, dx^k)$ is a form of type (1, 0). Then $d(p + iq) = (d' + d'')(p + iq) = d'(p + iq) + d''(p + iq)$ is a sum of a form of type (2, 0) and a form of type (1, 1). The condition $d(p + iq) = 0$ implies :

$$d'(p + iq) = 0, \quad d''(p + iq) = 0.$$

Since $d''(p + iq) = 0$ it follows that the coefficients of $p + iq$ are holomorphic functions on M. Thus, the forms $p = p_k \, dx^k$, $q = -F^j_k p_j \, dx^k$ are closed forms iff the form $p + iq$ is a closed form with holomorphic coefficients. If M is a compact manifold, then such a form is necessarily 0. Thus on compact complex manifolds there are not H-projective transformations of connections preserving the first Chern form.

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ON SUM THEOREMS

BY

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In this Note, we prove some general sum theorems which yields the locally finite sum theorem (Σ) in [5] as also the one on metacompact spaces due to H o d e l [5], and many others not hitherto considered, as immediate consequences. Further the approach here reveals a close relation between certain classes of continuous closed maps and sum theorems of this type.

Definition 1. *A topological property P is said to be additive if whenever each member of a nonvoid collection $\{(X_\alpha, \Gamma_\alpha), \alpha \in \Lambda\}$ of topological spaces has the property P, their (disjoint) topological sum $\oplus \{(X_\alpha, \Gamma_\alpha), \alpha \in \Lambda\}$ has the same property.*

Definition 2. *A collection $\{M_\alpha ; \alpha \in \Lambda\}$ of subsets of a topological space is said to be hereditarily closure preserving, if $C_\alpha \subseteq M_\alpha$ for each $\alpha \in \Lambda$ implies :*

$$\overline{\cup \{C_\alpha ; \alpha \in \Lambda\}} = \cup \{\overline{C}_\alpha ; \alpha \in \Lambda\},$$

where $\overline{C}$ denotes the closure of C.

Hereditarily closure-preserving family is certainly closure-preserving but not conversely.

A locally finite collection of subsets in a topological space is certainly hereditarily closure-preserving but not conversely. A closure-preserving family of closed subsets is locally finite iff it is point finite. For other definitions we refer to [3], [9], [10], [11], [13]. All topologies will be assumed to be T_2.

Theorem 1.1. (Sum Theorem A). *Let P be a topological property which is additive and invariant under continuous closed surjections. Then if $\{F_\alpha ; \alpha \in \Lambda\}$ is a hereditarily closure-preserving closed covering of a topological space (X, Γ), such that each F_α has the property P in its relative topology, then (X, Γ) has the property P.*

Proof : Let Γ_α denote the relative topology of F_α and let $F'_\alpha = \{\alpha\} \times F_\alpha$, with

$$\Gamma'_\alpha = \{\{\alpha\} \times U; \quad U \in \Gamma_\alpha\}.$$

Let

$$M = \oplus \{(F'_\alpha, \Gamma'_\alpha); \quad \alpha \in \Lambda\}$$